Funkcialaj Ekvacioj, **45** (2002) 291–318

Blowup Rate of Solutions to the Brezis-Nirenberg Equations with the Robin Condition

By

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1. Introduction

The aim of this paper is to investigate the blowup behavior of solutions to the Brezis-Nirenberg equation with the Robin condition. In our previous paper Kabeya, Yanagida and Yotsutani [10], we proved the range of λ for which a unique positive radial solution to

$$(1.1)\qquad \begin{cases} \Delta u+\lambda u+u^5=0 & \text{in } B=\{x\in \boldsymbol{R}^3 : |x|<1\}, \\ u>0 & \text{in } B, \\ \kappa\dfrac{\partial u}{\partial \nu}+u=0 & \text{on } \partial B, \end{cases}$$

exists, where ν is the outward unit normal vector to ∂B, $\lambda<\lambda_0$ (λ_0 is the first eigenvalue of $-\Delta$ with the homogeneous Robin condition on B, $\lambda_0=\pi^2$ if $n=3$ and $\kappa=0$) for each $\kappa\geq 0$.

When $\kappa=0$, in the three dimensional case, a solution to (1.1) exists for $\pi^2/4<\lambda<\pi^2$ while in the higher dimension, a solution does for $0<\lambda<\lambda_0$ (see e.g., Brezis and Nirenberg [4], Brezis and Peletier [5], or [10]). In this sense, the three dimensional case is an exceptional case and interesting phenomena occur in this case. So we concentrate on the three dimensional case.

Since our concern is on radial solutions, we consider the initial value problem of the ordinary differential equation

$$(1.2)\qquad \begin{cases} u_{rr}+\dfrac{2}{r}u_r+\lambda u+u_+^5=0, & 0<r<1, \\ u(0)=\alpha, \qquad u_r(0)=0, \end{cases}$$

and seek a suitable number $\alpha>0$ satisfying $u(r)>0$ on $(0,1)$ and

* Supported in part by the Grant-in-Aid for the Encouragement of Young Scientists (No. 13740116), Japan Society for the Promotion of Science

$$\kappa u_r(1) + u(1) = 0, \tag{1.3}$$

where $u_+ = \max\{u, 0\}$. Note that (1.2) has a solution for any $\alpha > 0$ and λ.

We introduce three numbers. Let $\mu_0 = \mu_0(\kappa) \in (0, \pi]$ be defined by

$$\begin{cases} \mu_0 = \pi, & \text{if } \kappa = 0, \\ 1 - \mu_0 \cot \mu_0 = \dfrac{1}{\kappa}, & \text{if } \kappa > 0. \end{cases}$$

Note that μ_0^2 is the first radial eigenvalue of $-\Delta$ subject to the boundary condition $\kappa \partial u/\partial \nu + u = 0$. For $0 \le \kappa \le 1$, define $\mu_1 = \mu_1(\kappa) \in [0, \pi/2]$ and $\zeta = \zeta(\kappa) \in [0, \infty)$ by

$$\begin{cases} \mu_1 = \dfrac{\pi}{2}, & \text{if } \kappa = 0, \\ 1 + \mu_1 \tan \mu_1 = \dfrac{1}{\kappa}, & \text{if } 0 < \kappa \le 1, \end{cases}$$

and

$$\begin{cases} \zeta = \infty, & \text{if } \kappa = 0, \\ \zeta \coth \zeta = \dfrac{1}{\kappa}, & \text{if } 0 < \kappa < 1, \\ \zeta = 0, & \text{if } \kappa = 1, \end{cases}$$

respectively. As we will see in Theorem B, $\lambda = \mu_1^2$ is a blowup point. Also note that

$$\begin{cases} \Delta u + \mu_1^2 u = 0 & \text{in } B, \\ \kappa u'(1) + u(1) = 0 \end{cases}$$

has a positive singular solution.

Let us recall our previous results (see also [4] for $\kappa = 0$).

Theorem A (Theorem 1.1 of [10]). *Let $n = 3$ and $0 \le \kappa \le 1$. If $\mu_1^2 < \lambda < \mu_0^2$, then* (1.1) *has a unique radial solution. If $-\zeta^2 \le \lambda \le \mu_1^2$, then* (1.1) *has no radial solution.*

By Theorem A, a mapping from λ to the initial value α is defined, that is, α is a function of $\lambda \in (\mu_1^2, \mu_0^2]$. Let us denote the unique solution by u_λ. We can draw the graph of $\alpha = \alpha(\lambda; \kappa)$. Concerning the graph of $\alpha = \alpha(\lambda; \kappa)$, we have the following global behavior.

Theorem B ((i) of Theorem 1.3 of [10]). *Let $0 \le \kappa \le 1$. Then the graph of $\alpha(\lambda; \kappa)$ is a continuous curve satisfying $\alpha(\lambda) \to 0$ as $\lambda \to \mu_0^2 - 0$ and $\alpha(\lambda; \kappa) \to \infty$ as $\lambda \to \mu_1^2 + 0$.*

We can see that $\lambda = \mu_1^2$ is the blowup point.

The purpose of this paper is to show the blowup order of $\alpha = \alpha(\lambda; \kappa)$ and an asymptotic behavior of a rescaled solution mainly following the method by Brezis and Peletier [5]. We utilize the Green function as in [5] and Rey [13] used for the Dirichlet problem.

First we consider the case $0 \leq \kappa < 1$.

Theorem 1.1. *Let* $0 \leq \kappa < 1$. *Then the asymptotic behavior of* $\alpha(\lambda; \kappa)$ *is*

$$\lim_{\lambda \to \mu_1^2 + 0} (\lambda - \mu_1^2)(\alpha(\lambda; \kappa))^2 = \frac{2\sqrt{3}\pi\mu_1^2\{(1-\kappa)^2 \sin \mu_1 + \kappa(1-\kappa)\mu_1 \cos \mu_1\}}{(1 - \kappa + \kappa^2\mu_1^2) \sin \mu_1}.$$

The blowup rate of $\alpha(\lambda; \kappa)$ as $\lambda \to \mu_1^2 + 0$ is known by [5] for $\kappa = 0$. Not to mention, our results covers that by [5]. In fact, if $\kappa = 0$, then $\mu_1 = \pi/2$ and the right-hand side is $\sqrt{3}\pi^3/2$, which is $\sqrt{3}$ times of that in [5]. Since Brezis and Peletier treated the equation $\Delta\bar{u} + \lambda\bar{u} + 3\bar{u}^5 = 0$, the scaling $u = \sqrt[4]{3}\bar{u}$ brings this difference. The difference of the coefficient also appears in the limiting behavior of a scaled function (see Theorem 1.3).

In Theorem 1.1, we exclude the case $\kappa = 1$. In this case, we see a different blowup order. Note that $\mu_1 = 0$ when $\kappa = 1$. The difference is affected by whether $\mu_1 = 0$ or not.

Theorem 1.2. *When* $\kappa = 1$, *then*

$$\lim_{\lambda \to +0} \lambda(\alpha(\lambda; 1))^4 = 3.$$

Similar to [5], the limiting behavior of a scaled function is obtained. Let us denote the Green function of $(-\Delta - \lambda)$ subject to $\kappa\partial u/\partial \nu + u = 0$ by $G^*_{\kappa,\lambda}(x, y)$ and the "reduced" Green function $G_{\kappa,\lambda}(x) := G^*_{\kappa,\lambda}(x, 0)$ for $x \neq 0$.

Theorem 1.3. *Let* u_λ *be the unique raidial solution to* (1.1). *Then the asymptotic behavior of* u_λ *is as follows.*

(i) *If* $0 \leq \kappa < 1$, *then*

$$\lim_{\lambda \to \mu_1^2 + 0} \frac{u_\lambda(x)}{\sqrt{\lambda - \mu_1^2}} = \frac{4}{\mu_1}\sqrt{\frac{\sqrt{3}\pi(1 - \kappa + \kappa^2\mu_1^2) \sin \mu_1}{2\{(1-\kappa)^2 \sin \mu_1 + \kappa(1-\kappa)\mu_1 \cos \mu_1\}}} G_{\kappa,\mu_1^2}(|x|). \tag{1.4}$$

for $x \neq 0$.

(ii) *If* $\kappa = 1$, *then*

$$\lim_{\lambda \to +0} \frac{u_\lambda(x)}{\sqrt[4]{\lambda}} = 4\sqrt[4]{3}\pi G_{1,0}(|x|) \tag{1.5}$$

for $x \neq 0$.

Besides $\mu_1 = 0$ or not, the difference between (i) and (ii) can be explained as the finite part of the reduced Green function $G_{\kappa,\mu_1^2}(|x|) - 1/(4\pi|x|)|_{x=0}$ is non-zero or not. See Theorem 2 of [5] for a similar result on the nearly critical growth.

In [10], the differential form of the Pohozaev identity plays a crucial role. However, to investigate the blowup nature, we need to use the integral form of the identity because it enables us to treat the Dirac δ function-like behavior.

As related topics, for the Neumann problem ($\kappa = \infty$), such a blowup behavior of solutions to the scalar-field equation with the critical Sobolev exponent is discussed in Budd, Knaap and Peletier [6] and there are many works on the nearly critical growth (see, e.g., Atkinson and Peletier [2], Han [8] or Pan and Wang [12]).

What will happen in the case where $1 < \kappa < \infty$? For $\kappa > 1$, the unique existence of a solution was also discussed in [10]. Moroever, similar results to Theorems 1.1 and 1.3 are obtained (the blowup rate is as in Theorem 1.1). The blowup point is a continuous function of $\kappa \in [0, \infty]$. These will be discussed in a forthcoming paper [9].

This paper is organized as follows. The Pohozaev identity for an auxiliary inhomogeneous linear problem is discussed in Sections 2 and 3. In Section 4, several properties of the accurately approximate solution, which are useful for proofs of Theorems 1.1, 1.2 and 1.3, are proved. Proofs of Theorem 1.1, 1.2 and 1.3 are given in Section 5.

