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Finite Dimensional Attractor for One-Dimensional Keller-Segel Equations

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1. Introduction

In this paper, we study the long time behavior of a one-dimensional reaction diffusion system appearing in mathematical biology by using the theory of infinite dimensional dynamical systems.

In 1970 Keller and Segel [9] have presented parabolic systems to describe the aggregation process of cellular slime mold by the chemical attraction. The system of a simplified form in the one-dimensional case is written as

$$\text{(KS)}\qquad \begin{cases} \dfrac{\partial u}{\partial t} = a\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial}{\partial x}\left(u\dfrac{\partial \chi}{\partial x}(\rho)\right), & (x,t) \in I \times (0,\infty), \\ \dfrac{\partial \rho}{\partial t} = b\dfrac{\partial^2 \rho}{\partial x^2} + cu - d\rho, & (x,t) \in I \times (0,\infty), \\ \dfrac{\partial u}{\partial x}(\alpha,t) = \dfrac{\partial u}{\partial x}(\beta,t) = \dfrac{\partial \rho}{\partial x}(\alpha,t) = \dfrac{\partial \rho}{\partial x}(\beta,t) = 0, & t \in (0,\infty), \\ u(x,0) = u_0(x), \quad \rho(x,0) = \rho_0(x), & x \in I. \end{cases}$$

Here, $I = (\alpha,\beta)$ is a bounded open interval. a, b, c and d are positive constants. The unknown functions $u = u(x,t)$ and $\rho = \rho(x,t)$ denote the concentration of amoebae and the concentration of chemical substance, respectively, in $I \times (0,\infty)$. The chemotactic term $-\partial(u\partial\chi(\rho)/\partial x)/\partial x$ indicates that the cells are sensitive to the chemicals and are attracted by them. $\chi(\rho)$ called the sensitivity function is a smooth function of $\rho \in (0,\infty)$ which describes cell's perception and response to the chemical stimulus ρ. Several normalized forms have been suggested: ρ, ρ^2, $\log\rho$, $\rho/(1+\rho)$ and $\rho^2/(1+\rho^2)$, etc., see [10] and [17]. In view of these forms, we will assume in this paper (except in the last section) that: (χ) $\chi(\rho)$ is a smooth function of $\rho \in (0,\infty)$ and its three derivatives satisfy the estimates

$$|\chi^{(i)}(\rho)| \le C\left(\rho + \frac{1}{\rho}\right)^r, \qquad 0 < \rho < \infty, \ i = 1,2,3$$

with some positive constant C and exponent r.

The system (KS) is called the Keller-Segel equations.

In these years the Keller-Segel equations attracted interests of many mathematicians. The local solutions were studied by the second author [23]. It was also suggested in [23] that, in the one-dimensional case, (KS) possesses a global solution and that, in the two-dimensional case, when $\chi(\rho) = k\rho$ (k being a positive constant) is a linear function, (KS) possesses a global solution for any sufficiently small initial function u_0. Afterward Nagai et al. [13] showed more strongly that the global solution exists if the norm $\|u_0\|_{L^1}$ is smaller than a specific number which is given from the coefficients of the equations. Recently, in the same case, Gajewski et al. [6] studied asymptotic behavior of the global solutions. On the other hand, Herrero et al. [7] showed that, when $\chi(\rho)$ is linear and the domain is a circular disc, there exist radial local solutions which blow up in a finite time. The blowup of non radial local solutions was shown recently by Horstmann et al. [8] and Nagai et al. [12]. For the study of stationary solutions, we refer to Ni et al. [14], Schaaf [17], Senba et al. [18] and Wiebers [22].

In this paper we are concerned with asymptotic behavior of the global solutions to the one-dimensinal Keller-Segel equations with a general sensitivity function satisfying (χ), and intend to construct an attractor set. In constructing the attractor we will use the theory of infinite dimensional dynamical systems for dissipative evolution equations which was developed in recent years by Temam [21] and by Eden, Foias, Nicolaenko and Temam [3, 4]. In order to use their theory, the first step is to formulate (KS) as a semilinear evolution equation in a suitable Hilbert space. We set the underlying space H as a product space of the pairs ${}^t(u, \rho)$ with $u \in L^2(I)$ and $\rho \in H^1(I)$, and will show that the nonlinear semigroup constructed on H satisfies some sufficient conditions which imply its crucial property called the squeezing property. As the result, will be shown existence of a compact set of finite fractal dimension which attracts solutions exponentially, such an attractor set is called the exponential attractor. However, we here notice that we can not expect any global compact attractor, for, since the norm $\|u(t)\|_{L^1} \equiv \|u_0\|_{L^1}$ is conserved for every $t \in [0, \infty)$, no compact set can attract every solution of (KS). Therefore, for each $\|u_0\|_{L^1} = \ell > 0$, we have to consider an underlying space like $K_\ell = \{{}^t(u, \rho) \in H; u \geq 0, \int_I u\,dx = \ell, \inf_{x \in I} \rho > 0\}$ to reset.

In the case where the sensitivity function is linear, $\chi(\rho) = k\rho$, we know the existence of a global Lyapunov functional. Thanks to this we can obtain a result of another type, that is, for any initial data ${}^t(u_0, \rho_0) \in K_\ell$, the ω-limit set of the solution to (KS) contains at least one stationary solution. For other typical cases of $\chi(\rho)$, however, we do not know whether such a Lyapunov functional exists or not.

This paper is organized as follows. In Section 2, we recall the definition of the exponential attractor and the existence theorem of exponential attractors in the book [4]. We list also some results on the Sobolev spaces which we need in this paper. In Section 3, the local solutions of (KS) are constructed by applying the Galerkin method. In Section 4, we establish various a priori estimates of the local solutions. Section 5 is devoted to estimating the lower bound of ρ. By using these estimates, the existence of global solutions is verified. In Section 6, we prove the main theorem of the paper. Finally, Section 7 is devoted to considering the case where the sensitivity function is linear.

Notations. $I = (\alpha, \beta)$, $-\infty < \alpha < \beta < \infty$, denotes an open interval in $\boldsymbol{R}$. For $1 \le p \le \infty$, $L^p(I)$ is the L^p space of measurable functions in I, its norm is denoted by $\|\cdot\|_{L^p}$. For $m = 0, 1, 2, \dots$, $H^m(I)$ is the real Sobolev space of exponent m, its norm is denoted by $\|\cdot\|_{H^m}$. More generally, $H^s(I)$ is the fractional Sobolev space which is the interpolation space between $H^m(I)$ and $H^{m+1}(I)$ for $m < s < m+1$, its norm is also denoted by $\|\cdot\|_{H^s}$. $\mathscr{C}(\bar{I})$ is the space of all continuous functions on $\bar{I}$.

