3-1 Bibliography and Recent Advances


Fisher-Bingham distribution and its MLE

\[ z(\Theta, \theta) = \int_{S^n} \exp(t^T \Theta t + \theta t) |dt| \]  

(1)

|dt|: the Haar measure on the hypersphere \( S^n \) over which \( t \) runs. 
\( \Theta \): \((n+1) \times (n+1) \) real symmetric matrix. \( \theta \): real vector of the length \( n+1 \). 

The rank of differential equations is \( 2n + 2 \).


3. T. Sei, A. Kume, Calculating the normalising constant of the Bingham distribution on the sphere using the holonomic gradient method, Statistics and Computing (2013, online). [Degenerated cases need independent discussions.]

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X, Θ: 3 × 3 real matrices. Θᵀ: the transpose of Θ. µ: the invariant measure on SO(3).

\[ z(Θ) = \int_{SO(3)} \exp(\text{Tr}(Θ^T X)) \, dµ(X). \]

Th The rank of differential equations is 4.

The cumulative distribution function for the first eigenvalue of Wishart matrices

Matrix hypergeometric integral \( _1F_1 \)

\[ X : m \times m \text{ real matrix.} \]

\[ \int_{0 < X < I_m} \exp(\text{Tr} XY)|X|^{a-(m+1)/2}|I_m - X|^{c-a-(m+1)/2} dX, \]

0 < \( X < I_m \) means that \( X \) and \( I_m - X \) are positive definite symmetric matrix. \( dX = \prod_{i \leq j} dx_{ij}. \) Th The rank of differential equations is \( 2^m. \)

\[ m = 10, n = 12, \beta = (1, 2, \ldots, 10) \]

Orthant probability

\[ z(\tau, \theta) = \int_0^\infty \cdots \int_0^\infty \exp \left( \sum_{i=1}^m \theta_i x_i + \sum_{i,j=1}^m x_i x_j \tau_{ij} \right) \, dx \]

\[ dx = dx_1 \cdots dx_m. \]

The rank of differential equations is \( 2^m \).

1. L. Schläfli, On the multiple integral \( \int^n dx dy \cdots dz \) whose limits are \( p_1 = a_1 x + b_1 y + \cdots + h_1 z > 0, \quad p_2 > 0, \ldots, p_n > 0 \) and \( x^2 + y^2 + \cdots + z^2 < 1 \), Quart. J. Pure Appl. Math. 2, 269–301; 3, 54–68, 97–108, 1858.


\[ z(\theta) = \int_C \exp\left(\sum_{j=1}^{n} \theta_j t^a_j \right) \prod_{i=1}^{d} t_i^{-b_i-1} dt, \quad dt = dt_1 \cdots dt_d \]

\( A = (a_{ij}) \): \( n \times d \) integer matrix. \( b_i \): real number. \( a_j \): the \( j \)-th column vector of the matrix \( A \). \( t^a_j = \prod_{i=1}^{d} t_i^{a_{ij}} \). Th The (holonomic) rank of the N.C. of the \( A \)-distribution associated to order polytopes has polynomial complexity.


**Mathematical byproduct**: construction of a basis of twisted or rapid decay cohomology groups for complement of the zero set of a generic polynomial is reduced to a question in combinatorial commutative algebra.

**More research topics will be in this conference!**