2. Pohozaev identity of the integral form

In this section, we give the Pohozaev identity of the integral form to the problem

$$(2.1) \qquad \begin{cases} \Delta u + \lambda u + f(|x|) = 0 & \text{in } B, \\ \kappa \dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial B, \end{cases}$$

with radial u, where B is the unit ball in $\boldsymbol{R}^3$ and $f \in L^\infty(B)$.

Proposition 2.1. *If there exists a radial solution $u \in H^2(B)$ to* (2.1), *then there holds*

$$4\pi(1 - \kappa + \lambda\kappa^2)(u'(1))^2 - 2\lambda \int_B u^2\,dx = -\int_B f(|x|)\{u + 2(x \cdot \nabla u)\}dx.$$

Remark. By the trace operator, we have $\partial u/\partial \nu \in L^2(B)$ for $u \in H^2(B)$. Moreover, since u is radial, $\partial u/\partial \nu = u'(1)$ is finite.

Proof. It suffices to prove the identity in the case where $u \in C^2(B) \cap C^1(\bar{B})$. Multiplying the both side of (2.1) by u and integrating it over B, we have

$$\int_B (|\nabla u|^2 - \lambda u^2)dx - \int_{\partial B} u\frac{\partial u}{\partial \nu}\,dS = \int_B fu\,dx. \tag{2.2}$$

On the other hand, multiplying the both side of (2.1) by $x \cdot \nabla u$ and integrating it over B, we have

$$\int_B (\Delta u + \lambda u)(x \cdot \nabla u)dx + \int_B f(|x|)(x \cdot \nabla u)dx = 0.$$

Since

$$\begin{aligned}
&\int_B \Delta u(x \cdot \nabla u)dx \\
&\quad = \int_B \{\mathrm{div}(\nabla u)\}(x \cdot \nabla u)dx = \int_{\partial B} \frac{\partial u}{\partial \nu}(x \cdot \nabla u)dS - \int_B \nabla u \cdot \nabla(x \cdot \nabla u)dx,
\end{aligned}$$

we have

$$\begin{aligned}
&\int_{\partial B} \frac{\partial u}{\partial \nu}(x \cdot \nabla u)dS - \int_B \nabla u \cdot \nabla(x \cdot \nabla u)dx + \frac{\lambda}{2}\int_B (x \cdot \nabla(u^2))dx \\
&\quad + \int_B f(|x|)(x \cdot \nabla u)dx = 0.
\end{aligned} \tag{2.3}$$

Moreover, since

$$\begin{aligned}
&\int_B \nabla u \cdot \nabla(x \cdot \nabla u)dx \\
&\quad = \int_B |\nabla u|^2 dx + \frac{1}{2}\int_B (x \cdot \nabla(|\nabla u|^2))dx \\
&\quad = \int_B |\nabla u|^2 dx + \frac{1}{2}\int_B \sum_{i=1}^{3} x_i\left(\frac{\partial}{\partial x_i}|\nabla u|^2\right)dx \\
&\quad = \int_B |\nabla u|^2 dx + \frac{1}{2}\int_B \mathrm{div}(x|\nabla u|^2)dx - \frac{3}{2}\int_B |\nabla u|^2 dx \\
&\quad = -\frac{1}{2}\int_B |\nabla u|^2 dx + \frac{1}{2}\int_{\partial B} (x \cdot \nu)|\nabla u|^2 dS,
\end{aligned}$$

and since

$$\int_B (x\cdot\nabla(u^2))dx$$
$$= \int_B \operatorname{div}(xu^2)dx - 3\int_B u^2\,dx = \int_{\partial B}(x\cdot\nu)u^2\,dS - 3\int_B u^2\,dx,$$

the equality (2.3) yields

$$(2.4)\qquad \int_{\partial B}\frac{\partial u}{\partial\nu}(x\cdot\nabla u)dS + \frac{1}{2}\int_B |\nabla u|^2 dx - \frac{1}{2}\int_{\partial B}(x\cdot\nu)|\nabla u|^2 dS$$
$$+\frac{\lambda}{2}\int_{\partial B}(x\cdot\nu)u^2\,dS - \frac{3}{2}\lambda\int_B u^2\,dx + \int_B f(|x|)(x\cdot\nabla u)dx = 0.$$

Substracting one half of (2.2) from (2.4), we get

$$(2.5)\qquad \int_{\partial B}\frac{\partial u}{\partial\nu}(x\cdot\nabla u)dS + \frac{1}{2}\left(\lambda\int_B u^2\,dx + \int_{\partial B} u\frac{\partial u}{\partial\nu}\,dS + \int_B f(|x|)u\,dx\right)$$
$$-\frac{1}{2}\int_{\partial B}(x\cdot\nu)|\nabla u|^2 dS + \frac{\lambda}{2}\int_{\partial B}(x\cdot\nu)u^2\,dS - \frac{3}{2}\lambda\int_B u^2\,dx$$
$$+\int_B f(|x|)(x\cdot\nabla u)dx = 0.$$

Since u is radial and B is the unit ball, we have

$$x\cdot\nabla u = \frac{\partial u}{\partial\nu}\qquad \text{on } \partial B.$$

Taking the boundary condition into account, we get

$$4\pi(1-\kappa+\lambda\kappa^2)(u'(1))^2 - 2\lambda\int_B u^2\,dx = -\int_B f(|x|)\{u + 2(x\cdot\nabla u)\}dx. \qquad \square$$

Remark. The identity (2.5) holds also for a bounded smooth domain $\Omega\subset\boldsymbol{R}^3$ with $u\in H^2(\Omega)$.

In order to apply Proposition 2.1 to (1.2), regarding $f = u^5$, we have

$$4\pi(1-\kappa+\lambda\kappa^2)(u'(1))^2 - 2\lambda\int_B u^2\,dx = -\int_B (u^6 + 2(x\cdot\nabla u)u^5)dx,$$

since u is radially symmetric. By the Gauss Theorem, we have

$$4\pi(u(1))^6 = \int_{\partial B} u^6\,dS = \int_B \operatorname{div}(xu^6)dx$$
$$= 3\int_B u^6\,dx + 6\int_B (x\cdot\nabla u)u^5\,dx,$$

and thus we obtain

$$\int_B (x \cdot \nabla u)u^5 \, dx = \frac{2}{3}\pi(u(1))^6 - \frac{1}{2}\int_B u^6 \, dx.$$

Thus we see that

$$(2.6) \qquad 4\pi(1-\kappa+\lambda\kappa^2)(u'(1))^2 - 2\lambda\int_B u^2 \, dx = -\frac{1}{3}\int_{\partial B} u^6 \, dS = -\frac{4\pi}{3}(u(1))^6.$$

From now on, we define

$$(2.7) \qquad J_\lambda(u) := (1-\kappa+\lambda\kappa^2)\int_{\partial B}\left(\frac{\partial u}{\partial \nu}\right)^2 dS - 2\lambda\int_B u^2 \, dx.$$

As in [5], we efficiently use $J_\lambda(u)$.

In order to calculate the blowup order, we need the reduced Green function $G_{\kappa,\lambda}(x)$ $(=G^*_{\kappa,\lambda}(x,0))$ of $-\Delta - \lambda$ subject to the boundary condition $\kappa \partial u/\partial \nu + u = 0$.

Lemma 2.1. *For $0 < \lambda < \mu_0^2$, the reduced Green function of $-\Delta - \lambda$ subject to the boundary condition $\kappa \partial u/\partial \nu + u = 0$ on ∂B is given by*

$$(2.8) \qquad G_{\kappa,\lambda}(r) = \frac{1}{4\pi(\sin\mu_1 + A\cos\mu_1)r}\{-\sin(\sqrt{\lambda}r - \mu_1) + A\cos(\sqrt{\lambda}r - \mu_1)\},$$

where $r = |x|$ and

$$(2.9) \qquad A = \frac{(1-\kappa)\sin(\sqrt{\lambda}-\mu_1) + \kappa\sqrt{\lambda}\cos(\sqrt{\lambda}-\mu_1)}{(1-\kappa)\cos(\sqrt{\lambda}-\mu_1) - \kappa\sqrt{\lambda}\sin(\sqrt{\lambda}-\mu_1)}.$$

Proof. As we see in [5], a function of the form

$$(2.10) \qquad E(r) = \frac{-D\sin(\sqrt{\lambda}r) + \cos(\sqrt{\lambda}r)}{4\pi r}$$

with a constant D is an elementary solution (i.e., E satisfies $-\Delta E - \lambda E = \delta$, but is not necessarily satisfies the boundary condition). We have only to find a suitable coefficient A in (2.8) so that $G_{\kappa,\lambda}$ satisfies the boundary condition. The phase shift in the desired form can be "absorbed" by taking suitable coefficient D in (2.10), we consider the desired function as in (2.8).

In fact, let $\tilde{G}(r) = 4\pi(\sin\mu_1 + A\cos\mu_1)G_{\kappa,\lambda}(r)$. Since

$$\tilde{G}_r(r) = -\frac{1}{r^2}\{-\sin(\sqrt{\lambda}r - \mu_1) + A\cos(\sqrt{\lambda} - \mu_1)\}$$
$$+\frac{\sqrt{\lambda}}{r}\{-\cos(\sqrt{\lambda}r - \mu_1) - A\sin(\sqrt{\lambda}r - \mu_1)\},$$

we have

$$r^2\tilde{G}_r(r) = \sin(\sqrt{\lambda}r - \mu_1) - A\cos(\sqrt{\lambda}r - \mu_1) \\ - \sqrt{\lambda}r\{\cos(\sqrt{\lambda}r - \mu_1) + A\sin(\sqrt{\lambda}r - \mu_1)\}.$$

Thus we obtain

$$\begin{aligned}(r^2\tilde{G}_r)_r &= \sqrt{\lambda}\{\cos(\sqrt{\lambda}r - \mu_1) + A\sin(\sqrt{\lambda}r - \mu_1)\} \\ &\quad - \sqrt{\lambda}\{\cos(\sqrt{\lambda}r - \mu_1) + A\sin(\sqrt{\lambda}r - \mu_1)\} \\ &\quad - \lambda r\{-\sin(\sqrt{\lambda}r - \mu_1) + A\cos(\sqrt{\lambda}r - \mu_1)\} \\ &= -\lambda r^2\tilde{G}\end{aligned}$$

for $r \neq 0$. So we have only to check the boundary condition and the intensity of $G_{\kappa,\lambda}$ at the origin.

