Notation. Let η be a smooth non-negative radial bump function which equals 1 when $|t| < 1$, equals 0 when $|t| > 2$, and smoothly interpolates between the two in the region $1 < |t| < 2$. We use L_x^p to denote the Lebesgue space with the norm

$$\|f\|_{L_x^p} = \left(\int_{-\infty}^{\infty} |f(x)|^p \, dx\right)^{\frac{1}{p}}$$

(when $p = 2$, $\|f\|_{L_x^2} = \|f\| = \sqrt{\langle f, f\rangle}$), and use $L_t^p L_x^p$ to denote the Lebesgue space with norm

$$\|f\|_{L_t^p L_x^p} := \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(t,x)|^p \, dxdt\right)^{\frac{1}{p}}.$$

Theorem 3.1. For any $a \in L_x^2$,we have

$$\|\eta S(\cdot)a\|_{L_t^4 L_x^4} \le C\|a\|_{L_x^2},$$

where C is a constant dependent on η.
Proof. We use $|\eta S(\cdot)a|^4 = |\eta S(\cdot)a|^2 |\eta S(\cdot)a|^2$ and reduce to showing

$$\|(\eta S(\cdot)a)(\eta S(\cdot)a)\|_{L_t^2 L_x^2} \le C\|a\|_{L_x^2}^2. \tag{0.1}$$

By the Perseval identity (in variables with x and t), we have

$$\begin{aligned}\|(\eta S(\cdot)a)(\eta S(\cdot)a)\|_{L_t^2 L_x^2} &= \|(\mathcal{F}_x \eta S(\cdot)a) *_\xi (\mathcal{F}_x \eta S(\cdot)a)\|_{L_t^2 L_\xi^2} \\ &= \|(\mathcal{F}_t\mathcal{F}_x \eta S(\cdot)a) *_{\tau,\xi} (\mathcal{F}_t\mathcal{F}_x \eta S(\cdot)a)\|_{L_\tau^2 L_\xi^2}.\end{aligned}$$

A computation shows that the Fourier transform (in x) of $\eta(t)S(t)a$ at ξ is $\eta(t)e^{8\pi^3 i\xi^3 t}\hat{a}(\xi)$, and its Fourier transform (in t) at τ is simply $\hat{\eta}(\tau - 4\pi^2\xi^3)\hat{a}(\xi)$. Then the right-hand side can be written

$$\left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\eta}(\tau - \tau_1 - 4\pi^2(\xi-\xi_1)^3)\hat{a}(\xi-\xi_1)\hat{\eta}(\tau_1 - 4\pi^2\xi_1^3)\hat{a}(\xi_1)\, d\xi_1 d\tau_1\right\|_{L_\tau^2 L_\xi^2}.$$

Integrating in τ_1, we have

$$\le C\left\|\int_{-\infty}^{\infty} |\eta_1(\tau - 4\pi^2(\xi-\xi_1)^3 - 4\pi^2\xi_1^3)||\hat{a}(\xi-\xi_1)||\hat{a}(\xi_1)|\, d\xi_1\right\|_{L_\tau^2 L_\xi^2}, \tag{0.2}$$

where η_1 is a smooth exponential decay function. Applying Cauchy-Schwarz inequality to the function inside the integral

$$\begin{aligned} &\left|\int_{-\infty}^{\infty} |\eta_1(\tau - 4\pi^2(\xi-\xi_1)^3 - 4\pi^2\xi_1^3)||\hat{a}(\xi-\xi_1)||\hat{a}(\xi_1)|\, d\xi_1\right| \\ \le\ & \left(\int_{-\infty}^{\infty} |\eta_1(\tau - 4\pi^2(\xi-\xi_1)^3 - 4\pi^2\xi_1^3)|\, d\xi_1\right)^{\frac{1}{2}} \times \\ & \times \left(\int_{-\infty}^{\infty} |\eta_1(\tau - 4\pi^2(\xi-\xi_1)^3 - 4\pi^2\xi_1^3)||\hat{a}(\xi-\xi_1)|^2|\hat{a}(\xi_1)|^2\, d\xi_1\right)^{\frac{1}{2}}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned} & \left\| \left(\int_{-\infty}^{\infty} |\eta_1(\tau - 4\pi^2(\xi-\xi_1)^3 - 4\pi^2\xi_1^3)| |\hat{a}(\xi-\xi_1)|^2 |\hat{a}(\xi_1)|^2 \, d\xi_1 \right)^{\frac{1}{2}} \right\|_{L^2_\tau L^2_\xi} \\ = \quad & \|\eta_1\|_{L^1_\tau}^{\frac{1}{2}} \|\hat{a}\|_{L^2_\xi}^2, \end{aligned}$$

so by Perseval's identity, (0.2) is

$$\leq C\|a\|_{L^2_x}^2 \sup_{\tau,\xi} \left(\int_{-\infty}^{\infty} |\eta_1(\tau - 4\pi^2(\xi-\xi_1)^3 - 4\pi^2\xi_1^3)| d\xi_1 \right)^{\frac{1}{2}}.$$

From the crude calculation, we conclude that the last term is bounded by some constant, hence (0.1) follows.