(Modifed) A-Hypergeometric Systems : a Computational Approach

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Computation and $\mathcal{A}$-Hypergeometric Systems

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1. Indicial polynomials, restriction, series solutions. T.Oaku, M.Saito, B.Sturmfels, N.Takayama. 1996–
2. Holonomic rank. [I.M.Gel'fand, M.Kapranov, A.Zelevinsky, A.Adolphson (before C)], E.Cattani, A.Dickenstein, C.D'Andrea, M.Saito, B.Sturmfels, N.Takayama, E.Miller, L.Matsusevich, U.Walther, G.Okuyama, K.Ohara. 1997–.
3. Slope(Gevrey class). F.Castro-Jimenez, N.Takayama, M.Hertillo, U.Walther, M.Schultz, M.Fernandez-Fernandez. 2001–

Basic Notions

- Let D be the ring of differential operators of n variables

$$D = \mathbf{C}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$$

where

$$x_i x_j = x_j x_i, \partial_i \partial_j = \partial_j \partial_i, \partial_i x_j = x_j \partial_i + \delta_{ij}.$$

Example: $\partial_1 x_1 = x_1 \partial_1 + 1$, $\partial_1 x_1^2 = x_1^2 \partial_1 + 2x_1$.

Initial ideal and indicial polynomial 1

- For $f \in D$ and $u, v \in \mathbf{R}^n$ such that $u_i + v_i \geq 0$, $\mathrm{in}_{(u,v)}(f)$ is the sum of the heighest weight terms of f with respect to (u, v). Here $u = (u_1, \ldots, u_n)$ stands for $x = (x_1, \ldots, x_n)$ and v stands for ∂.

Example: initial term

$$\mathrm{in}_{(0,0,1,1)}(x_1\partial_1 + x_2\partial_2 - 3) = x_1\xi_1 + x_2\xi_2.$$

$$\mathrm{in}_{(0,0,0,0,1,0,0,1)}(\partial_1\partial_4 - \partial_2\partial_3) = \xi_1\xi_4.$$

Example: initial term

$$\mathrm{in}_{(-1,-1,1,1)}(x_1\partial_1 + x_2\partial_2 - 3) = x_1\partial_1 + x_2\partial_2 - 3.$$

$$\mathrm{in}_{(-1,0,0,-1,1,0,0,1)}(\partial_1\partial_4 - \partial_2\partial_3) = \partial_1\partial_4.$$

$\mathrm{in}_{(u,v)}(I) = \langle \mathrm{in}_{(u,v)}(\ell) \mid \ell \in I \rangle$ is called the initial ideal.

Initial ideal and indicial polynomial 2

Let $w \in \mathbf{R}^n$ be a weight vector and I a left ideal in D. Put $\theta_i = x_i\partial_i$. The monic generator $b(s)$ of $\mathrm{in}_{(-w,w)}(I) \cap \mathbf{C}[s]$, $s = \sum_{i=1}^n w_i\theta_i$ is called the indicial polynomial of s with respect to w.
In case of ODE (with no x as a factor), $(-1,1)$-indicial polynomial is the characteristic polynomial at $x = 0$.
Example: (Modified Bessel function of the first kind)

$$\mathrm{in}_{(-1,1)}(x\partial_x^2 + (1-\beta)\partial_x - 1) = x\partial_x^2 + (1-\beta)\partial_x$$

$$x(x\partial_x^2 + (1-\beta)\partial_x) = \theta_x(\theta_x - 1) + (1-\beta)\theta_x = \theta_x(\theta_x - \beta).$$

Computation of indicial polynomials

The initial ideal and the indicial polynomial can be computed by Gröbner basis (T.Oaku, 1995).