Let X be a Banach space and let J be an interval in $\boldsymbol{R}$. $L^2(J; X)$ and $H^1(J; X)$ are X-valued L^2 and H^1 space in J, respectively. $\mathscr{C}(J; X)$ and $\mathscr{C}^m(J; X)$ $(m = 1, 2, 3, \dots)$ denote the space of X-valued continuous functions and m-times continuously differentiable functions in J, respectively.

For simplicity, we will use a single notation C to denote various constants determined by initial constants. C is therefore determined in each occurrence by $I = (\alpha, \beta), a, b, c, d$ and χ in a certain specific way. In a case where C depends also on some parameter, say ε, it will be denoted by C_ε.

2. Preliminary

Consider the Cauchy problem for a semilinear evolution equation

$$\text{(E)} \qquad \begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0 \end{cases}$$

in a separable Hilbert space H. Here, A is a positive definite self-adjoint linear operator in H, the inverse of which is a compact operator on H. $F(U)$ is a nonlinear operator from $\mathscr{D}(A)$ to H. U_0 is from K, where K denotes the space of initial data which is a connected subset of H. $U = U(t)$ is the unknown function.

We assume that the problem (E) is well posed in K, that is, for each $U_0 \in K$, there exists a unique global solution

$$U \in \mathscr{C}([0, \infty); K) \cap \mathscr{C}^1((0, \infty); H) \cap \mathscr{C}((0, \infty); \mathscr{D}(A)),$$

and the solution is continuous with respect to the initial value in the sense that, if $U_{0,n} \to U_0$ in K then the solution $U_n(t)$ converges to $U(t)$ in K for each fixed $t \in (0, \infty)$. This then allows us to construct a continuous semigroup $\{S(t)\}_{t\geq 0}$ on K such that, for $t > 0$, $S(t)$ maps K into $\mathscr{D}(A) \cap K$. Furthermore, we assume that there exists a compact absorbing set $\mathscr{B} \subset \mathscr{D}(A) \cap K$ for the semigroup $\{S(t)\}_{t\geq 0}$, that is, $\mathscr{B}$ is a compact subset of H and, for every bounded subset $B \subset K$, there is a time t_0 which may depend on B such that $\bigcup_{t\geq t_0} S(t)B \subset \mathscr{B}$. Then, according to Temam [21, Theorem 1.1], the ω-limit set $\mathscr{A}$ of $\mathscr{B}$ is a global attractor for $\{S(t)\}_{t\geq 0}$. We now consider the set

$$\mathscr{X} = \overline{\bigcup_{t\geq t_1} S(t)\mathscr{B}}$$

with fixed t_1 such that $\bigcup_{t\geq t_1} S(t)\mathscr{B} \subset \mathscr{B}$. It is easily observed that $\mathscr{X}$ is a compact subset of H such that $\mathscr{A} \subset \mathscr{X} \subset \mathscr{B}$ and $\mathscr{X}$ is absorbing and invariant for $\{S(t)\}_{t\geq 0}$. Therefore, $(\{S(t)\}_{t\geq 0}, \mathscr{X})$ defines a subdynamical system of $(\{S(t)\}_{t\geq 0}, K)$.

The exponential attractor is defined as follows, see Eden et al. [4].

Definition. A subset $\mathscr{M} \subset \mathscr{X}$ is called the exponential attractor for $(\{S(t)\}_{t\geq 0}, \mathscr{X})$ if: i) $\mathscr{A} \subset \mathscr{M} \subset \mathscr{X}$; ii) $\mathscr{M}$ is a compact subset of H and is an invariant set for $S(t)$; iii) $\mathscr{M}$ has finite fractal dimension $d_F(\mathscr{M})$; and iv) $h(S(t)\mathscr{X}, \mathscr{M}) \leq c_0 \exp(-c_1 t)$ for $t \geq 0$ with some constants $c_0, c_1 > 0$, where

$$h(B_0, B_1) = \sup_{U\in B_0} \inf_{V\in B_1} \|U - V\|_H$$

denotes the Hausdorff pseudodistance of two sets B_0 and B_1.

By virtue of [4, Theorem 3.1] we have the following theorem.

Theorem 2.1. *Let $F(U)$ satisfy the Lipschitz condition*

$$\text{(F)} \qquad \|F(U) - F(V)\|_H \leq C\|A^{1/2}(U - V)\|_H, \qquad U, V \in \mathscr{X},$$

and let the mapping $G(t, U_0) = S(t)U_0$ from $[0, T] \times \mathscr{X}$ into $\mathscr{X}$ satisfy the Lipschitz condition

$$\text{(G)} \qquad \|G(t, U_0) - G(s, V_0)\|_H \leq C_T\{|t - s| + \|U_0 - V_0\|_H\},$$
$$t, s \in [0, T], \quad U_0, V_0 \in \mathscr{X},$$

for each $T > 0$. Then, there exists an exponential attractor $\mathscr{M}$ for $(\{S(t)\}_{t\geq 0}, \mathscr{X})$.

By the Lipschitz condition (G) the proof of the theorem is reduced to constructing a similar exponential attractor for a discrete dynamical system $(\{S_*^n\}_{n\geq 0}, \mathscr{X})$, where $S_* = S(t_*)$ with a suitable time $t_* > 0$. For the discrete dynamical system, some condition on S_* called the squeezing property plays an important role: for some $\delta \in (0, 1/4)$, there exists an orthogonal projection P of finite rank N such that, for each pair $U, V \in \mathscr{X}$, either

$$\|S_* U - S_* V\|_H \leq \delta \|U - V\|_H$$

or

$$\|(I - P)(S_* U - S_* V)\|_H \leq \|P(S_* U - S_* V)\|_H.$$

In the case when the dynamical system is determined by a semilinear evolution equation like (E), this property can be verified from the Lipschitz condition (F), see [4, Proposition 3.1]. In fact the existence of an exponential attractor $\mathscr{M}_*$ for $(\{S_*^n\}_{n \geq 0}, \mathscr{X})$ is concluded, as well the dimension is estimated by

$$d_F(\mathscr{M}_*) \leq N \max\left\{1, \frac{\log(2L/\delta + 1)}{\log(1/4\delta)}\right\}, \tag{2.1}$$

where L is a Lipschitz constant of the mapping S_* from $\mathscr{X}$ into itself. The fractal dimension of $\mathscr{M}$ is then estimated by $d_F(\mathscr{M}) \leq d_F(\mathscr{M}_*) + 1$.

