Generating Hypergeometric Function Identities by Gröbner Basis and Polyhedral Geometry

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1. Basic Notions

2. Interplay of Convex Polytopes, Gröbner Bases and Hypergeometric Functions

   (a) Counting the number of solutions.

   (b) Construction of solutions.

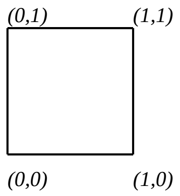
3. Projects in progress: Electronic Mathematical Formula Book and OpenXM, ...

1. Basic Notions

- $d \times n$ matrix $A = (a_1, \dots, a_n)$, $a_i \in \mathbf{Z}^d$. We regard A as the set of points $\{a_1, \dots, a_n\}$ in $\mathbf{Z}^d$.

Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$



- The ideal in $\mathbf{C}[\partial_1, \dots, \partial_n]$ generated by

$$\{\partial^u - \partial^v \mid Au = Av, \ \ u, v \in \mathbf{N}_0^n\}$$

is denoted by I_A and called the toric ideal.

Example: $I_A = \langle \partial_1\partial_4 - \partial_2\partial_3 \rangle$ for $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$

- Let D be the ring of differential operators of n variables

$$D = \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

where

$$x_i x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i, \partial_i x_j = x_j \partial_i + \delta_{ij}.$$

D acts on spaces of functions by

$$x_i \bullet F(x) = x_i F(x), \partial_i \bullet F(x) = \frac{\partial F}{\partial x_i}.$$

Example: $\partial_1 x_1 = x_1 \partial_1 + 1$, $\partial_1 x_1^2 = x_1^2 \partial_1 + 2x_1$.

- $R = \mathbf{C}(x_1, \ldots, x_n)\langle \partial_1, \ldots, \partial_n \rangle$.

Example: $\partial_1 \left(\frac{x_1}{1-x_2}\right) \partial_1 = \left(\frac{x_1}{1-x_2}\right) \partial_1^2 + \frac{1}{1-x_2} \partial_1$.

• For $f \in D$ and $w \in \mathbf{R}^n$, $\mathrm{in}_{(-w,w)}(f)$ is the sum of the highest weight terms of f with respect to $(-w, w)$. Here $-w = (-w_1, \ldots, -w_n)$ stands for $x = (x_1, \ldots, x_n)$ and w stands for ∂.
Example:

$$\mathrm{in}_{(-1,-1,1,1)}(x_1\partial_1 + x_2\partial_2 - 3) = x_1\partial_1 + x_2\partial_2 - 3.$$

$$\mathrm{in}_{(-1,0,0,-1,1,0,0,1)}(\partial_1\partial_4 - \partial_2\partial_3) = \partial_1\partial_4.$$

• A system of linear partial differential equations is a left ideal I in D or R.
The number $\dim_{\mathbf{C}(x_1,\ldots,x_n)} R/I$ is called the holonomic rank of I (multiplicity), which is denoted by $\mathrm{rank}(I)$. For $J \subseteq D$, the holonomic rank of J is that of RJ. It is known that the holonomic rank is equal to the dimension of the space of holomorphic functions at generic points.
Example:

$$n = 1, \ \mathrm{rank}(R \cdot \{x^2\partial_x^2 - 1/2\}) = 2.$$

$$n = 2, \ \text{rank}(R \cdot \{\partial_x - 1, \partial_y^2 + 1\}) = 2$$

where the solution space is spanned by $e^x \sin y$ and $e^x \cos y$.

- For a given set of points $A = \{a_1, \ldots, a_n\} \subset \mathbf{Z}^d$, a generic weight $w \in \mathbf{R}^n$ induces a triangulation of A, which is called a regular triangulation of A.

Example: For

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and $w = (1, 0, 0, 1)$, the regular triangulation induced by w is

$\{123, 234\}$:

- Let T be a regular triangulation. Put

$$C(T) = \{w \in \mathbf{R}^n \,|\, w \text{ induces the triangulation } T\},$$

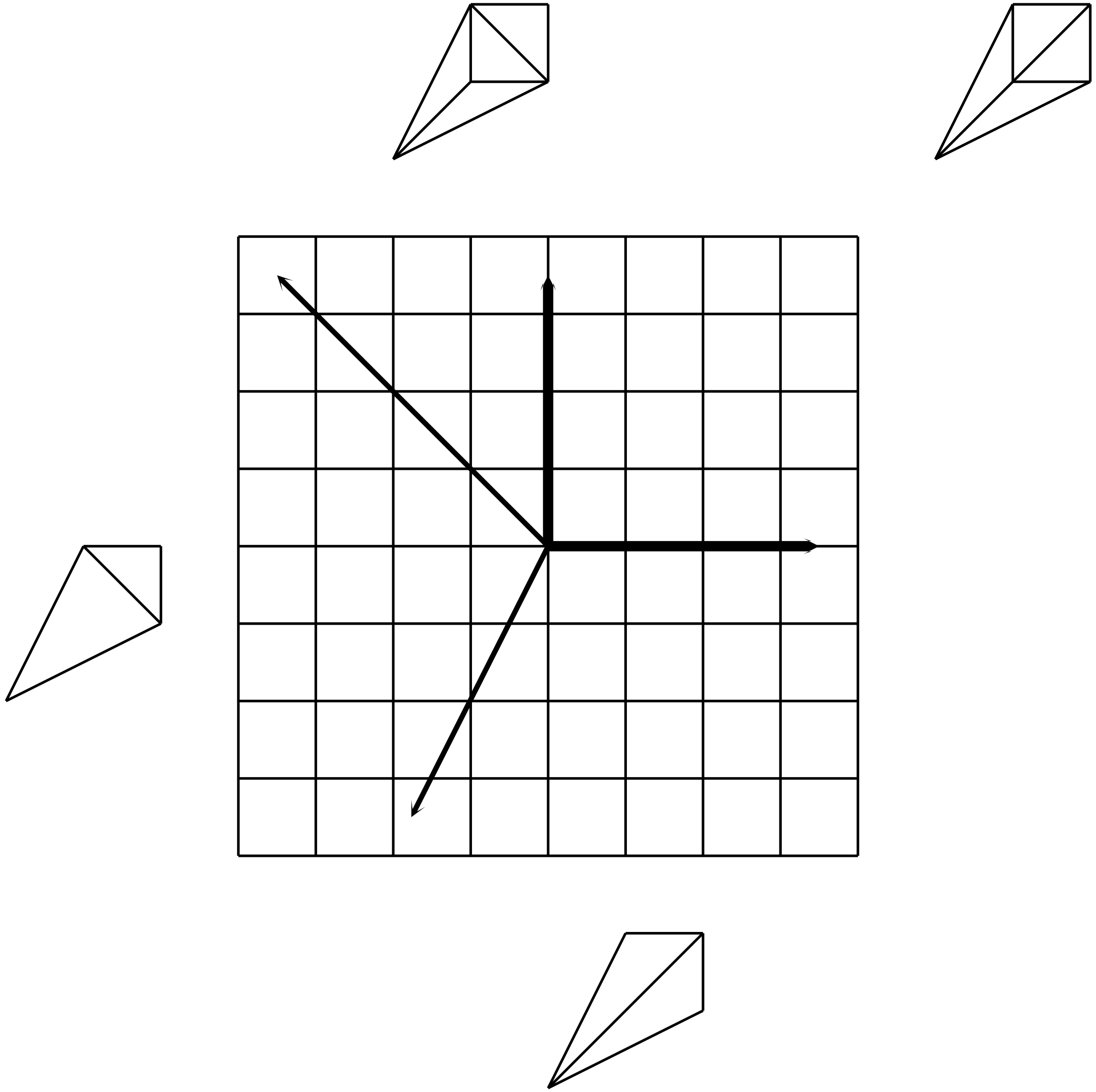
which is called the secondary cone associated to T. The collection of the secondary cones is called the secondary fan.

- Fix an ideal I in $\mathbf{C}[\partial_1, \ldots, \partial_n]$. Define $w \sim w' \in \mathbf{R}^n$ by $\mathrm{in}_w(I) = \mathrm{in}_{w'}(I)$. The division of the space of weights $\mathbf{R}^n$ by $\sim$ is called the Gröbner fan of I.

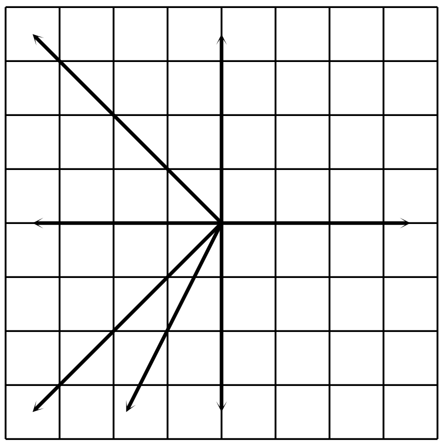
Example: For $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{pmatrix}$, I_A is generated by

$$-\partial_2\partial_5 + \partial_1\partial_3, \quad \partial_5^2 - \partial_2\partial_4$$

The Grönber fan consists of 7 maximal dimensional cones. Secondary fan consists of 4 maximal dimensional cones.