On the boundary, we get

$$\begin{aligned}&\kappa\tilde{G}_r(1) + \tilde{G}(1) \\ &\quad = \kappa\sin(\sqrt{\lambda} - \mu_1) - \kappa\sqrt{\lambda}\cos(\sqrt{\lambda} - \mu_1) - A\kappa\cos(\sqrt{\lambda} - \mu_1) \\ &\qquad - A\kappa\sqrt{\lambda}\sin(\sqrt{\lambda} - \mu_1) - \sin(\sqrt{\lambda} - \mu_1) + A\cos(\sqrt{\lambda} - \mu_1) \\ &\quad = (\kappa - 1)\sin(\sqrt{\lambda} - \mu_1) - \kappa\sqrt{\lambda}\cos(\sqrt{\lambda} - \mu_1) \\ &\qquad + A\{(1 - \kappa)\cos(\sqrt{\lambda} - \mu_1) - \kappa\sqrt{\lambda}\sin(\sqrt{\lambda} - \mu_1)\} = 0\end{aligned}$$

by the definition of A. As for the intensity at the origin, we have

$$\lim_{r\to 0} rG_{\kappa,\lambda}(r) = \frac{1}{4\pi}.$$

According to Theorem 8.2 of Mizohata [11], the Green function $G^*_{\kappa,\lambda}(x, y)$ must be expressed as

$$G^*_{\kappa,\lambda}(x, y) = G_{\kappa,\lambda}(|x - y|) + \Xi_{\kappa,\lambda}(x, y),$$

where $\Xi_{\kappa,\lambda}(x, y)$ satisfies

(i) $\Delta\Xi_{\kappa,\lambda} + \lambda\Xi_{\kappa,\lambda} = 0$ in B for any fixed $y \in B$,

(ii) $(1 - \alpha_\varepsilon(x - y))G^*_{\kappa,\lambda}(|x - y|) + \Xi_{\kappa,\lambda}(x, y)$ satisfies the boundary condition on ∂B for any fixed $y \in B_{1-2\varepsilon}$ with $\alpha_\varepsilon(r) \in C_0^\infty(B_{2\varepsilon})$ such that $0 \leq \alpha_\varepsilon \leq 1$ and $\alpha_\varepsilon \equiv 1$ on B_ε for sufficiently small $\varepsilon > 0$.

Since $G_{\kappa,\lambda}(x)$ satisfies the boundary condition on ∂B, taking especially $y = 0$ in (ii), we see that $\Xi_{\kappa,\lambda}(x, 0) \equiv 0$. Thus $G_{\kappa,\lambda}$ must be the reduced Green function of $(-\Delta - \lambda)$ with the boundary condition $\kappa\partial u/\partial\nu + u = 0$. □

Now we define the regular part of the reduced Green function by

$$g_{\kappa,\lambda}(r) = G_{\kappa,\lambda}(r) - \frac{1}{4\pi r},$$

i.e.,

$$g_{\kappa,\lambda}(r) = \frac{-\sin(\sqrt{\lambda}r - \mu_1) + A\cos(\sqrt{\lambda}r - \mu_1) - (\sin\mu_1 + A\cos\mu_1)}{4\pi(\sin\mu_1 + A\cos\mu_1)r}. \tag{2.11}$$

The regular part $g_{\kappa,\lambda}(r)$ as well as $G_{\kappa,\lambda}$ will play important roles to show the blowup rate.

Remark. Note that $G_{1,0}(r) = 1/(4\pi r)$ and $g_{1,0}(0) \equiv 0$. Indeed, $G'_{1,0}(1) = -1/(4\pi)$ and $G_{1,0}(1) = 1/(4\pi)$. $G_{1,0}(r)$ is the reduced Green function of $-\Delta$ on $\boldsymbol{R}^3$.

3. Pohozaev identity for the reduced Green function

Here we give a regularity lemma. The inequality which appears here gives us fundamental information on the estimate of $J_\lambda(u)$. We use the L^p-theoretic treatment of (pseudo) differential operators as in Taylor [14, Chapter XI]. For the Dirichlet problem, see Gilbarg and Trudinger [7, Chapter 8].

Lemma 3.1. *Let $u \in L^1(\Omega)$ with $\Delta u \in L^1(\Omega)$ be a solution to*

$$\begin{cases} -\Delta u - \lambda u = f & \text{in } \Omega, \\ \kappa\dfrac{\partial u}{\partial \nu} + u = b & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where Ω is a bounded smooth domain in $\boldsymbol{R}^3$, $f \in L^1(\Omega) \cap L^\infty(\Omega \backslash B_\rho)$ with B_ρ being a ball with radius $\rho > 0$ centered at the origin, $b \in C^2(\partial\Omega)$, and $\lambda \in (0, \mu_0^2)$. Then u satisfies

$$\|u\|_{W^{1,q}(\Omega)} + \|\nabla u\|_{C^{0,\xi}(\partial\Omega)} \le C(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)} + \|b\|_{C^{1,\xi}(\partial\Omega)}) \tag{3.2}$$

for any $q < 3/2$, $\xi \in (0,1)$ and any neighborhood ω of $\partial\Omega$ in Ω, with constant $C > 0$.

Proof. In view of the classical Schauder estimates, $\|b\|_{C^{1,\xi}(\partial\Omega)}$ in the right hand side follows if we see the $C^{1,\xi}$-regularity of the solution near $\partial\Omega$. To see the regularity at $\partial\Omega$ and the upper estimate by $\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)}$, we may assume $b = 0$. In this proof, C represents various constants.

First we claim that

$$\|u\|_{W^{1,q}(\Omega)} \le C\|f\|_{L^1(\Omega)}$$

for $q < 3/2$.

In case of $n=3$ and $p>3$, by the Sobolev embedding theorem $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$, we obtain

$$W_0^{1,p}(\Omega) \subset W^{1,p}(\Omega) \subset L^\infty(\Omega).$$

Thus the duality inclusion implies

$$L^1(\Omega) \subset (L^\infty(\Omega))^* \subset (W^{1,p}(\Omega))^* \subset (W_0^{1,p}(\Omega))^* = W^{-1,p/(p-1)}(\Omega).$$

Let $(-\Delta-\lambda)^{-1}$ be the inverse operator of $(-\Delta-\lambda)$ subject to the boundary condition $\kappa(\partial u/\partial v)+u=0$. Since $\lambda<\lambda_0$ (the first eigenvalue of $-\Delta$ with $\kappa\partial u/\partial v+u=0$ on $\partial\Omega$), $(-\Delta-\lambda)^{-1}$ exists. Then we have $u=(-\Delta-\lambda)^{-1}f$. Regarding $f\in W^{-1,p/(p-1)}(\Omega)$, we can express $f=(1-\Delta)^{1/2}g$ with $g\in L^{p/(p-1)}(\Omega)$ (see e.g., Ziemer [15] p. 185–187). We rewrite u as

$$\begin{aligned} u &= (-\Delta-\lambda)^{-1}(1-\Delta)^{1/2}g \\ &= (-\Delta-\lambda)^{-1}(1-\Delta)^{1/2}\{(1-\Delta)^{1/2}(1-\Delta)^{-1/2}g\} \\ &= \{(-\Delta-\lambda)^{-1}(1-\Delta)\}(1-\Delta)^{-1/2}g. \end{aligned}$$

Since $(-\Delta-\lambda)^{-1}(1-\Delta)\in \mathrm{OPS}^0_{1,0}$, $(-\Delta-\lambda)^{-1}(1-\Delta)$ is a bounded operator from $L^p(\Omega)$ to $L^p(\Omega)$ for any $p>1$ ([14] Chapter XI, Theorem 2.2, p. 267). Thus we see that

$$(-\Delta-\lambda)^{-1}L^1(\Omega) \subset (-\Delta-\lambda)^{-1}W^{-1,p/(p-1)}(\Omega) \subset W^{1,p/(p-1)}(\Omega)$$

in view of $(1-\Delta)^{-1/2}g\in W^{1,p/(p-1)}(\Omega)$. Hence $(-\Delta-\lambda)^{-1}$ is a bounded operator from $L^1(\Omega)$ to $W^{1,q}(\Omega)$ with $q:=p/(p-1)<3/2$. That is, for (3.1) with $b=0$, we have

$$\|u\|_{W^{1,q}(\Omega)} \le C\|f\|_{L^1(\Omega)}. \tag{3.3}$$

Next we claim that

$$\|\nabla u\|_{C^{0,\xi}(\partial\Omega)} \le C(\|f\|_{L^1(\Omega)}+\|f\|_{L^\infty(\omega)}).$$