Our goal is then apply Theorem 2.1 after formulating (KS) as an abstract equation of the form (E) in a suitable Hilbert space.

We list also some well-known theorems on the Sobolev spaces (cf. e.g. [2, 5, 11, 20]). Still I denotes an interval (α, β), $-\infty < \alpha < \beta < \infty$.

For $0 \leq s_0 < s < s_1 < \infty$, $H^s(I)$ coincides with the interpolation space $[H^{s_0}(I), H^{s_1}(I)]_\theta$, $s = (1 - \theta)s_0 + \theta s_1$, between $H^{s_0}(I)$ and $H^{s_1}(I)$, and the estimate

$$\|\cdot\|_{H^s} \leq C \|\cdot\|_{H^{s_0}}^{1-\theta} \|\cdot\|_{H^{s_1}}^{\theta} \tag{2.2}$$

holds.

When $s > 1/2$, $H^s(I) \subset \mathscr{C}(\bar{I})$ with

$$\|\cdot\|_{\mathscr{C}} \leq C_s \|\cdot\|_{H^s}, \qquad s > 1/2. \tag{2.3}$$

In particular, $H^1(I) \subset L^q(I)$ with

$$\|\cdot\|_{L^q} \leq C_{p,q} \|\cdot\|_{H^1}^{\gamma} \|\cdot\|_{L^p}^{1-\gamma}, \qquad 1 \leq p < q \leq \infty, \tag{2.4}$$

where $\gamma = (1/p - 1/q)/(1/p + 1/2)$. As usual we take the identification of $L^2(I)$ and its dual $L^2(I)'$ and consider that $H^1(I) \subset L^2(I) \subset H^1(I)'$. Then, (2.3) implies that, for any $s > 1/2$, $L^1(I) \subset H^s(I)'$ with

$$\|\cdot\|_{(H^s)'} \leq C_s \|\cdot\|_{L^1}, \qquad s > 1/2. \tag{2.5}$$

We shall use the following estimates for the product of functions. In view of (2.3), it is true that

$$\|uv\|_{H^m} \leq C \|u\|_{H^m} \|v\|_{H^m}, \qquad u, v \in H^m(I), \quad m = 1, 2. \tag{2.6}$$

If $u \in H^1(I)$ and $\chi \in H^2(I)$ with $(d\chi/dx)(\alpha) = (d\chi/dx)(\beta) = 0$, then

$$\left\langle \frac{d}{dx}\left\{u\frac{d\chi}{dx}\right\}, v\right\rangle_{(H^1)'\times H^1} = -\left(u\frac{d\chi}{dx}, \frac{dv}{dx}\right)_{L^2}, \qquad v \in H^1(I).$$

Therefore, by (2.3),

$$(2.7) \qquad \left\|\frac{d}{dx}\left\{x\frac{d\chi}{dx}\right\}\right\|_{(H^1)'} \leq \begin{cases} C\|u\|_{L^\infty}\|\chi\|_{H^1} \\ C\|u\|_{L^2}\left\|\dfrac{d\chi}{dx}\right\|_{L^\infty} \end{cases}, \qquad u \in H^1(I), \ \chi \in H^2_N(I),$$

where

$$\begin{cases} H^s_N(I) = \{u \in H^s(I); (du/dx)(\alpha) = (du/dx)(\beta) = 0\}, & s > 3/2, \\ H^s_{N^2}(I) = \{u \in H^s(I); (du/dx)(\alpha) = (du/dx)(\beta) = 0, \\ \qquad (d^3u/dx^3)(\alpha) = (d^3u/dx^3)(\beta) = 0\}, & s > 7/2. \end{cases}$$

By using (2.6) it is also verified that

$$(2.8) \qquad \left\|\frac{d}{dx}\left\{u\frac{d\chi}{dx}\right\}\right\|_{L^2} \leq C\|u\|_{H^1}\|\chi\|_{H^2}, \qquad u \in H^1(I), \ \chi \in H^2(I),$$

$$(2.9) \qquad \left\|\frac{d}{dx}\left\{u\frac{d\chi}{dx}\right\}\right\|_{H^1} \leq C\|u\|_{H^2}\|\chi\|_{H^3}, \qquad u \in H^2(I), \ \chi \in H^3(I).$$

Let $\chi(\cdot)$ be a smooth function defined in $(-\infty, \infty)$. Then, for each $m = 1, 2, 3$, $\chi(\rho)$ defines a bounded and continuous mapping from $H^m(I)$ into itself. Moreover, the following estimates are true:

$$(2.10) \qquad \|\chi(\rho)\|_{H^1} \leq p(\|\rho\|_{H^1}), \qquad \rho \in H^1(I);$$

$$(2.11) \qquad \|\chi(\rho)\|_{H^m} \leq p(\|\rho\|_{H^{m-1}})(\|\rho\|_{H^m} + 1), \qquad \rho \in H^m(I), \quad m = 2, 3;$$

$$(2.12) \qquad \|\chi(\rho) - \chi(\sigma)\|_{H^m} \leq p(\|\rho\|_{H^m} + \|\sigma\|_{H^m})\|\rho - \sigma\|_{H^m}, \qquad \rho, \sigma \in H^m(I), \quad m = 1, 2, 3;$$

where $p(\cdot)$ denotes some continuous increasing function determined from $\chi(\cdot)$.

