Counting self-dual codes over finite rings

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Recent results are based on joint work with **Hideo Wada** (Sophia University) and **Kiyoshi Nagata** (Daito Bunka University).
Coding theory: Background

Errors occur during the transmission of information.

Coding theory deals with mathematical methods used to package information so that transmission errors are detected and corrected.

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From its origins in engineering and information science, coding theory has also developed as an area of discrete mathematics, combining algebra, combinatorics, number theory and even geometry.
Coding theory: Background

Let $A$ be a set of symbols, $n \in \mathbb{N}$

$$A^n := \{(a_0, a_1, \ldots, a_{n-1}) \mid a_i \in A\}, \text{ the set of } n\text{-tuples over } A$$

**code of length $n$ over $A$: a subset of $A^n$**

**codeword**: element of a code

A distance function is usually defined on $A^n$

Example: Hamming distance between two $n$-tuples $a = (a_0, a_1, \ldots, a_{n-1})$ and $b = (b_0, b_1, \ldots, b_{n-1})$:

$$d_H(a, b) := \#\{i \mid a_i \neq b_i\}$$
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Parameters of a code: An \((n, k, d)_A\)-code is a code over \(A\) with
- length \(n\)
- size \(k\)
- minimum distance \(d\)

The goal of coding theory in engineering is the construction of codes with
- small \(n\)
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These are incompatible goals!
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Codes over fields

The traditional setting for codes: Galois fields $\mathbb{F}_q := GF(q)$

A (linear) code of length $n$ over $\mathbb{F}_q$: a subspace of $\mathbb{F}_q^n$

Examples: Binary codes
$(2^4, 2^{11}, 4)$ Reed-Muller
$(2^4, 2^8, 6)$ Nordstrom-Robinson (a non-linear binary code)
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Generalizations of Nordstrom-Robinson code: Preparata, Kerdock codes, etc.

Recent interest in codes over rings is due to the discovery that certain non-linear binary codes can be constructed as images of codes over the finite ring $\mathbb{Z}_4 := \mathbb{Z}/4\mathbb{Z}$.

**Definition.** The *Gray map* $\phi : \mathbb{Z}_4 \rightarrow \mathbb{Z}_2^2$ is given by

\[
\begin{align*}
0 & \mapsto 00, \quad 1 \mapsto 01, \quad 2 \mapsto 11, \quad 3 \mapsto 10.
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We can extend this to $\phi : \mathbb{Z}_4^n \rightarrow \mathbb{Z}_2^{2n}$. 
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We can extend this to $\phi : \mathbb{Z}_4^n \rightarrow \mathbb{Z}_2^{2n}$. 
Theorem. (Hammons, Kumar, Calderbank, Sloane and Solé, 1992) Let $\mathcal{O}$ be the linear $(2, 256, 6)_{\mathbb{Z}_4}$ code with generator matrix

$$G = \begin{bmatrix}
3 & 3 & 2 & 3 & 1 & 0 & 0 & 0 \\
3 & 0 & 3 & 2 & 3 & 1 & 0 & 0 \\
3 & 0 & 0 & 3 & 2 & 3 & 1 & 0 \\
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Then $\phi(\mathcal{O}) =$ Nordstrom-Robinson code. The non-linear binary codes are Gray map images of linear codes over $\mathbb{Z}_4$. There has been a lot of interest in codes over finite rings these last 15 years.
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Definition. Let $R$ be a finite ring. (e.g. $\mathbb{Z}_m := \mathbb{Z}/m\mathbb{Z}$)

1) code: an $R$-submodule of $R^n := \{(x_1, x_2, \ldots, x_n) \mid x_i \in R\}$
2) codeword: element of a code
3) Two vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are orthogonal if their Euclidean inner product is zero. i.e.

$$x \cdot y = \sum_i x_i y_i = 0$$
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**Self-dual codes**

**Definition.** Let $C$ be a code over a ring $R$.

1) *dual* of $C$:

\[ C^\perp := \{ y \in R^n \mid x \cdot y = 0, \forall x \in C \} \]

(Remark: $C^\perp$ is a code.)

2) If $C \subseteq C^\perp$, $C$ is *self-orthogonal*.

