Goal of today: Integration functor and de Rham cohomology groups in one dimensional case.

Let K be $\mathbf{C}$. Let $f = \sum_{j=0}^{m} f_j x^j$ be a polynomial in one variable x. $m = \mathrm{ord}_w(f)$, $w = (1)$.

$$x^i f(x) = \sum_{j=0}^{m} f_j x^{j+i}$$

$K[x]_k$ is the K vector space of the polynomials of which degree is less than or equal to k. Define a linear map (by the correspondence $e_i \Leftrightarrow x^i$)

$$K[x]_{k-m} \simeq K^{k-m+1} \ni e_i \mapsto \sum_{j=0}^{m} f_j e_{j+i} \in K^{k+1} \simeq K[x]_k.$$

The matrix representaion of this map is denoted by $M_k(f)$ called the Macaulay type matrix of the degree k.

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Example B1: $f(x) = x^2 + 1$.

$$e_0 \mapsto 1 \cdot (x^2+1) = e_2 + e_0, e_1 \mapsto x \cdot (x^2+1) = e_3 + e_1$$

$$M_3(f) = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$ Columns are indexed by e_0, e_1, ...

$$D = K\langle x, \partial \rangle$$

with the relation $\partial x = x\partial + 1$. $w = 1$, $\mathrm{ord}_{(-w,w)}(x^i \partial^j) = j - i$.
$f = \sum c_{ij} x^i \partial^j$ is called $(-w, w)$ homogeneous when $(j - i)$'s are the same value for all the terms.
Lemma B2: If we multiply two elements f and g in D which are $(-w, w)$ homogeneous, fg are also $(-w, w)$ homogeneous.
Proof(later). We have

$$\partial^p x^q = x^q \partial^p + pq x^{q-1} \partial^{p-1} + \frac{p(p-1)q(q-1)}{2!} x^{q-2} \partial^{p-2} + \cdots$$

Exercise B3: Prove it by induction. Since
$\mathrm{ord}_{(-w,w)}(x^{q-i} \partial^{p-i}) = (p - i) - (q - i) = p - q$, the lemma is shown in case that $f = x^p$, $g = \partial^q$ from the formula above.
General cases are reduced to this case. Q.E.D.
We denote by F_k the K vector space spanned by $x^i \partial^j$, $j - i \le k$ (the elements of which $(-w, w)$ order is less than or equal to k).
Note that $F_{-1} \subseteq xD$.

Lemma B4: Let $g \in D$ be a differential operator of $(-w, w)$ order k. For $f \in F_m \cap Dg$ and $m \geq k$, there exists $q \in F_{m-k}$ such that

$$f = qg$$

Proof(later). Since $f \in Dg$ (a left ideal generated by g), there exists q' such that $f = q'g$. Suppose that $r = \mathrm{ord}_{(-w,w)}(q') > m - k$ and $q' = q'_1 + q'_2$ where q'_1 is $(-w, w)$ homogeneous with the order r and the order of q'_2 is less than r. We decompose g in the same way as $g = g_1 + g_2$. $q'g = q'_1 g_1 + q'_1 g_2 + q'_2 g = f$. Since the top degree term is $q'_1 g_1$ which is $(-w, w)$ homogeneous, we have $q'_1 g_1 = 0$. Then $q'_1 = 0$. It is a contradiction. Q.E.D.

For an element f of D, the expression as $\sum c_{ij} x^i \partial^j$ (∂'s are collected to the right) is called the normally ordered expression and is denoted by $: f :$

Example. $: \partial x := x\partial_x + 1$.

Fix a natural number k. For $g \in D$, $\mathrm{ord}_{(-w,w)}(g) = j$, the operator g induces a linear map

$$K[\partial]_{k-j} \ni \partial^i \mapsto : \partial^i g : |_{x=0} \in K[\partial]_k$$

The matrix representation of this map is called the Macaulay type matrix for restriction of degree k and is denoted by $M_k(g)$.