Consider the sesquilinear form

$$a(u, v) = a_1 \int_I \frac{du}{dx}\frac{dv}{dx}\,dx + a_0 \int_I uv\,dx, \qquad u, v \in H^1(I),$$

where $a_0, a_1 > 0$ are positive constants. By means of the formula $a(u, v) = \langle \tilde{A}_0 u, v\rangle_{(H^1)'\times H^1}$, $v \in H^1(I)$, a bounded linear operator $\tilde{A}_0$ from $H^1(I)$ to $H^1(I)'$ is defined. $\tilde{A}_0$ is the second order differential operator $-a_1 d^2/dx^2 + a_0$ equipped with the Neumann boundary conditions at $x = \alpha, \beta$, (see Lions and Magenes

[11]). With the identification $L^2(I) = L^2(I)'$, the part $A_0 = \tilde{A}_0|_{L^2}$ of $\tilde{A}_0$ in $L^2(I)$ is defined. A_0 is shown to be a self-adjoint operator of $L^2(I)$ with the domain $\mathscr{D}(A_0) = H_N^2(I)$. Obviously, A_0 is positive definite and A_0^{-1} is a compact operator. For $\theta > 0$, we denote the fractional power of A_0 by A_0^θ. It is known that

$$(2.13) \qquad \mathscr{D}(A_0^\theta) = \begin{cases} H^{2\theta}(I), & 0 \le \theta < 3/4, \\ H_N^{2\theta}(I), & 3/4 < \theta < 7/4, \\ H_{N^2}^{2\theta}(I), & 7/4 < \theta \le 2. \end{cases}$$

3. Local solutions

We shall show the existence of local solutions to (KS) by using the Galerkin method. Since we have to use the method three times, it seems convenient to announce the result in a general form.

Let $\mathscr{H}$ be a separable Hilbert space, and $\mathscr{V}$ be another separable Hilbert space such that $\mathscr{V}$ is a dense subspace of $\mathscr{H}$ with compact embedding. Identifying $\mathscr{H}$ and its dual $\mathscr{H}'$, we have: $\mathscr{V} \subset \mathscr{H} \subset \mathscr{V}'$.

Consider the Cauchy problem in $\mathscr{V}'$

$$(3.1) \qquad \begin{cases} \dfrac{dU}{dt} + \mathscr{A}U = \mathscr{F}(U), & 0 < t < \infty, \\ U(0) = U_0. \end{cases}$$

Here, $\mathscr{A}$ is the bounded linear operator from $\mathscr{V}$ to $\mathscr{V}'$ defined by a symmetric sesquilinear form $a(\cdot,\cdot)$ on $\mathscr{V}$. $\mathscr{F}(\cdot)$ is a continuous mapping from $\mathscr{V}$ to $\mathscr{V}'$. $U_0 \in \mathscr{H}$ is an initial value. $U = U(t)$ is the unknown function.

We assume the following conditions:

$$(3.2) \qquad \begin{cases} |a(U,V)| \le M\|U\|_{\mathscr{V}}\|V\|_{\mathscr{V}}, & U, V \in \mathscr{V}, \\ a(U,U) \ge \gamma\|U\|_{\mathscr{V}}^2, & U \in \mathscr{V} \end{cases}$$

with some constants M and $\gamma > 0$;

$$(3.3) \qquad \|\mathscr{F}(U)\|_{\mathscr{V}'} \le \varepsilon\|U\|_{\mathscr{V}} + p_\varepsilon(\|U\|_{\mathscr{H}}), \qquad U \in \mathscr{V}$$

with an arbitrary constant $\varepsilon > 0$ and some continuous increasing function $p_\varepsilon(\cdot)$ depending on ε;

$$(3.4) \qquad \|\mathscr{F}(U) - \mathscr{F}(V)\|_{\mathscr{V}'} \le \varepsilon\|U - V\|_{\mathscr{V}} + (\|U\|_{\mathscr{V}} + \|V\|_{\mathscr{V}} + 1) \times p_\varepsilon(\|U\|_{\mathscr{H}} + \|V\|_{\mathscr{H}})\|U - V\|_{\mathscr{H}}, \qquad U, V \in \mathscr{V}$$

with an arbitrary constant $\varepsilon > 0$ and some continuous increasing function $p_\varepsilon(\cdot)$ depending on ε.

By the standard argument we can prove the following proposition. For the detailed proof, see [16, Theorem 2.1].

Proposition 3.1. *Assume the conditions* (3.2), (3.3) *and* (3.4). *For any* $U_0 \in \mathscr{H}$, *there exists a unique local solution to* (3.1) *such that*

$$U \in H^1(0, T_{U_0}; \mathscr{V}') \cap \mathscr{C}([0, T_{U_0}]; \mathscr{H}) \cap L^2(0, T_{U_0}; \mathscr{V}),$$

where T_{U_0} *is a positive number which is determined by* U_0. *Let* $B_r = \{U_0 \in \mathscr{H}; \|U_0\|_{\mathscr{H}} < r\}$, $r > 0$, *denote an open ball. In* B_r, T_{U_0} *can be taken uniformly as* $T_{U_0} \geq T_r > 0$, $U_0 \in B_r$. *Moreover, for each* $t \in [0, T_r]$, *the mapping:* $U_0 \mapsto U(t)$ *is continuous from* B_r *to* $\mathscr{H}$.

Let $\delta > 0$ be fixed arbitrarily. In order to apply Proposition 3.1 we introduce an auxiliary functions $\chi_\delta(\rho)$ which coincides with $\chi(\rho)$ in (KS) for every $\rho \in [\delta/2, \infty)$ and is a smooth function defined for every $\rho \in (-\infty, \infty)$.

We first set the spaces as

$$\mathscr{H} = L^2(I) \times H^1(I), \qquad \mathscr{V} = H^1(I) \times H^2_N(I).$$

Then, $\mathscr{V}'$ is identified with $\mathscr{V}' = H^1(I)' \times L^2(I)$ and the dual product between $\mathscr{V}'$ and $\mathscr{V}$ is given by

$$\left\langle \begin{pmatrix} f \\ \xi \end{pmatrix}, \begin{pmatrix} u \\ \rho \end{pmatrix} \right\rangle_{\mathscr{V}' \times \mathscr{V}} = \langle f, u \rangle_{(H^1)' \times H^1} + (\xi, A_2 \rho)_{L^2}, \qquad \begin{pmatrix} f \\ \xi \end{pmatrix} \in \mathscr{V}', \ \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathscr{V},$$

where $A_2 = -bd^2/dx^2 + d$ is a self-adjoint operator of $L^2(I)$ with the domain $\mathscr{D}(A_2) = H^2_N(I)$.

The operator $\mathscr{A}$ is defined by

$$\mathscr{A} = \begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & A_2 \end{pmatrix},$$

where $\tilde{A}_1 = -ad^2/dx^2 + 1$ equipped with the Neumann boundary conditions is a bounded operator from $H^1(I)$ to $H^1(I)'$. Obviously, $\mathscr{A}$ is determined by the sesquilinear form

$$a(U, V) = a \int_I \frac{du}{dx}\frac{dv}{dx}\, dx + \int_I uv\, dx + (A_2\rho, A_2\sigma)_{L^2}, \qquad U, V \in \mathscr{V},$$

where $U = {}^t(u, \rho)$, $V = {}^t(v, \sigma)$.