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Equivalent codes

Two codes of same length over $\mathbb{Z}_p^s$ are equivalent if one can be obtained from the other by permutation of coordinates, possibly followed by multiplication of some coordinates by $-1$.

$$C_1 \approx C_2 \iff \exists \text{ } n \times n \text{ matrix } P \text{ such that }$$

$$C_1 = C_2 P := \{ cP \mid c \in C_2 \}$$

where $P$ has exactly one entry $\pm 1$ in every row and in every column and all other entries are zero.
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Counting the number of codes

The number of codes *equivalent* to a code $C$ of length $n$ is

$$\frac{|E_n|}{|\text{Aut}(C)|},$$

where $E_n$ is the group of all sign-permutations and $\text{Aut}(C)$ is the automorphism group of $C$, i.e. the group of all sign-permutations that send $C$ to itself. Thus the number of *distinct* self-dual codes over $\mathbb{Z}_p^s$ of length $n$ is given by

$$N_{ps}(n) = \sum_{c} \frac{2^n n!}{|\text{Aut}(C)|},$$

where the sum runs over all inequivalent self-dual codes $C$. We wish to find a more explicit formula for $N_{ps}(n)$. This is called the mass formula.
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The number of codes equivalent to a code $\mathcal{C}$ of length $n$ is

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The mass formula

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is important for the computation of the number of inequivalent classes of self-dual codes over $\mathbb{Z}_{p^s}$ and the classification of such codes,...

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In 1993, Conway and Sloane classified all self-dual codes over $\mathbb{Z}_4$ up to length $n = 9$, without the aid of a mass formula.


Classification of all self-dual $\mathbb{Z}_4$-codes with $n \leq 15$ (Fields, Gaborit, Leon, Pless. IEEE Transactions Information Theory, 1998)
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Mass formulas for $\mathbb{Z}_{p^s}$


**Theorem.** Let $p$ be an odd prime. If $N_{p^2}(n)$ is the number of distinct self-dual codes over $\mathbb{Z}_{p^2}$ of length $n$ then

$$N_{p^2}(n) = \sum_{0 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor} \sigma_p(n, k) p^{\frac{k(k-1)}{2}},$$

where $\sigma_p(n, k)$ is the number of distinct self-orthogonal codes over $\mathbb{F}_p$ of dimension $k$.

- Classification of all self-dual codes over $\mathbb{Z}_9$ (for lengths $n \leq 8$ for $\mathbb{Z}_9$, $n \leq 7$ for $\mathbb{Z}_{25}$ and $n \leq 6$ for $\mathbb{Z}_{49}$)
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where $\sigma_p(n, k)$ is the number of distinct self-orthogonal codes over $\mathbb{F}_p$ of dimension $k$.

- Classification of all self-dual codes over $\mathbb{Z}_9$ (for lengths $n \leq 8$ for $\mathbb{Z}_9$, $n \leq 7$ for $\mathbb{Z}_{25}$ and $n \leq 6$ for $\mathbb{Z}_{49}$)
How is classification done?

To count the number of inequivalent codes of given length $n$:

1. Set $SUM = 0$
2. Find a self-dual code $C_1$ of length $n$
3. Compute $|Aut(C_1)|$, $SUM = SUM + \frac{2^n n!}{|Aut(C_1)|}$
4. For every $j = 2, 3, \ldots$, find a self-dual code $C_j$, not equivalent to $C_1, \ldots, C_{j-1}$, and compute $|Aut(C_j)|$, and $SUM = SUM + \frac{2^n n!}{|Aut(C_1)|}$
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An example

Classify self-dual codes of length \( n = 8 \) over \( \mathbb{Z}_9 \):

\[
N_9(8) = \sum_{0 \leq k \leq 4} \sigma_3(8, k) 3^{\frac{k(k-1)}{2}}
= 1 + 1120 + 36400 \cdot 3 + 44800 \cdot 3^3 + 2240 \cdot 3^6
= 2952881
\]

We can also compute

\[
\sum \frac{2^8 8!}{|Aut(C)|} = 1 + 224 + 4480 + 20160 + 26880 + 1680
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Codes over $\mathbb{Z}_p^3$, for primes $p$