Let χ be the characteristic function of ω and define $f_1=\chi f$ and $f_2=(1-\chi)f$. Let u_i be the solution of

$$\begin{cases} -\Delta u_i-\lambda u_i=f_i & \text{in } \Omega, \\ \kappa\dfrac{\partial u_i}{\partial v}+u_i=0 & \text{on } \partial\Omega, \end{cases}$$

for $i=1,2$. Note that the solution u to (3.1) satisfies $u=u_1+u_2$. Since $f_1\in L^\infty(\Omega)$, we have

$$\|u_1\|_{W^{2,p}(\Omega)} \le C\|f_1\|_{L^p(\Omega)} \le C\|f_1\|_{L^\infty(\Omega)} = C\|f_1\|_{L^\infty(\omega)}$$

by

$$u_1 = (-\Delta - \lambda)^{-1} f_1$$
$$= \{(-\Delta - \lambda)^{-1}(1 - \Delta)\}(1 - \Delta)^{-1} f_1 \in (1 - \Delta)^{-1} L^p(\Omega) = W^{2,p}(\Omega)$$

for any $p > 1$, as before. If we choose p sufficiently large, we conclude $u_1 \in C^{1,\xi}(\Omega)$ by the Sobolev embedding. Thus we have

$$\|\nabla u_1\|_{C^{0,\xi}(\partial\Omega)} \leq C\|f\|_{L^\infty(\omega)}. \tag{3.4}$$

As for u_2, we have

$$\|u_2\|_{W^{1,q}(\Omega)} \leq C\|f_2\|_{L^1(\Omega)} \leq C\|f\|_{L^1(\Omega)},$$

by (3.3). We note that $W^{1,q}(\Omega) \hookrightarrow L^t(\Omega)$ $(t = 3q/(3-q))$. If we choose q sufficiently close to $3/2$, then we have $W^{1,q}(\Omega) \hookrightarrow L^2(\Omega)$, that is,

$$\|u_2\|_{L^2(\Omega)} \leq C\|f_2\|_{L^1(\Omega)}.$$

Since u_2 satisfies $-\Delta u_2 - \lambda u_2 = 0$ in ω, we get

$$\|u_2\|_{C^{1,\xi'}(\bar{\omega}')} \leq C\|u_2\|_{L^2(\omega)},$$

where ω' $(\subset\omega)$ is a neighborhood of $\partial\Omega$ by the standard elliptic estimate with $0 < \xi' < 1$. Thus we have

$$\|\nabla u_2\|_{C^{0,\xi}(\partial\Omega)} \leq C\|f\|_{L^1(\Omega)}. \tag{3.5}$$

Combining (3.4) and (3.5), we obtain the desired conclusion together with the Schauder estimate on $\partial\Omega$. □

Using Lemma 3.1, we have a kind of the Pohozaev identity concerning $G_{\kappa,\lambda}$ and $g_{\kappa,\lambda}$.

Proposition 3.1. *Let $G_{\kappa,\lambda}(x)$ be defined by* (2.2) *and $g_{\kappa,\lambda}(x)$ be done by* (2.11). *Then there holds*

$$J_\lambda(G_{\kappa,\lambda}) = 4\pi(1 - \kappa + \kappa\lambda^2)(G'_{\kappa,\lambda}(1))^2 - 2\lambda \int_B G^2_{\kappa,\lambda}\, dx = -g_{\kappa,\lambda}(0).$$

Proof. Define

$$\delta_\rho(x) = \frac{1}{|B_\rho|}\chi_{B_\rho}(x)$$

for $\rho > 0$ with B_ρ being a ball centered at the origin and its radius ρ, and $\chi_{B_\rho}(x)$ is the characteristic function of B_ρ. Then $\delta_\rho(x)$ converges weakly in measure to the Dirac delta function δ. Let us define $v_\rho(x)$ by

$$v_\rho(x) = \begin{cases} -\dfrac{r^2}{8\pi\rho^3} + \dfrac{3}{8\pi\rho}, & 0 < r = |x| < \rho, \\ \dfrac{1}{4\pi r}, & \rho \le r. \end{cases}$$

That is, $v_\rho \in C^1(\boldsymbol{R}^3) \cap C^2(\boldsymbol{R}^3 \backslash \partial B_\rho)$ is a solution to

$$-\Delta v_\rho = \delta_\rho$$

in $\boldsymbol{R}^3$ such that $v_\rho(x) \to 0$ as $|x| \to \infty$. Note that $v_\rho \to 1/(4\pi|x|)$ in $L^2(B)$ as $\rho \to +0$. Let $u_\rho \in C^1(B) \cap C^2(B \backslash \partial B_\rho)$ be a radial solution of

$$\begin{cases} -\Delta u - \lambda u = \delta_\rho & \text{in } B, \\ \kappa \dfrac{\partial u}{\partial \nu} + u = 0 & \text{on } \partial B. \end{cases}$$

Then by Proposition 2.1, we have

$$4\pi(1 - \kappa + \lambda\kappa^2)(u_\rho'(1))^2 - 2\lambda \int_B u_\rho^2 \, dx = -\int_B \delta_\rho(x)\{u_\rho + 2(x \cdot \nabla u_\rho)\} dx. \tag{3.6}$$

By the a priori estimate (3.2) in Lemma 3.1, we have

$$\|u_\rho\|_{W^{1,q}(B)} + \|\nabla u_\rho\|_{C^{0,\xi}(\partial B)} \le C.$$

Since $\delta_\rho \rightharpoonup \delta$ in measure as $\rho \to 0$, choosing a subsequence if necessary (still denoted by $\{u_\rho\}$), we conclude that $u_\rho \to G_{\kappa,\lambda}$ in $L^2(B)$ and that $\partial u/\partial \nu$ to $\partial G_{\kappa,\lambda}/\partial \nu$ as $\rho \to +0$ in view of the compactness of the embeddings $W^{1,q}(B) \hookrightarrow L^2(B)$ and $C^{0,\xi}(\partial B) \hookrightarrow C^0(\partial B)$. Thus we have (along a subsequence)

$$\lim_{\rho \to +0} \left\{ 4\pi(1 - \kappa + \lambda\kappa^2)(u_\rho'(1))^2 - 2\lambda \int_B u_\rho^2 \, dx \right\}$$
$$= 4\pi(1 - \kappa + \lambda\kappa^2)(G_{\kappa,\lambda}'(1))^2 - 2\lambda \int_B G_{\kappa,\lambda}^2 \, dx.$$

On the other hand, there holds

$$\int_B \delta_\rho\{u_\rho + 2(x \cdot \nabla u_\rho)\} dx$$
$$= \int_B \delta_\rho\{(u_\rho - v_\rho) + 2x \cdot \nabla(u_\rho - v_\rho)\} dx + \int_B \delta_\rho\{v_\rho + 2x \cdot \nabla v_\rho\} dx.$$

As is shown in [5, p. 167, (4.12)] (direct computation), we have

$$\int_B \delta_\rho\{v_\rho + 2x \cdot \nabla v_\rho\} dx = 0.$$

By the definition of u_ρ and v_ρ, we have

$$(3.7) \qquad -\Delta(u_\rho - v_\rho) - \lambda(u_\rho - v_\rho) = \lambda v_\rho \quad \text{in } B.$$

The boundary condition yields

$$\kappa\frac{\partial}{\partial\nu}(u_\rho - v_\rho) + u_\rho - v_\rho\bigg|_{\partial B} = \frac{\kappa}{4\pi r^2} - \frac{1}{4\pi r}\bigg|_{r=1}.$$

Since

$$v_\rho \to \frac{1}{4\pi r} \qquad \text{in } L^2(B)$$

as $\rho \to +0$, and since $(-\Delta - \lambda)^{-1}$ is a bounded operator from $L^2(B)$ to $H^2(B)$, we have

$$(3.8) \qquad u_\rho - v_\rho \to g_{k,\lambda} \qquad \text{in } H^2(B)$$

as $\rho \to +0$ in view of (3.7), where $g_{\kappa,\lambda}$ is a unique solution of

$$(3.9) \qquad \begin{cases} -\Delta g - \lambda g = \dfrac{\lambda}{4\pi r} & \text{in } B, \\ \kappa\dfrac{\partial g}{\partial\nu} + g = \dfrac{1}{4\pi}\left(\dfrac{\kappa}{r^2} - \dfrac{1}{r}\right)\bigg|_{r=1} & \text{on } \partial B. \end{cases}$$

Here, $g_{\kappa,\lambda}$ is the finite part of the reduced Green function $G_{\kappa,\lambda}$. Since $B \subset \boldsymbol{R}^3$, we have $H^2(B) \hookrightarrow C(B)$ by the Sobolev embedding theorem. Thus $u_\rho - v_\rho$ converges to $g_{\kappa,\lambda}$ uniformly. Hence we get

$$\int_B \delta_\rho(x)(u_\rho - v_\rho)dx \to g_{\kappa,\lambda}(0)$$

as $\rho \to +0$.

Finally, concerning the remaining integral, we have

$$\int_B \delta_\rho(x)\{x \cdot \nabla(u_\rho - v_\rho)\}dx \le \frac{3}{4\pi\rho^2}\int_{B_\rho}|\nabla(u_\rho - v_\rho)|dx.$$

By (3.8), $\nabla(u_\rho - v_\rho)$ is bounded in $H^1(B)$, i.e., in $L^6(B)$. Hence we get

$$\int_{B_\rho}|\nabla(u_\rho - v_\rho)|dx \le |B_\rho|^{5/6}\|\nabla(u_\rho - v_\rho)\|_{L^6(B)} = \left(\frac{4}{3}\pi\right)^{5/6}\rho^{5/2}\|\nabla(u_\rho - v_\rho)\|_{L^6(B)}.$$

Thus we see

$$\int_B \delta_\rho(x)(x \cdot \nabla(u_\rho - v_\rho))dx = O(\rho^{1/2})$$

as $\rho \to +0$. Letting $\rho \to +0$ in (3.6), we obtain the conclusion

$$J_\lambda(G_{\kappa,\lambda}) = 4\pi(1-\kappa+\kappa\lambda^2)(G'_{\kappa,\lambda}(1))^2 - 2\lambda\int_B G_{\kappa,\lambda}^2\,dx = -g_{\kappa,\lambda}(0). \qquad \square$$

4. Approximating the solution

To prove Theorems 1.1–1.3, we need to approximate the solution very accurately. For $\lambda = \mu_1^2 + \varepsilon$, we denote the solution $u_{\mu_1^2+\varepsilon}$ to (1.2) by u_ε for simplicity and set

$$\beta := (u_\varepsilon(0))^{-2} = (\alpha(\lambda;\kappa))^{-2}$$

according to [5].