Example B5: $g = x\partial^2 + x\partial$, $k = 2$.

$$\begin{array}{rcl} 1 & \mapsto & : x\partial^2 + x\partial : |_{x=0} = 0 \\ \partial & \mapsto & : \partial g : |_{x=0} = x\partial^3 + \partial^2 + x\partial^2 + \partial \,|_{x=0} = \partial^2 + \partial \end{array}$$

$$M_2(g) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$K[\partial]_{k-1}$ is regarded as row vectors.

$$C^\bullet \,:\, 0 \stackrel{\varphi_{m+1}}{\to} K^{b_m} \stackrel{\varphi_m}{\to} K^{b_{m-1}} \stackrel{\varphi_{m-1}}{\to} \cdots \to K^{b_1} \stackrel{\varphi_1}{\to} K^{b_0} \stackrel{\varphi_0}{\to} 0$$

where φ_i's are K-linear maps is called a complex of vector spaces when $\varphi_i \circ \varphi_{i+1} = 0$ holds. Define

$$H^i(C^\bullet) = \frac{\mathrm{Ker}\,\varphi_i}{\mathrm{Im}\,\varphi_{i+1}}$$

which is a K-vector space.
Example B6: $g = x\partial^2 + x\partial$.

$$C^\bullet \,:\, 0 \to K^2 \stackrel{M_2(g)}{\longrightarrow} K^3 \to 0$$

$H^0(C^\bullet) \simeq K^2$, $H^1(C^\bullet) \simeq K$.
Let $p(x)$ be a rational function. The action $\bullet$ of D to p is defined by

$$x^i \partial^j \bullet p = x^i \frac{\partial^j p}{\partial x^j}$$

The formal Fourier transform of $x^i \partial^j$ is defined by $(-\partial)^i x^j$. It can be extended on D.

Theorem B7: Let $p(x)$ be a square free polynomial and suppose that $\mathrm{Ann}\,\frac{1}{p} = \{f \in D \,|\, f \bullet (1/p) = 0\}$ is generated by $\hat{g} \in D$. Let g be the formal Fourier transform of $\hat{g}$ and k_0 be the minimal integral root of the indicial polynomial (b-function) of g. For $k \geq k_0$, define the complex of vector spaces by

$$C^\bullet \,:\, 0 \to K^{k+1-\mathrm{ord}_{(-w,w)}(g)} \overset{M_k(g)}{\longrightarrow} K^{k+1} \to 0$$

Then, we have $H^i(C^\bullet) = H^{1-i}(\mathbf{C} \setminus V(p), \mathbf{C})$.

Example B8: $p(x) = x(1-x)$. g is the g of our running example B5, B6 and $k_0 = 1$.

Exercise B9: Compute $H^i(\mathbf{C} \setminus V(p))$ when $p(x) = x$ and $p(x) = x(x-1)(x-2)$. M2 codes in the last page.

Proof. The complex of K-vector spaces

$$0 \to D/xD \otimes_D D \xrightarrow{1 \otimes g} D/xD \otimes_D D \to 0$$

is rewritten as

$$G^\bullet \,:\, 0 \to D/xD \ni f \overset{\varphi}{\mapsto} :fg: |_{x=0} \in D/xD \simeq K[\partial] \to 0$$

We will prove that $H^j(G^\bullet) \simeq H^j(C^\bullet)$. Consider the case $j = 0$. Take non-zero element $f = \partial^i + \sum_{j<i} c_j \partial^j$ of $K[\partial]_k / \mathrm{Im}\, M_k(g)$ where $i \leq k$. If $f \in Dg + xD$ and $i \geq r = \mathrm{ord}_{(-w,w)}(g)$, then by the Lemma B4, there exists $q \in F_{i-r}$, $q' \in F_{i+1}$ such that $f - xq' = qg$. Therefore, we have $f =: qg :\in \mathrm{Im}\, M_k(g)$. It is a contraction, then we have $f \notin Dg + xD$ and consequently it is not in $: K[\partial]g :$. Therefore, the canonical K-linear map from $H^0(C^\bullet)$ to $H^0(G^\bullet)$ is injective.