$\mathrm{Im}\,{}^tA\oplus$



- Assume that $\mathbf{Q}$-row span of $a_1, \ldots, a_n$ contains the vector $(1, \ldots, 1)$. For $\beta \in \mathbf{C}^d$, let $H_A(\beta)$ be the ideal in D generated by I_A and $\sum_{i=j}^n a_{ij} x_i \partial_i - \beta_j$, $j = 1, \ldots, d$ $(A\theta - \beta,\ \theta_i = x_i \partial_i)$. The ideal is called the GKZ hypergeometric ideal. It was introduced by Gelfand, Kapranov, Zelevinsky in 1989.

Example: For $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{pmatrix}$ and $\beta = (-1, 0, 0)$, I_A is generated by

$$-\partial_2 \partial_5 + \partial_1 \partial_3, \quad \partial_5^2 - \partial_2 \partial_4$$

$H_A(\beta)$ is generated by I_A and

$$x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + x_5\partial_5 + 1,\ x_1\partial_1 + x_2\partial_2 - x_4\partial_4,$$

$$x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_4\partial_4$$

Example: For $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$ and $\beta = (\beta_1, \beta_2, \beta_3)$, $H_A(\beta)$ is generated by $\partial_1\partial_4 - \partial_2\partial_3$ and

$x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 - \beta_1,\ x_2\partial_2 + x_4\partial_4 - \beta_2, x_3\partial_3 + x_4\partial_4 - \beta_3$

There are two linearly independent solutions of the form

$$\begin{pmatrix} x^{\alpha}(1 + O(x_1x_4/x_2x_3)) \\ x^{\alpha'}(1 + O(x_1x_4/x_2x_3)) \end{pmatrix} = \begin{pmatrix} x^{\alpha}F(*,*,*;x_1x_4/x_2x_3) \\ x^{\alpha'}F(*,*,*;x_1x_4/x_2x_3) \end{pmatrix},$$

$\alpha = (0, \beta_1 - \beta_3, \beta_1 - \beta_2, \beta_2 + \beta_3 - \beta_1), \alpha' = (\beta_1 - \beta_2 - \beta_3, \beta_2, \beta_3, 0),$

where $F(a, b, c; z)$ is the Gauss hypergeometric function

$$F(a, b, c; x) = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(1)_n(c)_n} x^n,$$

$$(a)_n = a(a + 1) \cdots (a + n - 1).$$

(This solution is associated to the triangulation $\{123, 243\}$.)

2. Interplay of Convex Polytopes, Gröbner Bases, and Hypergeometric Functions.

• (GKZ, 1989). There is a correspondence between a regular triangulation and a set of series solution basis of $H_A(\beta)$ for generic β.

Example: See figure in a different sheet.

• (GKZ, 1989). The secondary cone of a regular triangulation T is the domain of convergence of the set of series solutions associated to T.

• (Sturmfels, 1990). The secondary fan of A is refined by the Gröbner fan of I_A.

• (B.Hubert and R.Thomas, 1998). A fast algorithm to get the Gröbner fan of I_A.

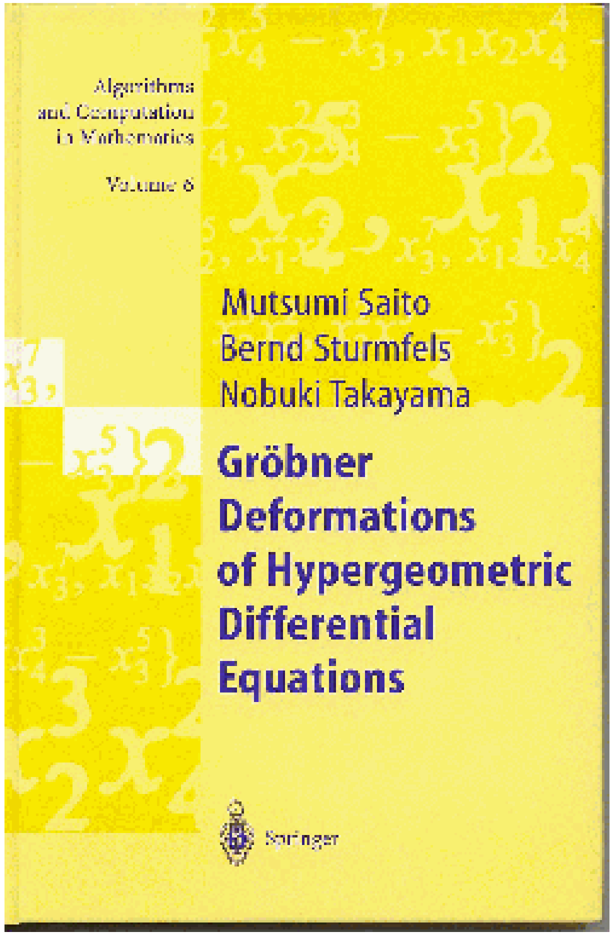
• (GKZ, 1989, Adolphson, 1994, SST=(Saito, Sturmfels, Takayama), 1998). The volume of A is equal to the holonomic rank of $H_A(\beta)$ for generic β, which is equal to the multiplicity of the zero-dimensional ideal $R \cdot H_A(\beta)$.