We define

$$U_\beta(x) := \frac{\sqrt{\beta}}{(\beta^2 + |x|^2/3)^{1/2}}.$$

$U_\beta(x)$ is a positive solution to

$$\Delta U + U^5 = 0 \qquad \text{in } \boldsymbol{R}^3,$$

with $U_\beta(0) = \beta^{-1/2}$ $(=u_\varepsilon(0))$. We set

$$g_{\kappa,\varepsilon} := g_{\kappa,\mu_1^2+\varepsilon}, \qquad \phi_\beta(x) := U_\beta + 4\sqrt{3}\pi\sqrt{\beta}g_{\kappa,\varepsilon}, \qquad \eta(x) := u_\varepsilon - \phi_\beta.$$

Note that η is a radially symmetric function. Then the "error" term η satisfies

$$\begin{aligned}
(4.1)\quad & -\Delta\eta - (\mu_1^2+\varepsilon)\eta \\
&= -\Delta(u_\varepsilon - U_\beta - 4\sqrt{3}\pi\sqrt{\beta}g_{\kappa,\varepsilon}) - (\mu_1^2+\varepsilon)(u_\varepsilon - U_\beta - 4\sqrt{3}\pi\sqrt{\beta}g_{\kappa,\varepsilon}) \\
&= -\Delta u_\varepsilon - (\mu_1^2+\varepsilon)u_\varepsilon + \Delta U_\beta + (\mu_1^2+\varepsilon)U_\beta \\
&\quad + 4\sqrt{3}\pi\sqrt{\beta}(\Delta g_{\kappa,\varepsilon} + (\mu_1^2+\varepsilon)g_{\kappa,\varepsilon}) \\
&= u_\varepsilon^5 - U_\beta^5 + (\mu_1^2+\varepsilon)\left(U_\beta - \frac{\sqrt{3}\sqrt{\beta}}{r}\right) =: F(x)
\end{aligned}$$

and

$$\begin{aligned}
\kappa\frac{\partial\eta}{\partial\nu} + \eta\bigg|_{r=1} &= \left(\kappa\frac{\partial}{\partial\nu} + 1\right)(u_\varepsilon - U_\beta - 4\sqrt{3}\pi\sqrt{\beta}g_{\kappa,\varepsilon})\bigg|_{r=1} \\
&= -\kappa\frac{\partial U_\beta}{\partial\nu} - U_\beta - \sqrt{3}\sqrt{\beta}\left(\frac{\kappa}{r^2} - \frac{1}{r}\right)\bigg|_{r=1} =: b,
\end{aligned}$$

by (3.9). Note that b is constant. Using these relations, we get the error term estimate.

Proposition 4.1. *The error function η satisfies*

$$\|\eta\|_{L^2(B)} + 2\sqrt{\pi}|\eta'(1)| \leq C_{1,\kappa}\beta^{5/2}|\log \beta|$$

as $\beta \to +0$ for $0 \leq \kappa < 1$ and

$$\|\eta\|_{L^2(B)} + 2\sqrt{\pi}|\eta'(1)| \leq C_{1,1}(\varepsilon\beta^{5/2}|\log \beta| + \beta^{5/2})$$

as $\beta \to +0$ for $\kappa = 1$, with constant $C_{1,\kappa} > 0$ independent of ε and β.

To prove Proposition 4.1, we need to apply Lemma 3.1. Hence the estimates of $|b|$ ($\|b\|_{C^{1,\alpha}(\partial B)}$), $\|F\|_{L^\infty(\omega)}$ and $\|F\|_{L^1(B)}$ are necessary.

Lemma 4.1. *The constant b satisfies the estimate*

$$|b| \ (=\|b\|_{C^{1,\alpha}(\partial B)}) \leq C_2\beta^{5/2} \qquad as\ \beta \to 0$$

for any $0 \leq \kappa \leq 1$ with $C_2 > 0$.

Proof. Since U_β is radial, we have

$$\begin{aligned} b &= \frac{1}{3}\kappa\sqrt{\beta}\left(\beta^2 + \frac{1}{3}\right)^{-3/2} - \sqrt{\beta}\left(\beta^2 + \frac{1}{3}\right)^{-1/2} - \sqrt{3}\sqrt{\beta}(\kappa - 1) \\ &= \frac{\kappa\sqrt{\beta}}{3(\beta^2 + 1/3)^{3/2}}\left\{1 - 3\sqrt{3}\left(\beta^2 + \frac{1}{3}\right)^{3/2}\right\} \\ &\quad + \frac{\sqrt{\beta}}{(\beta^2 + 1/3)^{1/2}}\{\sqrt{3}(\beta^2 + 1/3)^{1/2} - 1\} \\ &= \frac{3\sqrt{3}}{2}(1 - 3\kappa)\beta^{5/2} + O(\beta^{9/2}). \end{aligned}$$

Thus $|b|$ is of order $\beta^{5/2}$. Note that if $\kappa = 1/3$, then $|b| = O(\beta^{9/2})$. □

Since b is constant on ∂B, Lemma 4.1 is enough for (3.2).

Lemma 4.2. *Let ω be a neighorhood of ∂B in B, which does not contain the origin. Then the estimate*

$$\|F\|_{L^\infty(\omega)} \leq C_3\beta^{5/2}$$

as $\beta \to +0$ holds for any $0 \leq \kappa \leq 1$ with constant $C_3 > 0$.

Proof. $\sup_\omega|F|$ is obviously estimated as

$$\sup_{\omega} F \le \sup_{\omega} |u_\varepsilon^5 - U_\beta^5| + (\mu_1^2 + \varepsilon) \sup_{\omega} \left| U_\beta - \frac{\sqrt{3}\sqrt{\beta}}{r} \right|$$

with $r = |x|$ by (4.1). Since

$$\frac{\sqrt{\beta}}{(\beta^2 + r^2/3)^{1/2}} - \frac{\sqrt{3}\sqrt{\beta}}{r} = -\frac{3\beta^{5/2} + O(\beta^{9/2})}{2(\beta^2 + r^2/3)^{1/2} r^2},$$

we see that

$$\sup_{\omega} \left| U_\beta - \frac{\sqrt{3}\sqrt{\beta}}{r} \right| = O(\beta^{5/2}).$$

As for the first term, we have

$$\sup_{\omega} |u_\varepsilon^5 - U_\beta^5| \le \sup_{\omega} |u_\varepsilon|^5 + \sup_{\omega} |U_\beta|^5.$$

As is easily seen, we have

$$|U_\beta| \le C\sqrt{\beta}$$

in ω. Now we set

$$W_\beta(x) = \frac{\sqrt{\beta}}{[\beta^2 + \{1 + (\mu_1^2 + \varepsilon)\beta^2\} r^2/3]^{1/2}}. \tag{4.2}$$

Note that W_β is a solution to

$$\Delta w + (1 + (\mu_1^2 + \varepsilon)\beta^2) w^5 = 0. \tag{4.3}$$

Then by Lemma 1 (iii) of Atkinson and Peletier [1], we have

$$u_\varepsilon \le W_\beta \le C\sqrt{\beta} \tag{4.4}$$

in ω. By the proof of Lemma 1 in [1] (done by comparison of (4.3) with (1.1) via the change of variables), one sees that this estimate is not affected by the boundary condition. Thus we get the desired estimate. □

Remark. As we see from [1], $u_\varepsilon \le W_\beta$ holds for any $x \in B$. See also Remark 1 of [1] and (2.11) of [1].

Now we divide F into three parts: Let

$$f_{(1)} = (\mu_1^2 + \varepsilon)\left(U_\beta - \frac{\sqrt{3}\sqrt{\beta}}{r} \right),$$

$$f_{(2)} = u_\varepsilon^5 - W_\beta^5,$$

$$f_{(3)} = W_\beta^5 - U_\beta^5.$$

Lemma 4.3. *For $f_{(1)}$, the estimate*

$$\|f_{(1)}\|_{L^1(B)} \le C_{4,\kappa}\beta^{5/2}|\log\beta| \qquad \text{as } \beta \to +0$$

holds for $0 \le \kappa < 1$ and

$$\|f_{(1)}\|_{L^1(B)} \le C_{4,1}\varepsilon\beta^{5/2}|\log\beta| \qquad \text{as } \beta \to +0$$

for $\kappa = 1$, with constant $C_{4,\kappa}$ independent of ε and β.

Proof. Since $f_{(1)}$ is radial, we have

$$\begin{aligned}
\int_B |f_{(1)}|dx &= 4\pi(\mu_1^2+\varepsilon)\sqrt{\beta}\int_0^1\left(\frac{\sqrt{3}}{r} - \frac{1}{\sqrt{\beta^2+r^2/3}}\right)r^2\,dr \\
&= 12\sqrt{3}\pi(\mu_1^2+\varepsilon)\beta^{5/2}\int_0^{1/(\sqrt{3}\beta)}\left(\frac{1}{\rho} - \frac{1}{\sqrt{1+\rho^2}}\right)\rho^2\,d\rho \\
&= 12\sqrt{3}\pi(\mu_1^2+\varepsilon)\beta^{5/2}\int_0^{1/(\sqrt{3}\beta)}\left(\frac{\sqrt{1+\rho^2}-\rho}{\sqrt{1+\rho^2}}\right)\rho\,d\rho \\
&\le 12\sqrt{3}\pi(\mu_1^2+\varepsilon)\beta^{5/2}\int_0^{1/(\sqrt{3}\beta)}\frac{\rho}{1+\rho^2}\,d\rho \\
&\le c(\varepsilon)\beta^{5/2}|\log\beta|,
\end{aligned}$$

with $r = \sqrt{3}\beta\rho$ as $\beta \to 0$. When $0 \le \kappa < 1$, then $\mu_1 > 0$ and $c(\varepsilon)$ is not dependent on ε although it is of order ε when $\kappa = 1$ as seen from the above inequality since $\mu_1 = 0$ in this case. □

By (4.4), we have $f_{(2)} \le 0$. We need a lower estimate for u_ε. This originates from Lemma 2 of [1], Lemma 2.2 of Atkinson and Peletier [3].