Let $b(s)$ be the indicial polynomial for g. Suppose that $b(x\partial) \equiv x^r g \mod F_{-1}$. Suppose $k > k_0$. (This k is not k in the theorem statement in this paragraph.) Applying ∂^k to the both sides, we have $b(k)\partial^k + x(\cdots) \equiv \partial^k x^r g \mod F_{-1+k}$. Then, we have $b(k)\partial^k \equiv 0 \mod F_{-1+k} + Dg + xD$. It implies the surjectivity.

We prove that the canonical K-linear map from $\mathrm{Ker}\, M_k(g)$ to $\mathrm{Ker}\, \varphi$ is isomorphism. Let $\sum_{i \leq k-r} c_i \partial^i \neq 0$ belongs to the kernel of $M_k(g)$. In other words, we have : $(\sum c_i \partial^i) g : |_{x=0} = 0$. It implies that $\sum c_i \partial^i$ is a non-zero element of D/xD which belongs to the kernel of φ. Then, it is injective.
The case $r < 0$ can be shown analogously.

In order prove the surjectivity, we suppose that the highest $(-w, w)$ order terms of g is $\sum_{i=0}^{m} c_i x^i \partial^{r+i}$. Applying ∂^j to this sum, we have

$$\partial^j g = \sum_{i=0}^{m} c_i j(j-1)\cdots(j-i+1)\partial^{r+j} \mod F_{j+r-1} + xD$$

Suppose that $a\partial^j + \cdots$ belongs to the kernel of φ. We have

$$:(a\partial^j + \cdots)g : |_{x=0} = a(\sum_{i=0}^{m} c_i j(j-1)\cdots(j-i+1)))\partial^{r+j} + \cdots .$$

On the other hand, b-function of g is equal to

$$x^r \sum c_i x^i \partial^{r+i} = \sum c_i \theta(\theta-1)\cdots(\theta-r-i+1)$$

When $j > k - r$, we have $(j+r)\cdots(j+1)(\sum_{i=0}^{m} c_i j(j-1)\cdots(j-i+1))) \neq 0$. Therefore, $a = 0$, which yields the surjectivity.

It follows from the Grothendieck comparison theorem (algebraic de Rham vs analytic de Rham) that we have

$$H^i(\mathbf{C} \setminus V(p)) \simeq H^{1-i}(F^\bullet), \quad F^\bullet : 0 \to K[x, \frac{1}{p}] \xrightarrow{d} K[x, \frac{1}{p}] \to 0$$

Here, $K[x, 1/p]$ is regarded as a left D-module by the action $\bullet$ of ∂. Finally, we will show $H^i(G^\bullet) \simeq H(F^\bullet)$. The two complexes are border complexes of a double complex presented in the blackboard. Then, by the standard theorem of homology algebra (e.g., Kawada, Homology Algebra, Th 2.10 or Th 3.16), we obtain the conclusion. Q.E.D.

Inputs for Macaulay 2 (M2)

```
load "Dmodules.m2"
R=QQ[x,dx,WeylAlgebra=>{x=>dx}]
g = x*(1-x)
I = RatAnn(g)
Dintegration(I,{1})

rr=Dresolution(I)
rr.dd
p=rr.dd_1
p_0_0

QQ[x]
g=x*(1-x)
deRham(g)

QQ[x,y]
I = ideal(x*y-1,x^2-y^2,y^3-x)
syz gens I
res I
```

Exercise B10: Compute $H^i(\mathbf{C}^n \setminus V(p), \mathbf{C})$ by M2 for a polynomial p, e.g., $p = xy$, $n = 2$. Is the result compatible with geometric conclusion?
Note: (twisted) cohomology groups are used to derive several formulas for normalzing constants, which lie in holonomic D-modules, in statistics. Quiver D-modules (arxiv:0510451) may help to study it.