• (Batyrev 1993, ...). GKZ hypergeometric systems for (four dimensional) Reflexive polytopes describe prepotentials (=period maps =generating functions of quantum cohomology groups).

• (Hosono, Lian, S.T.Yau, 1996, 97) Construction of series solutions of GKZ systems for reflexive polytopes and a degenerate β. The main idea of the construction is the use of $\mathrm{in}_{(-w,w)}(I_A)$.

• (SST, 1998). The rank of $H_A(\beta)$ is equal to the rank of $\mathrm{in}_{(-w,w)}(H_A(\beta))$ ($\simeq D \cdot \{A\theta - \beta, \mathrm{in}_w(I_A)\}$) for any w and β.

• (SST, 2000). A construction algorithm of series solution basis of $H_A(\beta)$ for any β. The main idea of the construction is the use of $\mathrm{in}_{(-w,w)}(H_A(\beta))$.

Projects after 

1. e-formula book
2. Efficiency
3. Rank gap, isomorphism
4. Kummer type identity
5. Numerical evaluation
6. Connection formula and global behavior
7. Non-homogeneous I_A and irregular singularity

Projects in Progress

1. e-Bateman project (Electronic mathematical formula book)
First Step:
Mathematical formula book, e.g., Erdelyi: Higher Transcendental Functions

Formula (type A)

The solution space of the ordinary differential equation

$$x(1-x)\frac{d^2 f}{dx^2} - (c - (a+b+1)x)\,\frac{df}{dx} - abf = 0$$

is spanned by

$$F(a,b,c;x) = 1 + O(x),\ \ x^{1-c}F(a,b,c;x) = x^{1-c} + O(x^{2-c}))$$

when $c \notin \mathbf{Z}$.

Formula (type B)

$$F(a_1,a_2,b_2;z)\,F(-a_1,-a_2,2-b_2;z)$$

$$
\begin{aligned}
&+ \quad \frac{z}{e_2} F'(a_1, a_2, b_2; z)\, F(-a_1, -a_2, 2-b_2; z) \\
&- \quad \frac{z}{e_2} F(a_1, a_2, b_2; z)\, F'(-a_1, -a_2, 2-b_2; z) \\
&- \quad \frac{a_1 + a_2 - e_2}{a_1 a_2 e_2} z^2\, F'(a_1, a_2, b_2; z)\, F'(-a_1, -a_2, 2-b_2; z) \\
&= \quad 1
\end{aligned}
$$

where $e_2 = b_2 - 1$ and $a_1, a_2, e_2, e_2 - a_2 \notin \mathbf{Z}$.
(generalization of $\sin^2 x + \cos^2 x = 1$.)

We are trying to generate or verify type A formulas and type B formulas for GKZ hypergeometric systems.

	type A	type B
Algorithm	OK (SST book)	in progress
Implementation	partially done	NO

OpenXM version of Risa/Asir: `dsolv_starting_term()`

The OpenXM project is a child of the OpenMath project. OpenXM is taking a bottom-up approach; we are building applications for hypergeometric functions and parallel and distributed computation. OX-RFC's have been revised by experiments in these areas. Control integration is an important topic of OpenXM.

Our OpenXM servers `ox_asir`, `ox_sm1`, `ox_tigers`, `ox_gnuplot`, `ox_mathematica`, `OpenMathproxy` (JavaClasses), `ox_m2`, (`ox_phc`) are used to generate, verify and (present formulas) of type A for GKZ hypergeometric systems.

The life is too short to write an efficient system for hypergeometric functions from scratch. Integration of software systems is necessary.