Lemma 4.4. *u_ε satisfies the inequality*

$$u_\varepsilon(x) \ge \left(\frac{1}{1+\beta^2} - \frac{\mu_1^2+\varepsilon}{2}|x|^2\right)W_\beta(x)$$

on B.

Proof. The original proof is for the equation of the form $v'' + t^{-4}(v+v^5) = 0$ in $t > T$ with $T > 0$ such that $v(t) > 0$ on (T,∞). As in the proof of Lemma 4.2, we use the integral form of (4.3) and (4.4) to obtain the desired inequality. Since we consider the radial solution, this result can be translated into our problem by $u(r) = v(t)$ with $r = 1/t$. Moreover, the proof by [1] is not affected by the boundary condition. Though the original estimate is for the solution of the Dirichlet problem, the estimate is still valid for that of the third boundary value problem. So we omit the detail. □

This implies that u_ε blows up only at the origin and that η does not have any blowup points except the origin. Using Lemma 4.4, we have an estimate for $f_{(2)}$.

Lemma 4.5. *$f_{(2)}$ is estimated as*

$$\|f_{(2)}\|_{L^1(B)} \le C_{5,\kappa}\beta^{5/2}|\log \beta|$$

as $\beta \to +0$ if $0 \le \kappa < 1$, and

$$\|f_{(2)}\|_{L^1(B)} \le C_{5,1}(\varepsilon\beta^{5/2}|\log \beta| + \beta^{5/2})$$

as $\beta \to +0$ if $\kappa = 1$, where $C_{5,\kappa}$ is a constant independent of ε and β.

Proof. By the definition of W_β and U_β, we see $u_\varepsilon \le W_\beta \le U_\beta$ from (4.4). Moreover, by Lemma 4.4, we have

$$\begin{aligned} W_\beta^5 - u_\varepsilon^5 &= (W_\beta^4 + W_\beta^3 u_\varepsilon + W_\beta^2 u_\varepsilon^2 + W_\beta u_\varepsilon^3 + u_\varepsilon^4)(W_\beta - u_\varepsilon) \\ &\le 5W_\beta^4(W_\beta - u_\varepsilon) \le \frac{5}{2}(\mu_1^2 + \varepsilon)r^2 W_\beta^5 + \frac{5\beta^2}{1+\beta^2}W_\beta^5. \\ &\le \frac{5}{2}(\mu_1^2 + \varepsilon)r^2 U_\beta^5 + \frac{5\beta^2}{1+\beta^2}U_\beta^5 \end{aligned}$$

Thus we get

$$\begin{aligned} &\frac{5}{2}(\mu_1^2 + \varepsilon)\int_B |x|^2 U_\beta^5\,dx \\ &\le 10\pi(\mu_1^2 + \varepsilon)\beta^{5/2}\int_0^1 \frac{r^4}{(\beta^2 + r^2/3)^{5/2}}\,dr \\ &\le C(\varepsilon)\beta^{5/2}\int_0^{1/(\sqrt{3}\beta)} \frac{\rho^4}{(1+\rho^2)^{5/2}}\,d\rho \le C(\varepsilon)\beta^{5/2}|\log \beta|, \end{aligned}$$

and

$$\beta^2\int_B U_\beta^5\,dx \le 4\pi\beta^{9/2}\int_0^1 \frac{r^2}{(\beta^2 + r^2/3)^{5/2}}\,dr \le \tilde{C}\beta^{5/2}$$

with $r = \sqrt{3}\beta\rho$ as $\beta \to +0$ and $\tilde{C} > 0$ independent of β. As in the proof of Lemma 4.4, $C(\varepsilon)$ is independent of ε when $0 \le \kappa < 1$ and is of order ε when $\kappa = 1$. Thus we have the desired estimate. □

Finally, we prove the estimate for $f_{(3)}$.

Lemma 4.6. *$f_{(3)}$ is estimated as follows:*

$$\|f_{(3)}\|_{L^1(B)} \le C_{6,\kappa}\beta^{5/2}$$

as $\beta \to +0$ if $0 \le \kappa < 1$ and

$$\|f_{(3)}\|_{L^1(B)} \le C_{6,1}\varepsilon\beta^{5/2}$$

as $\beta \to +0$ if $\kappa = 1$, where $C_{6,\kappa}$ is a constant independent of ε and β.

Proof. We directly calculate $|W_\beta^5 - U_\beta^5|$. We have

$$\begin{aligned} |W_\beta^5 - U_\beta^5| &= \beta^{5/2}\left|\frac{1}{[\beta^2 + \{1 + (\mu_1^2 + \varepsilon)\beta^2\}r^2/3]^{5/2}} - \frac{1}{(\beta^2 + r^2/3)^{5/2}}\right| \\ &\le \frac{5}{6}\beta^{5/2}\frac{(\mu_1^2 + \varepsilon)\beta^2 r^2}{(\beta^2 + r^2/3)^{7/2}} \end{aligned}$$

by the mean value theorem. The ε-dependence comes from this estimate. Moreover, since

$$\int_0^1 \frac{r^4}{(\beta^2 + r^2/3)^{7/2}}\,dr \le \frac{3^{5/2}}{\beta^2}\int_0^{1/(\sqrt{3}\beta)} \frac{s^4}{(1 + s^2)^{7/2}}\,ds \le \frac{c}{\beta^2},$$

with $r = \sqrt{3}\beta s$, where c is independent of β, we obtain

$$\int_B |W_\beta^5 - U_\varepsilon^5|dx \le \begin{cases} C_{6,\kappa}\beta^{5/2}, & 0 \le \kappa < 1, \\ C_{6,1}\varepsilon\beta^{5/2}, & \kappa = 1, \end{cases}$$

with $C_{6,\kappa}$ independent of ε and β. □

Proof of Proposition 4.1. A proof is done by combining Lemmas 3.1, 4.1, 4.2, 4.3, 4.5 and 4.6. □

5. Proofs of theorems

In this section, first, we calculate $J_{\mu_1^2+\varepsilon}(u_\varepsilon)$ by using the results in Section 4. Now recall that

$$u_\varepsilon = \eta + \phi_\beta$$

with

$$\phi_\beta = U_\beta + 4\sqrt{3}\pi\sqrt{\beta} g_{\kappa,\varepsilon}$$

and recall (2.6) and (2.7), i.e.,

$$(5.1) \qquad J_{\mu_1^2+\varepsilon}(u_\varepsilon) = 4\pi\{1-\kappa+(\mu_1^2+\varepsilon)\kappa^2\}(u_\varepsilon'(1))^2 - 2(\mu_1^2+\varepsilon)\int_B u_\varepsilon^2\, dx$$
$$= -\frac{1}{3}\int_{\partial B} u_\varepsilon^6\, dS = -\frac{4}{3}\pi(u_\varepsilon(1))^6.$$

First we give a rough upper estimate for ϕ_β.

Lemma 5.1. *For ϕ_β, there exists $C_7 > 0$ independent of $\kappa \in [0,1]$ such that*

$$\int_B \phi_\beta^2\, dx \le C_7\beta \qquad \textit{and} \qquad (\phi_\beta'(1))^2 \le C_7\beta.$$

Proof. From the definition of ϕ_β, we have

$$\int_B \phi_\beta^2\, dx \le 2\int_B (U_\beta^2 + (4\sqrt{3}\pi)^2\beta(g_{\kappa,\varepsilon})^2)dx$$

and

$$|\phi_\beta'|^2 \le 2(|U_\beta'|^2 + (4\sqrt{3}\pi)^2\beta|g_{\kappa,\varepsilon}'|^2)$$

Since $g_{\kappa,\varepsilon}$ is the finite part of the reduced Green function, the second term of the right-hand side of each inequality is estimated from above by β. As for the first term of the right-hand side, we have $|U_\beta'|^2 \le C\beta$ on ∂B by the definition of U_β. Finally, concerning $\int_B U_\beta^2\, dx$, it is sufficient to show

$$(5.2) \qquad \int_{B_\sigma} U_\beta^2\, dx \le C\beta$$

with small $\sigma > 0$. We have readily seen that

$$\int_{B\setminus B_\sigma} U_\beta^2\, dx \le C\beta.$$

We prove (5.2). Indeed, we have

$$\int_{B_\sigma} U_\beta^2\, dx = 4\pi\beta\int_0^\sigma \frac{r^2}{(\beta^2+r^2/3)}\, dr = 12\sqrt{3}\pi\beta^2\int_0^{\sigma/(\sqrt{3}\beta)} \frac{\rho^2}{(1+\rho^2)}\, d\rho \le 12\pi\sigma\beta,$$

with $r = \sqrt{3}\beta\rho$. Thus we get the desired estimates. □

We have a useful approximation of $J_{\mu_1^2+\varepsilon}(u_\varepsilon)$.

Lemma 5.2. *$J_{\mu_1^2+\varepsilon}(u_\varepsilon)$ is approximated by $J_{\mu_1^2+\varepsilon}(\phi_\beta)$ as*

$$J_{\mu_1^2+\varepsilon}(u_\varepsilon) = J_{\mu_1^2+\varepsilon}(\phi_\beta) + R_{\varepsilon,\beta}$$

with $|R_{\varepsilon,\beta}| \le C\beta^3|\log\beta|$ if $0 \le \kappa < 1$ and $|R_{\varepsilon,\beta}| \le C\varepsilon\beta^3|\log\beta|$ if $\kappa = 1$.

