Let $F(a,b,c;z)$ be the Gauss hypergeometric function

$$F(a,b,c;x) = \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(1)_n(c)_n} x^n,$$

$$(a)_n = a(a+1)\cdots(a+n-1).$$

Everybody knows some hypergeometric functions very well:
Example:

$$\arcsin x = xF(1/2, 1/2, 3/2; x^2)$$

Example:

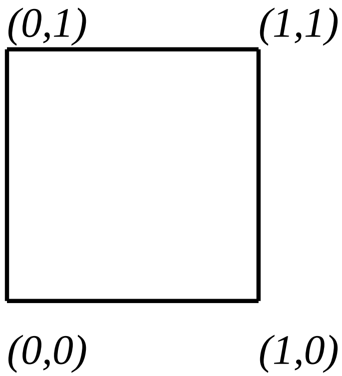
$$-\frac{x_2}{2x_1} \pm \left(\frac{x_2}{2x_1} - \sum_{m=0}^{\infty} \frac{1}{m+1} \binom{2m}{m} \frac{x_1^m x_3^{m+1}}{x_2^{2m+1}} \right) \quad = \quad \frac{-x_2 \pm \sqrt{x_2^2 - 4x_1x_3}}{2x_1}.$$

This is the familiar *quadratic formula* for expressing the two zeros of a quadratic polynomial $p(z) = x_1z^2 + x_2z + x_3$ in terms of its three coefficients. The quadratic formula satisfies the GKZ hypergeometric system

$$(\partial_1\partial_3 - \partial_2^2)f = 0, \ (x_1\partial_1 + x_2\partial_2 + x_3\partial_3)f = 0, \ (x_2\partial_2 + 2x_3\partial_3 - 1)f = 0.$$

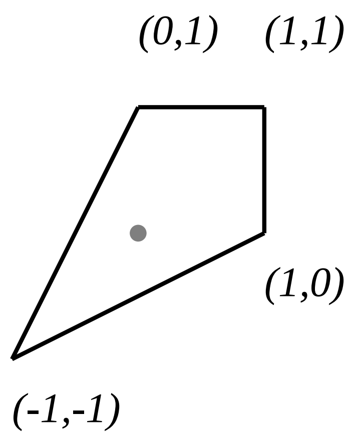
Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$



Example:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{pmatrix}$$



Example: (Left ideal and system of linear differential equations)

$$n = 1, \ \text{rank}(R \cdot \{x^2\partial_x^2 - 1/2\}) = 2.$$

The ideal is the differential equation

$$x^2\frac{\partial^2 F}{\partial x^2} - \frac{1}{2}F = 0.$$

Example:

$$n = 2, \ \text{rank}(R \cdot \{\partial_x - 1, \partial_y^2 + 1\}) = 2.$$

The ideal is the system of differential equations

$$\frac{\partial F}{\partial x} - F = 0, \frac{\partial^2 F}{\partial^2 y} + F = 0.$$

Summary of notions and theorems.

Example: 7 solutions attached to each Gröbner fan. The domain of convergence is the corresponding secondary cone.

$H_A(\beta)$ for

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & -1 & 0 \end{pmatrix}, \ \beta = (-\frac{1}{2} \quad \frac{1}{3} \quad -\frac{1}{5})$$

is generated by

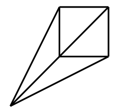
$$-\partial_2\partial_5 + \partial_1\partial_3, \quad \partial_5^2 - \partial_2\partial_4$$

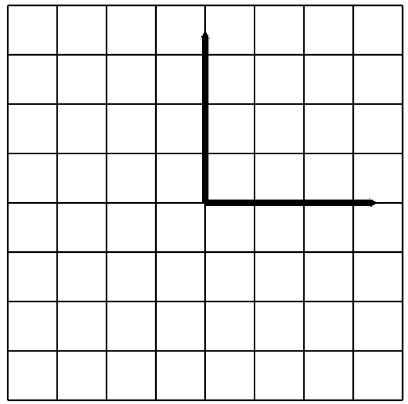
$$x_1\partial_1 + x_2\partial_2 + x_3\partial_3 + x_4\partial_4 + x_5\partial_5 + 1/2, \ x_1\partial_1 + x_2\partial_2 - x_4\partial_4 - 1/3,$$

$$x_1\partial_1 + x_2\partial_2 + x_3\partial_3 - x_4\partial_4 - 1/5$$

(We will denote by $A\theta - \beta$ the last 3 equations. Put $\theta = {}^t(x_1\partial_1, \ldots, x_5\partial_5)$.)