On the other hand, in view of (2.7), the nonlinear mapping $\mathscr{F}(U)$ is defined by

$$\mathscr{F}(U) = \begin{pmatrix} -(d/dx)(u(d/dx)(\chi_\delta(\rho))) + u \\ cu \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathscr{V}. \tag{3.5}$$

Then we verify the following result.

Theorem 3.2. *Let* $U_0 = {}^t(u_0, \rho_0) \in L^2(I) \times H^1(I)$. *Then there exists a unique local soluion* $U = {}^t(u, \rho)$ *to* (3.1) *on an interval* $[0, T_{u_0,\rho_0}]$ *such that*

$$u \in H^1(0, T_{u_0,\rho_0}; H^1(I)') \cap \mathscr{C}([0, T_{u_0,\rho_0}]; L^2(I)) \cap L^2(0, T_{u_0,\rho_0}; H^1(I)),$$

$$\rho \in H^1(0, T_{u_0,\rho_0}; L^2(I)) \cap \mathscr{C}([0, T_{u_0,\rho_0}]; H^1(I)) \cap L^2(0, T_{u_0,\rho_0}; H^2_N(I)),$$

where T_{u_0,ρ_0} *is determined by the sum of norms* $\|u_0\|_{L^2} + \|\rho_0\|_{H^1}$.

Proof. It is sufficient to observe that (3.3) and (3.4) are fulfilled.

By (2.7), (2.4) and (2.10) we have:

$$\left\|\frac{d}{dx}\left\{u\frac{d}{dx}\chi_\delta(\rho)\right\}\right\|_{(H^1)'} \le C\|u\|_{L^\infty}\|\chi_\delta(\rho)\|_{H^1} \le C\|u\|_{L^2}^{1/2}\|u\|_{H^1}^{1/2}p(\|\rho\|_{H^1})$$

$$\le \varepsilon\|u\|_{H^1} + C_\varepsilon\|u\|_{L^2}p(\|\rho\|_{H^1})$$

with an arbitrary $\varepsilon > 0$. Therefore, (3.3) holds. Similarly, by (2.7), (2.11) and (2.12) we have:

$$\left\|\frac{d}{dx}\left\{u\frac{d}{dx}\chi_\delta(\rho) - v\frac{d}{dx}\chi_\delta(\sigma)\right\}\right\|_{(H^1)'}$$

$$\le C[\|u - v\|_{L^2}\|\chi_\delta(\rho)\|_{H^2} + \|v\|_{H^1}\|\chi_\delta(\rho) - \chi_\delta(\sigma)\|_{H^1}]$$

$$\le C[\|u - v\|_{L^2}p(\|\rho\|_{H^1})(\|\rho\|_{H^2} + 1) + \|v\|_{H^1}p(\|\rho\|_{H^1} + \|\sigma\|_{H^1})\|\rho - \sigma\|_{H^1}]$$

$$\le (\|U\|_{\mathscr{V}} + \|V\|_{\mathscr{V}} + 1)p(\|U\|_{\mathscr{H}} + \|V\|_{\mathscr{H}})\|U - V\|_{\mathscr{H}}, \qquad U, V \in \mathscr{V},$$

where $U = {}^t(u, \rho)$, $V = {}^t(v, \sigma)$. Therefore, (3.4) holds also.

Let us next set the spaces as

$$\mathscr{H} = H^1(I) \times H^2_N(I), \qquad \mathscr{V} = H^2_N(I) \times H^3_N(I).$$

Then, $\mathscr{V}' = L^2(I) \times H^1(I)$ with the dual product

$$\left\langle \begin{pmatrix} f \\ \xi \end{pmatrix}, \begin{pmatrix} u \\ \rho \end{pmatrix} \right\rangle_{\mathscr{V}' \times \mathscr{V}} = (f, A_1 u)_{L^2} + (A_2^{1/2}\xi, A_2^{3/2}\rho)_{L^2}.$$

The operator $\mathscr{A}$ is given by $\mathscr{A} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$, where A_2 is regarded as a bounded operator from $H^3_N(I)$ to $H^1(I)$. $\mathscr{A}$ is determined by the sesquilinear form

$$a(U, V) = (A_1 u, A_1 v)_{L^2} + (A_2^{3/2}\rho, A_2^{3/2}\sigma)_{L^2}, \qquad U, V \in \mathscr{V},$$

where $U = {}^t(u, \rho)$, $V = {}^t(v, \sigma)$. In view of (2.8), the nonlinear mapping $\mathscr{F}(U)$ is defined as before by (3.5).

In this setting we verify the following result.

Theorem 3.3. *Let* $U_0 = {}^t(u_0, \rho_0) \in H^1(I) \times H_N^2(I)$. *Then there exists a unique local soluion to* (3.1) *on an interval* $[0, T_{u_0,\rho_0}]$ *such that*

$$u \in H^1(0, T_{u_0,\rho_0}; L^2(I)) \cap \mathscr{C}([0, T_{u_0,\rho_0}]; H^1(I)) \cap L^2(0, T_{u_0,\rho_0}; H_N^2(I)),$$
$$\rho \in H^1(0, T_{u_0,\rho_0}; H^1(I)) \cap \mathscr{C}([0, T_{u_0,\rho_0}]; H_N^2(I)) \cap L^2(0, T_{u_0,\rho_0}; H_N^3(I)),$$

where T_{u_0,ρ_0} *is determined by* $\|u_0\|_{H^1} + \|\rho_0\|_{H^2}$.

Proof. As before we observe that (3.3) and (3.4) are fulfilled. By (2.8) and (2.11) we have:

$$\left\|\frac{d}{dx}\left\{u\frac{d}{dx}\chi_\delta(\rho)\right\}\right\|_{L^2} \le C\|u\|_{H^1}p(\|\rho\|_{H^1})(\|\rho\|_{H^2}+1), \qquad \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathscr{V}. \tag{3.6}$$

Therefore, (3.3) holds. Similarly, by (2.8),

$$\left\|\frac{d}{dx}\left\{u\frac{d}{dx}\chi_\delta(\rho) - v\frac{d}{dx}\chi_\delta(\sigma)\right\}\right\|_{L^2} \le C[\|u-v\|_{H^1}\|\chi_\delta(\rho)\|_{H^2} + \|v\|_{H^1}\|\chi_\delta(\rho)-\chi_\delta(\sigma)\|_{H^2}]. \tag{3.7}$$

This shows that (3.4) is also valid.