Proof. Using Proposition 4.1 and Lemma 5.1, we have

$$\begin{aligned}
|R_{\varepsilon,\beta}| &= |J_{\mu_1^2+\varepsilon}(\phi_\beta + \eta) - J_{\mu_1^2+\varepsilon}(\phi_\beta)| \\
&\le 4\pi\{1 - \kappa + (\mu_1^2 + \varepsilon)\kappa^2\}\{2|\phi_\beta'(1)|\,|\eta'(1)| + (\eta'(1))^2\} \\
&\quad + 2(\mu_1^2 + \varepsilon)\int_B (2|\phi_\beta\eta| + \eta^2)dx \\
&\le 4\pi\{1 - \kappa + (\mu_1^2 + \varepsilon)\kappa^2\}\{2|\phi_\beta'(1)|\,|\eta'(1)| + (\eta'(1))^2\} \\
&\quad + 2(\mu_1^2 + \varepsilon)(2\|\phi_\beta\|_{L^2(B)}\|\eta\|_{L^2(B)} + \|\eta\|_{L^2(B)}^2) \\
&\le C(\varepsilon)\beta^3|\log\beta|,
\end{aligned}$$

where the constant $C(\varepsilon)$ is independent of ε if $0 \le \kappa < 1$ and is of order ε if $\kappa = 1$ since $\mu_1 = 0$ if $\kappa = 1$. Thus we get the conclusion. □

Our aim is to get the behavior of $J_{\mu_1^2+\varepsilon}(u_\varepsilon)$ as $\beta \to +0$. To this end, we investigate $J_{\mu_1^2+\varepsilon}(\phi_\beta)$. We need to estimate $\|\phi_\beta\|^2$ more accurately than in Lemma 5.1.

Lemma 5.3. *$J_{\mu_1^2+\varepsilon}(\phi_\beta)$ is expanded as*

$$J_{\mu_1^2+\varepsilon}(\phi_\beta) = -48\pi^2\beta g_{\kappa,\varepsilon}(0) + 12\sqrt{3}\pi^2(\mu_1^2 + \varepsilon)\beta^2 + O(\beta^3|\log\beta|)$$

as $\beta \to +0$.

Proof. Since

$$\phi_\beta = U_\beta + 4\sqrt{3}\pi\sqrt{\beta}\left(G_{\kappa,\mu_1^2+\varepsilon} - \frac{1}{4\pi|x|}\right),$$

we have

$$\left.\frac{\partial\phi_\beta}{\partial\nu}\right|_{\partial B} = 4\sqrt{3}\pi\sqrt{\beta}G'_{\kappa,\mu_1^2+\varepsilon}(1) + O(\beta^{5/2})$$

by

$$\frac{\partial}{\partial\nu}\left(\frac{1}{(\beta^2 + |x|^2/3)^{1/2}} - \frac{\sqrt{3}}{|x|}\right) = O(\beta^2)$$

on ∂B. Thus we get

$$4\pi(\phi_\beta'(1))^2 = 192\pi^3\beta(G'_{\kappa,\mu_1^2+\varepsilon}(1))^2 + O(\beta^3).$$

Next as for $\int_B \phi_\beta^2\,dx$, we have

$$\begin{aligned}
\int_B \phi_\beta^2\,dx &= \int_B (U_\beta + 4\sqrt{3}\pi\sqrt{\beta}g_{\kappa,\varepsilon})^2 dx \\
&= \int_B U_\beta^2\,dx + 8\sqrt{3}\pi\sqrt{\beta}\int_B U_\beta g_{\kappa,\varepsilon}\,dx \\
&\quad + 48\pi^2\beta\int_B \left(G_{\kappa,\mu_1^2+\varepsilon} - \frac{1}{4\pi|x|}\right)^2 dx \\
&= \int_B U_\beta^2\,dx + 8\sqrt{3}\sqrt{\beta}\pi\int_B U_\beta g_{\kappa,\varepsilon}\,dx + 48\pi^2\beta\int_B G^2_{\kappa,\mu_1^2+\varepsilon}\,dx \\
&\quad - 24\pi\beta\int_B \left(\frac{g_{\kappa,\varepsilon}}{|x|} + \frac{1}{4\pi|x|^2}\right)dx + 3\beta\int_B \frac{1}{|x|^2}\,dx \\
&= 48\pi^2\beta\int_B G^2_{\kappa,\mu_1^2+\varepsilon}\,dx + 8\sqrt{3}\pi\beta\int_B \left\{\frac{1}{(\beta^2+|x|^2/3)^{1/2}} - \frac{\sqrt{3}}{|x|}\right\}g_{\kappa,\varepsilon}\,dx \\
&\quad - 3\beta\int_B \frac{1}{|x|^2}\,dx + \int_B U_\beta^2\,dx \\
&= 48\pi^2\beta\int_B G^2_{\kappa,\mu_1^2+\varepsilon}\,dx \\
&\quad - 24\sqrt{3}\pi\beta^3\int_B \frac{g_{\kappa,\varepsilon}}{|x|(\beta^2+|x|^2/3)^{1/2}\{|x| + \sqrt{3}(\beta^2+|x|^2/3)^{1/2}\}}\,dx \\
&\quad + \beta\int_B \left(\frac{1}{\beta^2+|x|^2/3} - \frac{3}{|x|^2}\right)dx.
\end{aligned}$$

As for the second term, since $g_{\kappa,\varepsilon} \in L^\infty(B)$, we have

$$\begin{aligned}
&\int_B \frac{1}{|x|(\beta^2+|x|^2/3)^{1/2}\{|x| + (\beta^2+|x|^2/3)^{1/2}\}}\,dx \\
&\quad \le \int_B \frac{dx}{|x|^2(\beta^2+|x|^2/3)^{1/2}} = 4\pi\int_0^1 \frac{1}{(\beta^2+r^2/3)^{1/2}}\,dr \\
&\quad = 4\sqrt{3}\pi\int_0^{1/(\sqrt{3}\beta)} \frac{1}{(1+s^2)^{1/2}}\,ds \le c|\log\beta|,
\end{aligned}$$

with $r = |x|$ and $r = \sqrt{3}\beta s$. Similarly, as for the third term, we get

$$\int_B \left(\frac{1}{\beta^2 + |x|^2/3} - \frac{3}{|x|^2}\right) dx = -12\pi\beta^2 \int_0^1 \frac{dr}{\beta^2 + r^2/3}$$

$$= -12\sqrt{3}\pi\beta \int_0^{1/(\sqrt{3}\beta)} \frac{ds}{1+s^2} = -12\sqrt{3}\pi\beta\left(\tan^{-1}\frac{1}{\sqrt{3}\beta}\right)$$

$$= -12\sqrt{3}\pi\beta\left(\frac{\pi}{2} + O(\beta)\right) = -6\sqrt{3}\pi^2\beta + O(\beta^2).$$

Thus we obtain

$$\int_B \phi_\beta^2 \, dx = 48\pi^2\beta \int_B G_{\kappa,\mu_1^2+\varepsilon}^2 \, dx - 6\sqrt{3}\pi^2\beta^2 + O(\beta^3|\log\beta|).$$

Hence we have

$$J_{\mu_1^2+\varepsilon}(\phi_\beta) = 192(1 - \kappa + (\mu_1^2+\varepsilon)\kappa^2)\pi^3\beta(G'_{\kappa,\mu_1^2+\varepsilon}(1))^2$$
$$- 96\pi^2(\mu_1^2+\varepsilon)\beta \int_B G_{\kappa,\mu_1^2+\varepsilon}^2 \, dx + 12\sqrt{3}\pi^2(\mu_1^2+\varepsilon)\beta^2$$
$$+ O(\beta^3|\log\beta|)$$
$$= 48\pi^2\beta J_{\mu_1^2+\varepsilon}(G_{\kappa,\mu_1^2+\varepsilon}^2) + 12\sqrt{3}\pi^2(\mu_1^2+\varepsilon)\beta^2 + O(\beta^3|\log\beta|)$$
$$= -48\pi^2\beta g_{\kappa,\varepsilon}(0) + 12\sqrt{3}\pi^2(\mu_1^2+\varepsilon)\beta^2 + O(\beta^3|\log\beta|)$$

by Proposition 3.1. □

Using Lemmas 5.2 and 5.3, we prove Thereom 1.1.

Proof of Theorem 1.1. First we note that

$$4\pi(u_\varepsilon(1))^6 = \int_{\partial B} u_\varepsilon^6 \, dS = O(\beta^3)$$

by (4.4). Thus we have

$$J_{\mu_1^2+\varepsilon}(u_\varepsilon) = O(\beta^3)$$

by (5.1). From Lemmas 5.2 and 5.3, we obtain

$$48\pi^2\beta g_{\kappa,\varepsilon}(0) = 12\sqrt{3}\pi^2(\mu_1^2+\varepsilon)\beta^2 + O(\beta^3|\log\beta|) \tag{5.3}$$

as $\beta \to +0$. Note that we here treat the case $0 \le \kappa < 1$. Recalling the definition of g_λ (see (2.11)), and using l'Hospital's rule, we get

$$g_{\kappa,\varepsilon}(0) = \frac{\sqrt{\mu_1^2+\varepsilon}(-\cos\mu_1 + A\sin\mu_1)}{4\pi(\sin\mu_1 + A\cos\mu_1)}, \tag{5.4}$$

where A is defined as

$$A = \frac{(1-\kappa)\sin\left(\sqrt{\mu_1^2+\varepsilon}-\mu_1\right) + \kappa\sqrt{\mu_1^2+\varepsilon}\cos\left(\sqrt{\mu_1^2+\varepsilon}-\mu_1\right)}{(1-\kappa)\cos\left(\sqrt{\mu_1^2+\varepsilon}-\mu_1\right) - \kappa\sqrt{\mu_1^2+\varepsilon}\sin\left(\sqrt{\mu_1^2+\varepsilon}-\mu_1\right)}. \tag{5.5}$$