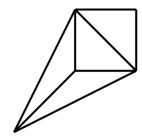
$$\begin{pmatrix} x_5^{-\frac{7}{10}} x_3^{-\frac{2}{15}} x_2^{\frac{1}{3}} \\ x_5^{-\frac{1}{30}} x_4^{-\frac{1}{3}} x_3^{-\frac{2}{15}} \\ x_5^{-\frac{5}{6}} x_2^{\frac{1}{5}} x_1^{\frac{2}{15}} \\ x_5^{-\frac{13}{30}} x_4^{-\frac{1}{5}} x_1^{\frac{2}{15}} \end{pmatrix} \left(1 + O\left(\frac{x_1 x_3}{x_2 x_5}, \frac{x_2 x_4}{x_5^2}\right)\right)$$

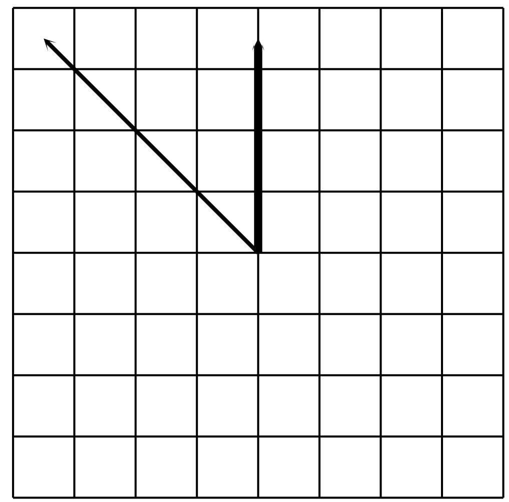




$$\mathrm{in}_{(-w,w)}(H_A(\beta)) = D \cdot \{A\theta - \beta, \theta_1\theta_3, \theta_2\theta_4\}$$

where $\theta_i = x_i \partial_i$.

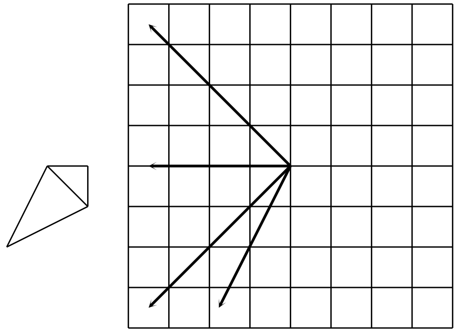




$$\begin{pmatrix} x_3^{-\frac{5}{6}} x_2^{\frac{31}{30}} x_1^{-\frac{7}{10}} \\ x_5^{-\frac{13}{30}} x_4^{-\frac{1}{5}} x_1^{\frac{2}{15}} \\ x_5^{-\frac{1}{30}} x_4^{-\frac{1}{3}} x_3^{-\frac{2}{15}} \\ x_5^{-\frac{31}{30}} x_3^{\frac{1}{5}} x_1^{\frac{1}{3}} \end{pmatrix} \left(1 + O\left(\frac{x_2 x_5}{x_1 x_3}, \frac{x_2 x_4}{x_5^2}, \frac{x_1 x_3 x_4}{x_5^3}\right)\right)$$

$$\mathrm{in}_{(-w,w)}(H_A(\beta)) = D \cdot \{A\theta - \beta, \theta_2\theta_5, \theta_2\theta_4, \theta_1\theta_3\theta_4\}$$

where $\theta_i = x_i \partial_i$.

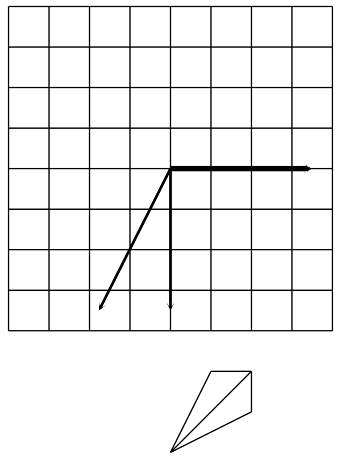


$$\begin{pmatrix} x_3^{-\frac{5}{6}} x_2^{\frac{31}{30}} x_1^{-\frac{7}{10}} \\ x_4^{-\frac{31}{90}} x_3^{-\frac{13}{90}} x_1^{-\frac{1}{90}} \\ x_4^{-\frac{91}{90}} x_3^{-\frac{73}{90}} x_1^{-\frac{61}{90}} x_5^2 \\ x_4^{-\frac{61}{90}} x_3^{-\frac{43}{90}} x_1^{-\frac{31}{90}} x_5 \end{pmatrix} \left(1 + O\left(\frac{x_2 x_5}{x_1 x_3}, \frac{x_2 x_4}{x_5^2}, \frac{x_5^3}{x_1 x_3 x_4}\right)\right)$$