- kan/sm1:

```
(cohom.sm1) run  (ox.sm1) run (intw.sm1) run

[ [(x Dx^2+ (1-5) Dx - 1)] [(x)] [(x) -1 (Dx) 1]] wbf /rr set  ;
rr 0 get fctr message ;

[[1,5,3,7]] appell1 /ff set ;
[ ff 0 get, ff 1 get [(x1) -1 (Dx1) 1]] wbf /rr2 set ;
rr2 0 get fctr message ;
```

0, 5

$F_1(1,3,7,5;x_1,x_2)$, **0, 3**.

- Risa/Asir(faster algorithm by M.Noro, 2002):

```
F=sm1.gkz([ [[-1,1]], [-3]]);
B=generic_bfct(F[0],F[1],[dx1,dx2],[1,0]);
print(fctr(B));
B=generic_bfct(F[0],F[1],[dx1,dx2],[0,1]);
print(fctr(B));
```

$$(-\theta_1 + \theta_2 - \beta)f = 0, \quad \theta_i = x_i \partial_i \tag{1}$$

$$(\partial_1 \partial_2 - 1)f = 0 \tag{2}$$

$s(s-3)$, $s(s+3)$

Assume that $a_i \in \mathbf{N}_0^d$. Integer programming problem (IPA):
Minimize u_n subject to $Au = \beta$, $u \in \mathbf{N}_0^n$.

Theorem

The set of parametric solutions of the integer programming program IPA w.r.t. $\beta \simeq$ the set of (parametric) roots of the indicial polynomial w.r.t $x_n = 0$.

As to an exact formulation, see Saito, Sturmfels, Takayama, Hypergeometric polynomials and integer programming, Compo. Math. (1999), 185–204.

We had made computer experiments before conjecturing and proving this theorem.

For modified systems, we have determined the indicial polynomial.

Gröbner basis is essential both for experiments and our proof.

Finding roots of indicial polynomial is a first step to draw graphs.

Pictures of (modified) $\mathcal{A}$-HG functions 1 (Bessel)

$A = (-1, 1)$.

$$(-\theta_1 + \theta_2 - \beta)f = 0, \quad \theta_i = x_i\partial_i \tag{3}$$

$$(\partial_1\partial_2 - 1)f = 0 \tag{4}$$

$x_1x_2 = 0$ is the singular locus.

$$x_1^{-\beta}(x_1x_2)^{\beta/2}I_\beta(-\beta, 2\sqrt{x_1x_2}) \simeq x_1^{-\beta}(1 + O(x_1, x_2))$$

$$x_1^{-\beta}(x_1x_2)^{\beta/2}I_\beta(\beta, 2\sqrt{x_1x_2}) \simeq x_2^{\beta}(1 + O(x_1, x_2))$$

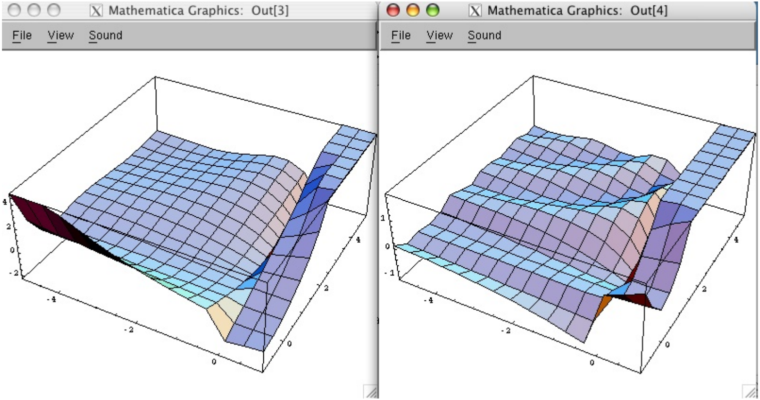
$I_\nu(z)$ is the ν-th modified Bessel function of the first kind.

$$I_\nu(z) = (z/2)^\nu \sum_{k=0}^{\infty} \frac{(z^2/4)^k}{k!\Gamma(\nu + k + 1)}.$$

Pictures of (modified) $\mathcal{A}$-HG functions 2 (Bessel)