A code $C$ of length $n$ over $\mathbb{Z}_p^3$ has a “generator matrix" which can be written as

$$G = \begin{bmatrix}
l_k & A_2 & A_3 & A_4 \\
0 & pl_l & pB_3 & pB_4 \\
0 & 0 & p^2 l_m & p^2 C_4 \\
\end{bmatrix} \begin{bmatrix}
A \\
pB \\
p^2 C \\
\end{bmatrix}$$

$l_i$: $i \times i$ identity matrix

$A_3 = A_{30} + pA_{31}$

$B_4 = B_{40} + pB_{41}$

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$A_2, B_3, C_4, A_{ij}$ and $B_{ij}$ have entries from $\{0, 1, \ldots, p - 1\}$

Columns have sizes $k, l, m$ and $h$, with $n = k + l + m + h$.

$C$ has $p^{3k+2l+m}$ codewords.
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Thus: whenever $C = C^\perp$, $k = h$ and $l = m$.
A self-dual code then is of even length $n = 2(k + l)$. 
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We can characterize self-dual codes:

**Proposition.** Let $C$ be a code over $\mathbb{Z}_{p^3}$. Then $C$ is a self-dual code if and only if $k = h$, $l = m$ and the following hold:

\begin{align*}
AA^t &\equiv 0 \pmod{p^3} \quad (1) \\
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Proposition. Let $p$ be an odd prime. A self-dual code over $\mathbb{Z}_{p^3}$ can be induced from a self-dual code $C_1$ over $\mathbb{Z}_p$; there are $p^{k\left(\frac{n}{2}-1\right)}$ self-dual codes over $\mathbb{Z}_{p^3}$ corresponding to each subspace of $C_1$ of dimension $k$ ($0 \leq k \leq \frac{n}{2}$).

Proposition. Define $\varepsilon$ as follows: 1) if $\vec{1}_n \in A$ and $8 \mid n$, then $\varepsilon = 1$; 2) if $\vec{1}_n \notin A$, then $\varepsilon = 0$. Any self-dual code over $\mathbb{Z}_{2^3}$ is induced from a self-dual code $C_1$ over $\mathbb{Z}_2$. There are $2^{kl+k^2+\varepsilon}$ self-dual codes over $\mathbb{Z}_{2^3}$ corresponding to each subspace of dimension $k$ ($0 \leq k \leq \frac{n}{2}$) of $C_1$. 
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The number of underlying self-dual codes over $\mathbb{Z}_p$

**Lemma.** (Pless, 1965) Let $p$ be an odd prime and $\sigma_p(n, k)$ the number of self-orthogonal codes of even length $n$ and dimension $k$ over $\mathbb{Z}_p$. Then:

1. If $(-1)^{n/2}$ is a square,

$$\sigma_p(n, k) = \frac{(p^{n-k} - p^{n/2-k} + p^{n/2} - 1) \prod_{i=1}^{k-1} (p^{n-2i} - 1)}{\prod_{i=1}^{k} (p^i - 1)}, \quad k \geq 1.$$  

2. If $(-1)^{n/2}$ is not a square,

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\]
The number of subspaces

**Lemma.** Let $V$ be an $n$-dimensional vector space over the integers modulo $p$. The number $\rho(n, k)$ of subspaces $T \subset V$ of dimension $k \leq n$ is given by

$$\rho(n, k) = \frac{(p^n - 1)(p^n - p)\ldots(p^n - p^{k-1})}{(p^k - 1)(p^k - p)\ldots(p^k - p^{k-1})}. $$
Main results

**Theorem.** Let $N_{p^3}(n)$ denote the number of distinct self-dual codes of even length $n$ over $\mathbb{Z}_{p^3}$.

1. If $p$ is odd then

$$N_{p^3}(n) = \left(1 + \left(-\frac{1}{p}\right)^{\frac{n}{2}}\right) \prod_{i=1}^{\frac{n}{2}-1} \frac{p^{n-2i} - 1}{p^i - 1} \sum_{k=0}^{\frac{n}{2}} \left(\prod_{i=0}^{k-1} \frac{p^{n-i} - 1}{p^{k-i} - 1}\right) p^{k(n^{2}-1)}.$$