We finally set the spaces as

$$\mathscr{H} = H_N^2(I) \times H_N^3(I), \qquad \mathscr{V} = H_N^3(I) \times H_{N^2}^4(I),$$

the dual space $\mathscr{V}'$ being identified with $\mathscr{V}' = H^1(I) \times H_N^2(I)$.

Then as before the following result is verified.

Theorem 3.4. *Let* $U_0 = {}^t(u_0, \rho_0) \in H_N^2(I) \times H_N^3(I)$. *Then there exists a unique local soluion to* (3.1) *on an interval* $[0, T_{u_0,\rho_0}]$ *such that*

$$u \in H^1(0, T_{u_0,\rho_0}; H^1(I)) \cap \mathscr{C}([0, T_{u_0,\rho_0}]; H_N^2(I)) \cap L^2(0, T_{u_0,\rho_0}; H_N^3(I)),$$
$$\rho \in H^1(0, T_{u_0,\rho_0}; H_N^2(I)) \cap \mathscr{C}([0, T_{u_0,\rho_0}]; H_N^3(I)) \cap L^2(0, T_{u_0,\rho_0}; H_{N^2}^4(I)),$$

where T_{u_0,ρ_0} *is determined by* $\|u_0\|_{H^2} + \|\rho_0\|_{H^3}$.

Proof. Essential thing is to verify (3.3) and (3.4) in the spaces $\mathscr{H}$, $\mathscr{V}$ and $\mathscr{V}'$ announced above. But, as before, these are readily verified by using (2.9), (2.11) and (2.12).

We are now in a position to state the main theorem of this section.

Theorem 3.5. *Let* $0 \le u_0 \in L^2(I)$ *and* $\rho_0 \in H^1(I)$ *with* $\inf_{x\in I}\rho_0 \ge \delta > 0$. *Then, there exists a unique local solution to* (KS) *on an interval* $[0, \tilde{T}_{u_0,\rho_0}]$ *such that*

$$u \in H^1(0, \tilde{T}_{u_0,\rho_0}; H^1(I)') \cap \mathscr{C}([0, \tilde{T}_{u_0,\rho_0}]; L^2(I)) \cap L^2(0, \tilde{T}_{u_0,\rho_0}; H^1(I)),$$

$$\rho \in H^1(0, \tilde{T}_{u_0,\rho_0}; L^2(I)) \cap \mathscr{C}([0, \tilde{T}_{u_0,\rho_0}]; H^1(I)) \cap L^2(0, \tilde{T}_{u_0,\rho_0}; H^2_N(I))$$

and that

$$(3.8) \qquad u(t) \geq 0 \quad \text{and} \quad \rho(t) \geq \delta e^{-dt}, \qquad 0 \leq t \leq \tilde{T}_{u_0,\rho_0}.$$

Here, $\tilde{T}_{u_0,\rho_0}$ *is determined by the sum of norms* $\|u_0\|_{L^2} + \|\rho_0\|_{H^1}$ *and the constant* δ. *Let*

$$\tilde{B}_{r,\delta} = \{{}^t(u_0, \rho_0) \in L^2(I) \times H^1(I); \|u_0\|_{L^2} + \|\rho_0\|_{H^1} < r, u_0 \geq 0, \inf_{x \in I} \rho_0 \geq \delta\},$$

then the mapping: ${}^t(u_0, \rho_0) \mapsto {}^t(u(t), \rho(t))$ *is continuous from* $\tilde{B}_{r,\delta}$ *to* $L^2(I) \times H^1(I)$ *for each* $t \in [0, \tilde{T}_{r,\delta}]$, *where* $\inf_{u_0,\rho_0} \tilde{T}_{u_0,\rho_0} \geq \tilde{T}_{r,\delta} > 0$.

Proof. Let u, ρ be the solution obtained in Theorem 3.2. In order to complete the proof of the theorem it suffices to verify (3.8). In fact, if u and ρ satisfy (3.8) for $0 \leq t \leq T_{u_0,\rho_0}$, then u, ρ is obviously a unique local solution to (KS) for $0 \leq t \leq \tilde{T}_{u_0,\rho_0}$, where $\tilde{T}_{u_0,\rho_0} = \min\{T_{u_0,\rho_0}, (\log 2)/d\}$.

Consider first the case when $u_0 \in H^2_N(I)$ and $\rho_0 \in H^3_N(I)$. In this case (3.8) is verified by the truncation method (cf. [23, Theorem 2.1]). In fact, let $H(u)$ be a decreasing $\mathscr{C}^3$ function defined for $u \in (-\infty, \infty)$ such that $H(u) > 0$ for $u < 0$ and $H(u) = 0$ for $u \geq 0$. Moreover, let $H(u)$ satisfy the following conditions:

$$\begin{cases} 0 \leq H(u) \leq Cu^2, & u \in (-\infty, \infty), \\ 0 \leq H'(u)u \leq CH(u), & u \in (-\infty, \infty), \\ 0 \leq H''(u)u^2 \leq CH(u), & u \in (-\infty, \infty), \\ \left|\dfrac{d(H'')^{1/2}}{du}(u)u\right| \leq C(H''(u))^{1/2}, & u \in (-\infty, \infty) \end{cases}$$

with some constant C. ($H(u)$ is for example constructed by $H(u) = \int_0^u H_1(u)du$, $H_1(u) = \int_0^u H_0(u)du$, where $H_0(u)$ is some decreasing continuous function in $(-\infty, \infty)$ such that $H_0(u) = 1$ for $u \leq -1$, $H_0(u) = u^4$ for $-1/2 \leq u \leq 0$, and $H_0(u) = 0$ for $u \geq 0$.)

We consider the function

$$\varphi(t) = \int_I H(u(x,t))dx, \qquad 0 \leq t < T_{u_0,\rho_0}.$$

Clearly, $\varphi(t)$ is a nonnegative $\mathscr{C}^1$ function with the derivative

$$\varphi'(t) = -a\int_I H''(u)\left(\frac{\partial u}{\partial x}\right)^2 dx + \int_I H''(u)u\frac{\partial u}{\partial x}\frac{\partial \chi_\delta}{\partial x}\, dx.$$

Moreover, it is easily seen that $\varphi'(t) \le C_{u_0,\rho_0}\varphi(t)$ with some constant C_{u_0,ρ_0}. Since $\varphi(0) = 0$, it follows that $\varphi(t) = 0$ for every $t \in [0, T_{u_0,\rho_0}]$, that is, $u(t) \ge 0$ for every $t \in [0, T_{u_0,\rho_0}]$. By comparison theorem this implies that $\rho(t) \ge \delta e^{-dt}$ for $t \in [0, T_{u_0,\rho_0}]$.