$$\begin{pmatrix} x_4^{-\frac{1}{90}} x_3^{-\frac{73}{90}} x_1^{-\frac{61}{90}} x_2 \\ x_4^{-\frac{31}{90}} x_3^{-\frac{13}{90}} x_1^{-\frac{1}{90}} \\ x_3^{-\frac{5}{6}} x_2^{\frac{31}{30}} x_1^{-\frac{7}{10}} \\ x_4^{-\frac{61}{90}} x_3^{-\frac{43}{90}} x_1^{-\frac{31}{90}} x_5 \end{pmatrix} \left(1 + O\left(\frac{x_2 x_5}{x_1 x_3}, \frac{x_5^2}{x_2 x_4}, \frac{x_2^2 x_4}{x_1 x_3 x_5}\right)\right)$$

$$\begin{pmatrix} x_4^{-\frac{31}{90}} x_3^{-\frac{13}{90}} x_1^{-\frac{1}{90}} \\ x_4^{\frac{29}{90}} x_3^{-\frac{133}{90}} x_1^{-\frac{121}{90}} x_2^2 \\ x_3^{-\frac{5}{6}} x_2^{\frac{31}{30}} x_1^{-\frac{7}{10}} \\ x_4^{-\frac{1}{90}} x_3^{-\frac{73}{90}} x_1^{-\frac{61}{90}} x_2 \end{pmatrix} \left(1 + O\left(\frac{x_2 x_5}{x_1 x_3}, \frac{x_5^2}{x_2 x_4}, \frac{x_1 x_3 x_5}{x_2^2 x_4}, \frac{x_2^3 x_4}{x_1^2 x_3^2}\right)\right)$$

$$\begin{pmatrix} x_4^{-\frac{11}{12}} x_2^{-\frac{103}{60}} x_1^{\frac{17}{15}} x_3 \\ x_4^{-\frac{7}{20}} x_2^{-\frac{1}{60}} x_3^{-\frac{2}{15}} \\ x_4^{-\frac{17}{20}} x_3^{\frac{13}{15}} x_2^{-\frac{91}{60}} x_1 \\ x_4^{-\frac{5}{12}} x_2^{-\frac{13}{60}} x_1^{\frac{2}{15}} \end{pmatrix} \left(1 + O\left(\frac{x_2 x_5}{x_1 x_3}, \frac{x_5^2}{x_2 x_4}, \frac{x_1 x_3 x_5}{x_2^2 x_4}, \frac{x_1^2 x_3^2}{x_2^3 x_4}\right)\right)$$



$$\begin{pmatrix} x_2^{-\frac{43}{60}} x_4^{-\frac{11}{12}} x_1^{\frac{2}{15}} x_5 \\ x_2^{-\frac{31}{60}} x_4^{-\frac{17}{20}} x_3^{-\frac{2}{15}} x_5 \\ x_4^{-\frac{5}{12}} x_2^{-\frac{13}{60}} x_1^{\frac{2}{15}} \\ x_4^{-\frac{7}{20}} x_2^{-\frac{1}{60}} x_3^{-\frac{2}{15}} \end{pmatrix} \left(1 + O\left(\frac{x_1 x_3}{x_2 x_5}, \frac{x_5^2}{x_2 x_4}\right)\right)$$

- Consider n points

$$\begin{pmatrix} a_1 \\ w_1 \end{pmatrix}, \dots, \begin{pmatrix} a_n \\ w_n \end{pmatrix}$$

in $\mathbf{R}^{d+1}$. Consider the projection

$$p \; : \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \\ y_{n+1} \end{pmatrix} \mapsto \begin{pmatrix} y_1 \\ y_2 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{pmatrix}.$$

Projecting down the lower convex hull of the n points by the projection p, we obtain a triangulation of A.

Let Δ_p be the set of points

$$e_0 = 0, e_1, \ldots, e_p \in \mathbf{Z}^p.$$

We denote by $\Delta_p \times \Delta_q$ the set of points $e_i \times e_j$ put on the hyperplane $x_1 = 1$.

Example:

$$\Delta_1 \times \Delta_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$