If $0 \le \kappa < 1$, since

$$\sqrt{\mu_1^2+\varepsilon} = \mu_1 + \frac{\varepsilon}{2\mu_1} + O(\varepsilon^2),$$

we have

$$\begin{aligned} A &= \frac{(1-\kappa)\dfrac{\varepsilon}{2\mu_1} + \kappa\left(\mu_1 + \dfrac{\varepsilon}{2\mu_1}\right) + O(\varepsilon^2)}{(1-\kappa) - \kappa\left(\mu_1 + \dfrac{\varepsilon}{2\mu_1}\right)\dfrac{\varepsilon}{2\mu_1} + O(\varepsilon^2)} \\ &= \frac{\kappa\mu_1 + \dfrac{\varepsilon}{2\mu_1} + O(\varepsilon^2)}{(1-\kappa) - \dfrac{\kappa}{2}\varepsilon + O(\varepsilon^2)}. \end{aligned}$$

If $0 \le \kappa < 1$, we get

$$A = \frac{\kappa\mu_1}{1-\kappa} + \frac{1-\kappa+\kappa^2\mu_1^2}{2(1-\kappa)^2\mu_1}\varepsilon + O(\varepsilon^2).$$

Thus we have

$$\begin{aligned} g_{\kappa,\varepsilon}(0) &= \frac{\mu_1 + \dfrac{\varepsilon}{2\mu_1} + O(\varepsilon^2)}{4\pi\left(\sin\mu_1 + \left\{\dfrac{\kappa\mu_1}{1-\kappa} + \dfrac{1-\kappa+\kappa^2\mu_1^2}{2(1-\kappa)^2\mu_1}\varepsilon\right\}\cos\mu_1\right)} \\ &\quad\times\left(-\cos\mu_1 + \frac{\kappa\mu_1}{1-\kappa}\sin\mu_1 + \frac{(1-\kappa+\kappa^2\mu_1^2)\varepsilon}{2(1-\kappa)^2\mu_1}\sin\mu_1 + O(\varepsilon^2)\right) \\ &= \frac{(1-\kappa+\kappa^2\mu_1^2)(\sin\mu_1)\varepsilon}{8\pi\{(1-\kappa)^2\sin\mu_1 + \kappa(1-\kappa)\mu_1\cos\mu_1\}} + O(\varepsilon^2) \end{aligned} \tag{5.6}$$

by noting $\kappa\mu_1\sin\mu_1/(1-\kappa) = \cos\mu_1$.

Substituting (5.6) for (5.3), we have

$$\frac{6\pi(1-\kappa+\kappa^2\mu_1^2)(\sin\mu_1)\beta\varepsilon}{(1-\kappa)^2\sin\mu_1+\kappa(1-\kappa)\mu_1\cos\mu_1}+O(\beta\varepsilon^2)$$
$$=12\sqrt{3}\pi^2(\mu_1^2+\varepsilon)\beta^2+O(\beta^3|\log\beta|).$$

Dividing the both sides by β^2, we get

$$\frac{\varepsilon}{\beta}=\frac{2\sqrt{3}\pi(\mu_1^2+\varepsilon)\{(1-\kappa)^2\sin\mu_1+\kappa(1-\kappa)\mu_1\cos\mu_1\}}{(1-\kappa+\kappa^2\mu_1^2)\sin\mu_1}$$
$$+O(\beta|\log\beta|)+\varepsilon\cdot O\left(\frac{\varepsilon}{\beta}\right).$$

Letting $\varepsilon\to 0$, we obtain

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\beta}=\frac{2\sqrt{3}\pi\mu_1^2\{(1-\kappa)^2\sin\mu_1+\kappa(1-\kappa)\mu_1\cos\mu_1\}}{(1-\kappa+\kappa^2\mu_1^2)\sin\mu_1} \tag{5.7}$$

Since $\beta=(u_\varepsilon(0))^{-2}$,

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\beta}=\lim_{\varepsilon\to 0}\varepsilon(\alpha(\mu_1^2+\varepsilon;\kappa))^2=\frac{2\sqrt{3}\pi\mu_1^2\{(1-\kappa)^2\sin\mu_1+\kappa(1-\kappa)\mu_1\cos\mu_1\}}{(1-\kappa+\kappa^2\mu_1^2)\sin\mu_1}.$$

Thus we get the desired value. □

Proof of Theorem 1.2. When $\kappa=1$, then (2.6) yields

$$4\pi\varepsilon(u_\varepsilon'(1))^2-2\varepsilon\int_B u_\varepsilon^2\,dx=-\frac{4}{3}\pi u_\varepsilon^6(1). \tag{5.8}$$

Unlike the proof of Theorem 1.1, ε appears directly in this relation when $\kappa=1$. By (4.4) and Lemma 4.4, we see that

$$(u_\varepsilon(1))^6=27\beta^3+O(\varepsilon\beta^3+\beta^5).$$

Directly substituting $\mu_1=0$ for (5.5) and (2.11), we have

$$A=-\cot\sqrt{\varepsilon},\qquad g_{\kappa,\varepsilon}(r)=\frac{\tan\sqrt{\varepsilon}\sin(\sqrt{\varepsilon}r)}{4\pi r}+\frac{\cos(\sqrt{\varepsilon}r)-1}{4\pi r}+O(\varepsilon).$$

Since $u_\varepsilon=U_\beta+4\sqrt{3}\pi\sqrt{\beta}g_{\kappa,\varepsilon}+\eta=\phi_\beta+\eta$, we get

$$\int_B u_\varepsilon^2\,dx=\int_B U_\beta^2\,dx+O(\varepsilon\beta+\varepsilon\beta^3|\log\beta|+\beta^3)$$

by Proposition 4.1 and Lemma 5.1. The relation

$$\int_B U_\beta^2\,dx = 4\pi\int_0^1 \frac{\beta r^2}{\beta^2 + r^2/3}\,dr = 12\sqrt{3}\beta^2\int_0^{1/(\sqrt{3}\beta)}\left(1 - \frac{1}{1+\rho^2}\right)d\rho$$

$$= 12\pi\beta - 12\pi\beta^2\tan^{-1}\frac{1}{\sqrt{3}\beta}$$

with $r = \sqrt{3}\beta\rho$ yields

$$\int_B u_\varepsilon^2\,dx = 12\pi\beta + O(\varepsilon\beta + \beta^2). \tag{5.9}$$

Similarly, we have

$$u_\varepsilon'(1) = -\sqrt{3}\sqrt{\beta} + O(\varepsilon\sqrt{\beta} + \varepsilon\beta^{5/2}|\log\beta| + \beta^{5/2})$$

in view of

$$(g_{\kappa,\varepsilon})'(1) = \frac{1}{4\pi}\{(\tan\sqrt{\varepsilon})(\sqrt{\varepsilon}\cos\sqrt{\varepsilon} - \sin\sqrt{\varepsilon}) - \sqrt{\varepsilon}\sin\sqrt{\varepsilon} + (1 - \cos\sqrt{\varepsilon})\}$$

$$= -\frac{\varepsilon}{8\pi} + O(\varepsilon^2)$$

and Proposition 4.1. Thus we get

$$(u_\varepsilon'(1))^2 = 3\beta + O(\varepsilon\beta + \varepsilon\beta^3|\log\beta| + \beta^3). \tag{5.10}$$

Substituting (5.9) and (5.10) for (5.8), we obtain

$$-12\pi\varepsilon\beta + O(\varepsilon^2\beta + \varepsilon\beta^2) = -36\pi\beta^3 + O(\varepsilon\beta^3 + \beta^5).$$

Dividing the both sides by β^3, we see that

$$\lim_{\varepsilon\to 0}\frac{\varepsilon}{\beta^2} = 3 \qquad \text{i.e.} \qquad \lim_{\lambda\to 0}\lambda u_\lambda(0)^4 = 3. \qquad \square$$

Proof of Theorem 1.3. As stated in the top of this section, we have

$$\beta^{-1/2}u_\varepsilon = \beta^{-1/2}\eta + \beta^{-1/2}U_\beta + 4\sqrt{3}\pi\left(G_{\kappa,\mu_1^2+\varepsilon}(x) - \frac{1}{4\pi|x|}\right)$$

for $x \neq 0$. Since $\beta^{-1/2}\eta \to 0$ as $\beta \to 0$ by Proposition 4.1 (η can be estimated as in Proposition 4.1 pointwise if $x \neq 0$), and since $\beta^{-1/2}U_\beta \to \sqrt{3}/|x|$, we have

$$\lim_{\varepsilon\to 0}\beta^{-1/2}u_\varepsilon = 4\sqrt{3}\pi G_{\kappa,\mu_1^2}$$

for $x \neq 0$. Thus by (5.7), we obtain

$$\lim_{\varepsilon\to 0} \frac{u_\lambda}{\sqrt{\lambda-\mu_1^2}} = \frac{4}{\mu_1}\sqrt{\frac{\sqrt{3}\pi(1-\kappa+\kappa^2\mu_1^2)\sin\mu_1}{2\{(1-\kappa)^2\sin\mu_1+\kappa(1-\kappa)\mu_1\cos\mu_1\}}}G_{\kappa,\mu_1^2}$$

for $0 \leq \kappa < 1$ and

$$\lim_{\lambda\to 0}\frac{u_\lambda}{\sqrt[4]{\lambda}} = 4\sqrt[4]{3}\pi G_{1,0}$$

for $\kappa = 1$ by Theorem 1.2. □

Acknowledgment. The ealier version of this work was finished while the author was visiting the University of Minnesota in 1996. He would like to thank Professor Wei-Ming Ni of the University of Minnesota for his warm hospitality.

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(Ricevita la 30-an de julio, 2001)
(Reviziita la 9-an de novembro, 2001)