2. If $n \equiv 2, 6 \pmod{8}$ then

$$N_8(n) = \sum_{k=0}^{\frac{n}{2}-1} \left(\prod_{i=0}^{k-1} \frac{2^{n-2i-2} - 1}{2^{i+1} - 1}\right) \left(\prod_{i=k}^{\frac{n}{2}-2} \frac{2^{n-2i-2} - 1}{2^{i+1-k} - 1}\right) 2^{kn^{2}}.$$
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Main results

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\]

4. If \( n \equiv 0 \pmod{8} \) then

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\[+ \sum_{k=1}^{\frac{n}{2}} \left( \prod_{i=0}^{k-2} \frac{2^{n-2i-3} + 2^{\frac{n}{2}-i-2} - 1}{2i+1 - 1} \right) \left( \prod_{i=k}^{\frac{n}{2}-1} \frac{2^{n-2i} - 1}{2i+1-k - 1} \right) 2^{\frac{kn}{2}+1}.\]
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4. If \( n \equiv 0 \pmod{8} \) then

\[
N_8(n) = \sum_{k=0}^{\frac{n}{2}-1} \left( \prod_{i=0}^{k-1} \frac{2^{n-2i-2} + 2^{\frac{n}{2}-i-1} - 2}{2^{i+1} - 1} \right) \left( \prod_{i=k}^{\frac{n}{2}-2} \frac{2^{n-2i-2} - 1}{2^{i+1-k} - 1} \right) 2^{\frac{kn}{2}}
+ \sum_{k=1}^{\frac{n}{2}} \left( \prod_{i=0}^{k-2} \frac{2^{n-2i-3} + 2^{\frac{n}{2}-i-2} - 1}{2^{i+1} - 1} \right) \left( \prod_{i=k}^{\frac{n}{2}-1} \frac{2^{n-2i} - 1}{2^{i+1-k} - 1} \right) 2^{\frac{kn}{2}+1}.
\]
A (partial) classification of self-dual codes over $\mathbb{Z}_8$ and $\mathbb{Z}_9$ by Gulliver, et.al.

Dougherty, Gulliver and Wong. *Self-dual codes over $\mathbb{Z}_8$ and $\mathbb{Z}_9$.* Designs, Codes and Cryptography 41 (Nov 2006):

- $n = 2$. There is only one self-dual code over $\mathbb{Z}_8$ of length 2.

$$G_2 = \begin{pmatrix} 2 & 2 \\ 0 & 4 \end{pmatrix}$$

- $n = 4$. There is only one self-dual code over $\mathbb{Z}_8$ of length 4.

$$G_4 = \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$
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- $n = 6$. One self-dual code over $\mathbb{Z}_8$ of length 6.

$$G_4 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 2 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$
Example: self-dual codes over $\mathbb{Z}_8$ with $n = 6$, $k = 2$, $\lambda = 1$.

We start with a self-dual binary code

$$
\begin{bmatrix}
A \\
B
\end{bmatrix} = 
\begin{bmatrix}
1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0
\end{bmatrix}
$$

with

$$A_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad A_{30} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} , \quad A_{40} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$
Example: self-dual codes over $\mathbb{Z}_8$ with $n = 6$, $k = 2$, $l = 1$.

\[
C = \begin{bmatrix}
1 & 0 & 1 & 1 + 2x & 1 + 2y + 4z & 2 + 4z' \\
0 & 1 & 1 & 3 - 2x & 2 + 4(z' + y + y') & 1 + 2y' + 4z'' \\
0 & 0 & 2 & 2 & 4(1 - x) & 4x \\
0 & 0 & 0 & 4 & 4 & 4
\end{bmatrix},
\]

where $x, y, y', z, z', z''$ are arbitrary elements of $\mathbb{F}_2$.

The code $C$ is self-dual over $\mathbb{Z}_8$. 
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0 & 0 & 0 & 4 & 4 & 4 \\
\end{bmatrix},
\]

where $x, y, y', z, z', \text{ and } z''$ are arbitrary elements of $\mathbb{F}_2$. 

The code $C$ is self-dual over $\mathbb{Z}_8$. 

what next

- Classification for $\mathbb{Z}_{p^3}$ codes of moderate lengths; develop efficient methods for computing automorphism groups

- Generalize to $\mathbb{Z}_{p^s}$

- Explore other rings: Galois rings, finite chain rings, Frobenius rings

- another track: Generalization of Hammons, et. al. result for other ring settings
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Thank you.