Sage Quick Reference:
Elementary Number Theory

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`http://wiki.sagemath.org/quickref`

Everywhere $m, n, a, b, etc.$ are elements of `ZZ`

`ZZ` = $\mathbf{Z}$ = all integers

Integers

$\ldots, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots$

n divided by m has *remainder* `n % m`

`gcd(n,m)`, `gcd(`*list*`)`

extended gcd $g = sa + tb = \gcd(a, b)$: `g,s,t=xgcd(a,b)`

`lcm(n,m)`, `lcm(`*list*`)`

binomial coefficient $\binom{m}{n}$ = `binomial(m,n)`

digits in a given base: `n.digits(`*base*`)`

number of digits: `n.ndigits(`*base*`)`

(*base* is optional and defaults to 10)

divides $n \mid m$: `n.divides(m)` if $nk = m$ some k

divisors – all d with $d \mid n$: `n.divisors()`

factorial – $n!$ = `n.factorial()`

Prime Numbers

$2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, \ldots$

factorization: `factor(n)`

primality testing: `is_prime(n)`, `is_pseudoprime(n)`

prime power testing: `is_prime_power(n)`

$\pi(x) = \#\{p : p \leq x \text{ is prime}\}$ = `prime_pi(x)`

set of prime numbers: `Primes()`

$\{p : m \leq p < n \text{ and } p \text{ prime}\}$ =`prime_range(m,n)`

prime powers: `prime_powers(m,n)`

first n primes: `primes_first_n(n)`

next and previous primes: `next_prime(n)`, `previous_prime(n)`, `next_probable_prime(n)`

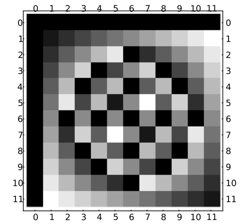
prime powers: `next_prime_power(n)`, `pevious_prime_power(n)`

Lucas-Lehmer test for primality of $2^p - 1$

```
def is_prime_lucas_lehmer(p):
    s = Mod(4, 2^p - 1)
    for i in range(3, p+1): s = s^2 - 2
    return s == 0
```

Modular Arithmetic and Congruences

`k=12; m = matrix(ZZ, k, [(i*j)%k for i in [0..k-1] for j in [0..k-1]]); m.plot(cmap='gray')`



Euler's $\phi(n)$ function: `euler_phi(n)`

Kronecker symbol $\left(\frac{a}{b}\right)$ = `kronecker_symbol(a,b)`

Quadratic residues: `quadratic_residues(n)`

Quadratic non-residues: `quadratic_residues(n)`

ring $\mathbf{Z}/n\mathbf{Z}$ = `Zmod(n)` = `IntegerModRing(n)`

a modulo n as element of $\mathbf{Z}/n\mathbf{Z}$: `Mod(a, n)`

primitive root modulo n = `primitive_root(n)`

inverse of $n \pmod{m}$: `n.inverse_mod(m)`

power $a^n \pmod{m}$: `power_mod(a, n, m)`

Chinese remainder theorem: `x = crt(a,b,m,n)`

finds x with $x \equiv a \pmod{m}$ and $x \equiv b \pmod{n}$

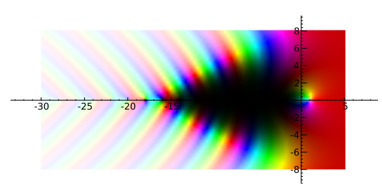
discrete log: `log(Mod(6,7), Mod(3,7))`

order of $a \pmod{n}$ =`Mod(a,n).multiplicative_order()`

square root of $a \pmod{n}$ =`Mod(a,n).sqrt()`

Special Functions

`complex_plot(zeta, (-30,5), (-8,8))`



$\zeta(s) = \prod_p \frac{1}{1-p^{-s}} = \sum \frac{1}{n^s}$ = `zeta(s)`

$\mathrm{Li}(x) = \int_2^x \frac{1}{\log(t)} dt$ = `Li(x)`

$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt$ = `gamma(s)`

Continued Fractions

`continued_fraction(pi)`

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cdots}}}}$$

continued fraction: `c=continued_fraction(x,` *bits*`)`

convergents: `c.convergents()`

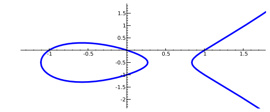
convergent numerator p_n = `c.pn(n)`

convergent denominator q_n = `c.qn(n)`

value: `c.value()`

Elliptic Curves

`EllipticCurve([0,0,1,-1,0]).plot(plot_points=300,thickness=3)`



`E = EllipticCurve(`$[a_1, a_2, a_3, a_4, a_6]$`)`

$$y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6$$

conductor N of E =`E.conductor()`

discriminant Δ of E =`E.discriminant()`

rank of E = `E.rank()`

free generators for $E(\mathbf{Q})$ = `E.gens()`

j-invariant = `E.j_invariant()`

$N_p = \#\{\text{solutions to } E \text{ modulo } p\}$ = `E.Np(`*prime*`)`

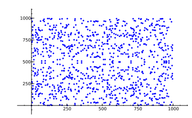
$a_p = p + 1 - N_p$ =`E.ap(`*prime*`)`

$L(E, s) = \sum \frac{a_n}{n^s}$ = `E.lseries()`

$\mathrm{ord}_{s=1} L(E, s)$ = `E.analytic_rank()`

Elliptic Curves Modulo p

`EllipticCurve(GF(997), [0,0,1,-1,0]).plot()`



`E = EllipticCurve(GF(p),` $[a_1, a_2, a_3, a_4, a_6]$`)`

$\#E(\mathbf{F}_p)$ = `E.cardinality()`

generators for $E(\mathbf{F}_p)$ = `E.gens()`

$E(\mathbf{F}_p)$ =`E.points()`