For general initial functions $0 \le u_0 \in L^2(I)$ and $\delta \le \rho_0 \in H^1(I)$, we take sequences $\{u_{0,n}\}$ and $\{\rho_{0,n}\}$ such that $0 \le u_{0,n} \in H_N^2(I)$ and $\delta \le \rho_{0,n} \in H_N^3(I)$ which converge to u_0 in $L^2(I)$ and to ρ_0 in $H^1(I)$, respectively. We already know that, for each initial data $u_{0,n}$, $\rho_{0,n}$, there exists a solution u_n, ρ_n to (KS) with the estimate (3.8) on some interval $[0, T_{u_{0,n},\rho_{0,n}}]$. Since $T_{u_{0,n},\rho_{0,n}}$ is determined by $\|u_{0,n}\|_{L^2} + \|\rho_{0,n}\|_{H^1}$ and δ, we have: $T_0 = \inf_n T_{u_{0,n},\rho_{0,n}} > 0$. On the other hand, the solution is continuous with respect to the initial data. Therefore, as $n \to \infty$, $u_n \to u$ in $\mathscr{C}([0, T_0]; L^2(I))$ and $\rho_n \to \rho$ in $\mathscr{C}([0, T_0]; H^1(I))$. Hence, u and ρ satisfy (3.8) for $0 \le t \le T_0$. But, by the uniqueness of local solution, (3.8) is verified for every $t \in [0, \tilde{T}_{u_0,\rho_0}]$.

Remark 3.6. Let u, ρ be a local solution of (KS) on an interval $[0, T_{u,\rho})$ for initial functions $u_0 \in L^2(I)$ and $\rho_0 \in H^1(I)$ with $u_0 \ge 0$ and $\inf_{x \in I} \rho_0(x) \ge \delta > 0$ which has the regularity

$$u \in H^1(0, T_{u,\rho}; H^1(I)') \cap \mathscr{C}([0, T_{u,\rho}); L^2(I)) \cap L^2(0, T_{u,\rho}; H^1(I)),$$

$$\rho \in H^1(0, T_{u,\rho}; L^2(I)) \cap \mathscr{C}([0, T_{u,\rho}); H^1(I)) \cap L^2(0, T_{u,\rho}; H_N^2(I)).$$

Then, by the similar argument as in the proof of Theorem 3.5, it is proved that (3.8) is valid for every $t \in [0, T_{u,\rho})$.

4. A priori estimates and global solutions

In this section, we shall establish a priori estimates of the local solutions, and obtain the existence of global solutions.

Proposition 4.1. *Let* $0 \le u_0 \in H_N^2(I)$ *and* $\rho_0 \in H_N^3(I)$ *with* $\inf_{x \in I} \rho_0 > 0$. *Let* u, ρ *be a local solution of* (KS) *on an interval* $[0, T_{u,\rho})$ *such that*

$$0 \le u \in H^1(0, T_{u,\rho}; H^1(I)) \cap \mathscr{C}([0, T_{u,\rho}); H_N^2(I)) \cap L^2(0, T_{u,\rho}; H_N^3(I)),$$

$$0 < \rho \in H^1(0, T_{u,\rho}; H_N^2(I)) \cap \mathscr{C}([0, T_{u,\rho}); H_N^3(I)) \cap L^2(0, T_{u,\rho}; H_{N^2}^4(I)),$$

and let

$$\inf_{x \in I} \rho(t) \ge \delta > 0 \qquad \text{for every } 0 \le t < T_{u,\rho}. \tag{4.1}$$

Then, with some increasing continuous function $p_\delta(\cdot)$ *dependent on* δ *but independent of* $T_{u,\rho}$, *the estimate*

$$\|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \le p_\delta(\|u_0\|_{H^2} + \|\rho_0\|_{H^3}), \qquad 0 \le t < T_{u,\rho}$$

is true.

Proof. The proof will consist of several steps. Throughout the proof we denote by C, $p(\cdot)$ and η some constants, some increasing functions and some positive exponents respectively, which are determined by the initial constants and are independent of individual solutions with the exception that they may depend on δ.

Step 1. Integrating the first equation of (KS) in x yields that

$$\frac{d}{dt}\int_I u\, dx = 0.$$

Therefore, since $u(t) \ge 0$,

$$\|u(t)\|_{L^1} = \|u_0\|_{L^1}, \qquad 0 \le t < T_{u,\rho}. \tag{4.2}$$

Step 2. Let us regard $\rho(t)$ as a solution of the equation

$$\frac{d\rho}{dt} + \tilde{A}_2\rho = cu(t), \qquad 0 < t < T_{u,\rho}$$

in the space $H^1(I)'$, where $\tilde{A}_2 = -b(\partial^2/\partial x^2) + d$ equipped with the Neumann boundary conditions is a bounded operator from $H^1(I)$ to $H^1(I)'$. $-\tilde{A}_2$ is the generator of an analytic semigroup on $H^1(I)'$ with the estimate $\|e^{-t\tilde{A}_2}\|_{\mathscr{L}((H^1)')} \le Ce^{-dt}$, $0 \le t < \infty$. By the theory of semigroups, $\rho(t)$ is written in the form

$$\rho(t) = e^{-t\tilde{A}_2}\rho_0 + c\int_0^t e^{-(t-s)\tilde{A}_2}u(s)ds,$$

so that

$$\tilde{A}_2\rho(t) = e^{-t\tilde{A}_2}\tilde{A}_2\rho_0 + c\int_0^t \tilde{A}_2^{7/8}e^{-(t-s)\tilde{A}_2}\tilde{A}_2^{1/8}u(s)ds.$$

We here note a fact that

$$\mathscr{D}(\tilde{A}_2^{1/8}) = [H^1(I)', H^1(I)]_{1/8} = H^{3/4}(I)'.$$

Then, from (2.5) and (4.2),

$$\|\tilde{A}_2\rho(t)\|_{(H^1)'} \le C\left[e^{-dt}\|\tilde{A}_2\rho_0\|_{(H^1)'} + \int_0^t (t-s)^{-7/8}e^{-d(t-s)}ds\|u_0\|_{L^1}\right].$$