$A = (-1, 1), \beta = 1.$

Left: $\mathrm{Re}\,\sqrt{y/x} I_{-1}(2\sqrt{xy}), -5 \le x \le 1, -1 \le y \le 5$, $x^{-1}(1 + O(x, y))$



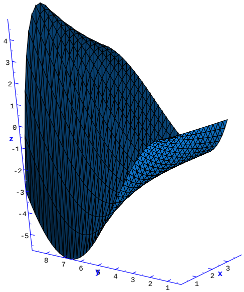
$x \mapsto tx,\ y \mapsto ty.$

Right: $\mathrm{Re}\,\sqrt{y/x} I_{-1}(2\sqrt{t^2 y})$, $-5 \le x \le 1, -1 \le t \le 5, y = 1$, $t^{-2}x^{-1}(1 + O(t, x))$.

Bessel function in two variables (H.Kimura, K.Okamoto)

$$f(a;x,y) = \int_C \exp(-\frac{1}{4}t^2 - xt - y/t)t^{-a-1}dt,$$

where $C = \vec{01} + \{e^{2\pi\sqrt{-1}\theta} \mid \theta \in [0, 2\pi]\} + \vec{10}$. The function $f(a;x,y)$ satisfies the holonomic system $\partial_x\partial_y + 1, \partial_x^2 - 2x\partial_x + 2y\partial_y + 2a, 2y\partial_y^2 + 2(a+1)\partial_y - \partial_x + 2x$. The rank of the system is 3. Take $a = 1/2$. It admits a unique solution of the form $y^{-a}g(x,y)$ such that g is holomorphic at the origin and



$g(0,0) = 1$. $A = (-1, 1, 2)$.

Some Open Questions: 1. Slope and series solutions

Question

1. Determine the slope (Gevrey class) along $t = 0$.
2. Study analytic property of series solutions standing for roots of the indicial polynomial w.r.t. t of the modifed $\mathcal{A}$-hypergeometric system.

References: The first question is solved in case of $\mathcal{A}$-hypergeometric system along $x_i = 0$.

1. F.Castro-Jimenez, N.Takayama, Slopes of a Hypergeometric System Associated to a Monomial Curve, Trans. of AMS (2003), 3761–3775.
2. M.Schulze and U.Walther, Irregularity of hypergeometric systems via slopes along coordinate subspaces. Duke Math.J. 465–509 (2008).

Some works by T.Koike (Kobe) for the question 2.

Some Open Questions: 2. Minimal extension

Question

Does the minimal extension of the D-module $(D_n/H_A(\beta) \oplus D_1/D_1\partial_s)'$ on $\mathbf{C}^* \times \mathbf{C}^n$ to $\mathbf{C}^{n+1}$ agree with the modifed $D_{n+1}/H_{A,w}(\beta)$?

References: We denote by i the inclusion. Minimal extension is defined by $\mathbb{D}i_*\mathbb{D}$ where $\mathbb{D}$ is the dualizing functor. Computation of the dual of $\mathscr{A}$-hypergeometric system is studied in

1. U.Walther, Duality and monodromy reducibility of A-hypergeometric systems, Math.Ann. (2007) 55–74.

Some Open Questions: 3. Restriction

Question

1. Determine the restriction of $D_{n+1}/H_{A,w}(\beta)$ to $t = 0$ or $x_i = 0$.
2. (More difficult) Determine the restriction to irreducible factors of the principal $\mathcal{A}$-discriminant.

Reference: An algorithm of computing the restriction to non-singular variety Z is given by Oaku-Takayama (2001) (Prop. 7.4). It is implemented on Macaulay 2 as the command `localCohom(ZZ,Ideal,Module)`

```
I=AppellF1{1/2,1/2,1/2,1};
W=ring I;
J=ideal(x-y);
-- 0-th restriction to x-y
localCohom(1,J,module W/I);
```

Formulas for F_4 by Raimundas Vidunas.