1-2 Pfaffian System

A system of differential equations of the form

$$\frac{\partial F}{\partial x_i} = P_i F, \quad i = 1, \ldots, n, \tag{1}$$

where P_i is an $r \times r$ square matrix whose entries are rational functions satisfying the integrability condition and F is an unknown vector valued function of size r, is called a Pfaffian system. Solving the Pfaffian system means to determine the vector valued function F under a given initial condition $F = F_0$ at $x = a$.

Example 1 In case of $n = 1$, a Pfaffian system is a system of first order ordinary differential equations.

Example 2

$$\frac{\partial F}{\partial x} = \begin{pmatrix} 0 & y/x \\ -xy & 1/x \end{pmatrix} F, \ \frac{\partial F}{\partial y} = \begin{pmatrix} 0 & 1 \\ -x^2 & 0 \end{pmatrix} F.$$

$F = (\cos(xy), -x\sin(xy))^T$ is a solution of the system

Singular locus, integrability condition

When the all P_i's are non-sigular at $x = a$, the Pfaffian system is called non-sigular at $x = a$. If not, $x = a$ is called the singular point of the Pfaffian system. The set of the singular points is called the singular locus. Example 2 (continued) The singular locus is $x = 0$. We assume that

$$\frac{\partial P_i}{\partial x_j} + P_i P_j = \frac{\partial P_j}{\partial x_i} + P_j P_i \qquad (2)$$

holds for any i, j. The condition is called the integrability condition.

Theorem

If $x = a$ is out of the singular locus, then there exist r linearly independent holomorphic solutions of the Pfaffian system.

Exercise 1.2.1 Derive the integrability condition by assuming the existence of r-linearly independent smooth solutions. Hint: we have $\partial_i \partial_j \bullet F = \partial_j \partial_i \bullet F$ for any smooth solution F.

From a left ideal I to a Pfaffian system

Assume $n = 1$ and a left ideal I is generated by $\partial^2 + a_1(x)\partial + a_0(x)$. $\{1, \partial\}$ is a base set of standard monomials. Then, we have

$$\partial \begin{pmatrix} 1 \\ \partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} \begin{pmatrix} 1 \\ \partial \end{pmatrix} \mod I$$

The associated Pfaffian system is

$$\frac{\partial F}{\partial x} = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix} F$$

When f is a solution of I, $F = (1 \bullet f, \partial \bullet f)^T$ is a solution of the Pfaffian system

- We can make an analogous construction of a Pfaffian system for a Gröbner basis of I and a base set of standard monomials.

$r = \dim_{\mathbf{C}(x)} R_n/I$ is called the holonomic rank[1] of a left ideal I.

Theorem

Let r be the holonomic rank of a left ideal I in R. The set of $r \times r$ matrices P_i constructed by the method above satisfies the integrability condition.

[1] R_n/I is regarded as $\mathbf{C}(x)$-vector space.

Exercise

Exercise 1.2.2 Obtain the reduced Gröbner basis for the left ideal I generated by $\partial^4 - 1$ and $\partial^2 - 1$ in R_1 $(\partial = \partial_1)$. How much is the holonomic rank of I? Give a base set of standard monomials and derive a Pfaffian sytem for I.

Exercise 1.2.3 Derive a Pfaffian system for the system of differential equations

$$g_1 = \underline{-x\partial_x} + y\partial_y + 1, g_2 = \underline{\partial_x\partial_y} - 1, g_3 = \underline{y\partial_y^2} + 2\partial_y - x.$$

It is already checked that $\{g_1, g_2, g_3\}$ is a Gröbner basis with the graded lexicographic order such that $\partial_x > \partial_y$. This is a system for the normalizing constant $z(x, y) = \int_0^{+\infty} \exp\left(-\frac{x}{t} - yt\right) dt$ where $x, y > 0$.

Reference: Chapter 6 of "Edited by Hibi, Gröbner Bases: Statistics and Software Systems, Springer, 2014".