2-1 Holonomic Functions

2-1 Holonomic function in one variable

Let $f(x)$ be a smooth (C^∞) function defined on an open interval U in $\mathbf{R}$. The function f or its analytic continuation is called a holonomic (analytic) function when $\exists L \in \mathbf{C}(x)\langle\partial\rangle$ such that $L \bullet f = 0$ (L is called an annihilator of f).

Example. $\int_0^{+\infty} \exp(-x - \theta x^3) dx$ is a holonomic function, because it is annihilated by an ordinary differential operator with polynomial coefficients.

Theorem

The sum and the product of holonomic functions are holonomic functions. The derivative of any holonomic function is a holonomic function.

Holonomic function of several variables

Let $f(x) = f(x_1, \dots, x_n)$ be a smooth function defined on an open set U in $\mathbf{R}^n$. The function f or its analytic continuation is called a holonomic (analytic) function when there exist n-differential operators L_i, $i = 1, \dots, n$ of the form

$$L_i = a^i_{m_i}(x)\partial_i^{m_i} + a^i_{m_i-1}(x)\partial_i^{m_i-1} + \cdots + a^i_0(x) \tag{1}$$

where $a^i_j(x) \in \mathbf{C}(x)$ which annihilate the function f. The following important theorem follows from the D-module theory.

Theorem (Zeilberger, 1990)

If $f(x_1, \dots, x_n)$ is a holonomic function in x, then the integral $\int_\Omega f(x)dx_n$ is a holonomic function in $(x_1, \dots, x_{n-1})$ (under some conditions on the set Ω).

Advanced: If Ω is a twisted cycle [Aomoto-Kita] or a rapid decay cycle [Hien], then the integral is holonomic.

Examples of holonomic functions

Exercise 2.1.1 Which are holonomic (analytic) functions?

1. $\exp(f(x))$ where f is a rational function,
2. $\frac{1}{\sin x}$ [Hint] Use Th: Any solution of the ordinary differential equation $(a_m(x)\partial^m + \cdots + a_0(x)) \bullet f = 0$, $a_i \in \mathbf{C}[x]$, is holomorphic out of the singular locus $\{x \mid a_m(x) = 0\}$.
3. $\Gamma(x)$, [Hint] $\Gamma(x)$ has poles at $x = -n$, $n \in \mathbf{N}_0$.
4. 2^x,
5. $H(x)$ (Heaviside function),
6. x^a where a is a constant,
7. $|x|$,
8. $\int_{-\infty}^{+\infty} \exp(-xt^6 - t)dt$, $x > 0$.

Weyl algebra and holonomic ideal

Let D_n be the ring of differential operators of polynomial coefficients. D_n is a subring of R_n. For $L = \sum_{(\alpha,\beta)\in E} a_{\alpha,\beta} x^\alpha \partial^\beta \in D_n$, we define

$$\mathrm{ord}_{(u,v)}(L) = \max_{(\alpha,\beta)\in E} (u\alpha + v\beta) \quad (2)$$

$$\mathrm{in}_{(u,v)}(L) = \sum_{\mathrm{ord}(x^\alpha\partial^\beta)=\mathrm{ord}(L),(\alpha,\beta)\in E} a_{\alpha,\beta} x^\alpha \xi^\beta \in \mathbf{C}[x,\xi] \quad (3)$$

where $u = (1,\ldots,1)$ and $v = (1,\ldots,1)$. Example $L = (x_1 - x_2)\partial_1\partial_2 + \partial_1 + \partial_2$. We have $\mathrm{ord}_{(u,v)}(L) = 3$ and $\mathrm{in}_{(u,v)}(L) = (x_1 - x_2)\xi_1\xi_2$ ▮

For a left ideal I of D_n, define $\mathrm{in}_{(u,v)}(I) = \langle \mathrm{in}_{(u,v)}(L) \mid L \in I \rangle$, which is called the (u,v)-initial ideal of I. I is called a holonomic ideal when the (Krull) dimension of $\mathrm{in}_{(u,v)}(I)$ is n.

Some important theorems on holonomic ideal

Exercise 2.1.2 Show that when J is a holonomic left ideal, then $I = R_n J$ is a zero-dimensional ideal in R_n. (In other words, the holonomic rank $\dim_{\mathbf{C}(x)} R_n/I$ of I is finite) When J is a holonomic left ideal such that $J \neq D_n$, we have

Theorem (Bernstein inequality)

$(\mathrm{Krull})\dim \mathrm{in}_{(u,v)}(J) \geq n$.

Theorem (Cor. of Kashiwara 1978)

If I is a zero-dimensional left ideal in R_n, then, $I \cap D_n$ is a holonomic ideal.

Note The ideal $I \cap D_n$ is called the Weyl closure of I. An algorithm to construct generators of the Weyl closure from generators of I was given by H.Tsai (2002). It is implemented in Macaulay 2 (`WeylClosure`).

Supplemental Exercise

Exercise 2.1.3

1. For $f = \exp(1/(x_1^3 - x_2^2 x_3^2))$, define polynomials p_i and q_i by
$$p_i/q_i = (\partial f/\partial x_i)/f.$$
We have
$q_1 = q_2 = q_3(x_1^3 - x_2^2 x_3^2)^2, p_1 = -3x_1^2, p_2 = 2x_2 x_3^2, p_3 = 2x_2^2 x_3.$
Show that
$$q_i \partial_i - p_i, \quad i = 1, 2, 3$$
generate a zero dimensional ideal I in R_3 but they do not generate a holonomic ideal in D_3.
2. Compute the Weyl closure of I.

Holonomic Schwartz distribution

A distribution f on $\mathbf{R}^n$ is called a holonomic (Schwartz) distribution when it is annihilated by a holonomic ideal.

Theorem

When $f(x_1, \ldots, x_n)$ is a holonomic distribution, $\int_{\mathbf{R}^{n-m}} f(x) dx_{m+1} \cdots dx_n$ is a holonomic distribution of m-variables (under some conditions).

Note Condition: the support of f with respect to $x_{m+1}, \ldots, x_n$ is compact.

Exercise 2.1.4 Which are holonomic distributions?

1. $\frac{1}{\sin x}$,
2. $H(x)$ (Heaviside function),
3. $|\sin x|$,
4. $|x|$,
5. $\int_0^1 exp(-x^3 y + x) dx = \int_{\mathbf{R}} exp(-x^3 y + x) H(x) H(1-x) dx$,
6. $\frac{1}{\pi} \int_{-\infty}^{\infty} e^{i\xi x} \frac{\sin \xi}{\xi} d\xi$ [Hint] $2 \sin \xi / \xi$ is the Fourier transform of $H(1-x)H(1+x)$ as a distribution.