2-2 Integration algorithm

Let I be a holonomic ideal in D_n.

Theorem (Bernstein, 1972)

$(I + \partial_n D_n) \cap D_{n-1}$, which is called the integration ideal of I, is a holonomic left ideal in D_{n-1}.

Proof that the integral of holonomic function is holonomic Let f be a holonomic function of n-variables annihilated by a holonomic ideal I. An element of the integration ideal $L \in D_{n-1}$ is written as

$$L = L_1 + \partial_n L_2, \quad L_i \in D_n, L_1 \bullet f = 0. \tag{1}$$

Then, we have $L \bullet \int_C f dx_n = \int_C (L_1 + \partial_n L_2) \bullet f dx_n = [L_2 f]_C$. The last term is equal to 0 (under some conditions). Since L's[1] generate a holonomic ideal and the integral is analytic (under some conditions), the integral is a holonomic function.

[1] L_1 and L_2 may be called the telescoping operator and the certifier in the method of creative telescope.

Zeilberger (1990, 1991) gave some algorithms to find elements of the integration ideal. Oaku (1997) gave an algorithm to find generators of the integration ideal. The first step of the algorithm is to compute a Gröbner basis of a given ideal I with the order defined by the weight vector $(w, -w) = (0, \ldots, 0, 1; 0, \ldots, 0, -1)$ (1 stands for x_n (integration variable) and -1 stands for ∂_n).

Example 1 For $f = \exp(-t^2/(2\theta^2))$, we have $\partial_t \bullet f = -t/\theta^2 f$ and $\partial_\theta \bullet f = t^2/\theta^3 f$. We have the following annihilators of f. $P_1 = \underline{t} + \theta^2\partial_t$, $P_2 = \underline{t^2} - \theta^3\partial_\theta$. The underlined terms are the leading terms defined by $(w, -w)$ (t is the integration variable). Dividing P_2 by P_1, we have $P_2 - tP_1 = -\theta^3\partial_\theta - t\theta^2\partial_t$. This has an expression $-\theta^3\partial_\theta + \theta^2 - \partial_t(\theta^2 t)$. Then, $\theta^2(-\theta\partial_\theta + 1)$ belongs to the integration ideal ∎

Summary

When un unnormalized probability distribution function $u(\theta, t)$ is holonomic in θ, t, then the normalizing constant $z(\theta) = \int_\Omega u(\theta, t)dt$ is holonomic in θ (under some conditions). The vector of the function z and its suitable derivatives satisfies a Pfaffian system.

Reference Holonomic Gradient Descent and its Application to the Fisher-Bingham Integral (2011)

Exercise 2.2.1 Find differential equations for the following functions (by performing the first step of computing the integration ideal).

1. $\int_{-\infty}^{+\infty} \exp\left(\frac{-t^2}{2\theta^2}\right) dt$,
2. $\int_{\mathbf{R}} \exp(-ix\xi)H(1-x)H(1+x)dx$,
3. $\int_0^{+\infty} \exp(-x/t - yt)dt$,
4. $\int_0^{+\infty} \exp(-t - \theta t^3)dt$,
5. $\int_{\mathbf{R}^2} \exp(x_1 t_1 + x_2 t_2)\delta(t_1^2 + t_2^2 - 1)dt_1 dt_2$ (Von Mises distribution)