Therefore we obtain that

$$\|\rho(t)\|_{H^1} \le C[e^{-dt}\|\rho_0\|_{H^1} + \|u_0\|_{L^1}], \qquad 0 \le t < T_{u,\rho}. \tag{4.3}$$

Step 3. Multiply the first equation in (KS) by u and integrate the product in x. Then,

$$\frac{1}{2}\frac{d}{dt}\int_I u^2\,dx + a\int_I\left(\frac{\partial u}{\partial x}\right)^2 dx = \int_I u\frac{\partial u}{\partial x}\frac{\partial \chi}{\partial x}\,dx$$
$$\leq \frac{a}{2}\int_I\left(\frac{\partial u}{\partial x}\right)^2 dx + \frac{1}{2a}\int_I u^2\chi'(\rho)^2\left(\frac{\partial \rho}{\partial x}\right)^2 dx.$$

Similarly, multiplying the second equation in (KS) by $\partial^2\rho/\partial x^2$ and integrating the product in x yield that

$$\frac{1}{2}\frac{d}{dt}\int_I\left(\frac{\partial \rho}{\partial x}\right)^2 dx + b\int_I\left(\frac{\partial^2 \rho}{\partial x^2}\right)^2 dx + d\int_I\left(\frac{\partial \rho}{\partial x}\right)^2 dx = -c\int_I u\frac{\partial^2\rho}{\partial x^2}\,dx$$
$$\leq \frac{b}{2}\int_I\left(\frac{\partial^2 \rho}{\partial x^2}\right)^2 dx + \frac{c^2}{2b}\int_I u^2\,dx.$$

Here, from (4.1) and (χ) and with the use of (2.3) and (2.4),

$$\int_I u^2\chi'(\rho)^2\left(\frac{\partial \rho}{\partial x}\right)^2 dx \leq C\|u\|_{L^4}^2(\|\rho\|_{L^\infty}+1)^{2r}\left\|\frac{\partial\rho}{\partial x}\right\|_{L^4}^2$$
$$\leq C\|u\|_{H^1}\|u\|_{L^1}\|\rho\|_{H^2}^{1/2}(\|\rho\|_{H^1}+1)^{2r+3/2}$$
$$\leq \varepsilon(\|u\|_{H^1}^2+\|\rho\|_{H^2}^2) + C_\varepsilon\|u\|_{L^1}^4(\|\rho\|_{H^1}+1)^{8r+6}$$

with an arbitrary $\varepsilon > 0$. Similarly,

$$\int_I u^2\,dx \leq C\|u\|_{H^1}^{2/3}\|u\|_{L^1}^{4/3} \leq \varepsilon\|u\|_{H^1}^2 + C_\varepsilon\|u\|_{L^1}^2.$$

Therefore, we observe that

$$\frac{d}{dt}\int_I\left\{u^2+\left(\frac{\partial\rho}{\partial x}\right)^2\right\}dx + \int_I\left\{\frac{a}{2}\left(\frac{\partial u}{\partial x}\right)^2+\frac{b}{2}\left(\frac{\partial^2\rho}{\partial x^2}\right)^2\right\}dx$$
$$+\int_I\left\{u^2+d\left(\frac{\partial\rho}{\partial x}\right)^2\right\}dx \leq p(\|u\|_{L^1}+\|\rho\|_{H^1}).$$

Therefore, solving this in $\|u\|_{L^2}^2+\|\partial\rho/\partial x\|_{L^2}^2$, we conclude that

$$\text{(4.4)}\quad \|u(t)\|_{L^2}^2 \leq C\Big[e^{-\eta t}(\|u_0\|_{L^2}^2+\|\rho_0\|_{H^1}^2)$$
$$+\int_0^t e^{-\eta(t-s)}p(\|u(s)\|_{L^1}+\|\rho(s)\|_{H^1})ds\Big],\qquad 0\leq t< T_{u,\rho},$$

(4.5) $\int_0^t (\|u(s)\|_{H^1}^2 + \|\rho(s)\|_{H^2}^2) ds$

$$\leq C\left[\|u_0\|_{L^2}^2 + \|\rho_0\|_{H^1}^2 + \int_0^t p(\|u(s)\|_{L^1} + \|\rho(s)\|_{H^1}) ds\right], \qquad 0 \leq t < T_{u,\rho}.$$

From (4.2) and (4.3),

(4.6) $$\|u(t)\|_{L^2} \leq p(\|u_0\|_{L^2} + \|\rho_0\|_{H^1}), \qquad 0 \leq t < T_{u,p},$$

(4.7) $$\int_0^t (\|u(s)\|_{H^1}^2 + \|\rho(s)\|_{H^2}^2) ds$$

$$\leq C[\|u_0\|_{L^2}^2 + \|\rho_0\|_{H^1}^2 + tp(\|u_0\|_{L^1} + \|\rho_0\|_{H^1})], \qquad 0 \leq t < T_{u,\rho}.$$

Step 4. Multiplying the first equation in (KS) by $\partial^2 u/\partial x^2$ and integrating the product in x, we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_I \left(\frac{\partial u}{\partial x}\right)^2 dx + a\int_I \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx$$

$$= \int_I \left\{\frac{\partial}{\partial x}\left(u\frac{\partial \chi}{\partial x}\right)\right\}\frac{\partial^2 u}{\partial x^2} dx$$

$$\leq \frac{a}{2}\int_I \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx + C\int_I \left\{\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial \chi}{\partial x}\right)^2 + u^2\left(\frac{\partial^2 \chi}{\partial x^2}\right)^2\right\} dx.$$

Here we observe that

$$\frac{\partial^2 \chi}{\partial x^2} = \chi''(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2 + \chi'(\rho)\frac{\partial^2 \rho}{\partial x^2}.$$

Furthermore from (4.1) and (χ),

(4.8) $$\int_I \left\{\chi'(\rho)^2\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial \rho}{\partial x}\right)^2 + \chi'(\rho)^2 u^2\left(\frac{\partial^2 \rho}{\partial x^2}\right)^2 + \chi''(\rho)^2 u^2\left(\frac{\partial \rho}{\partial x}\right)^4\right\} dx$$

$$\leq \varepsilon(\|u\|_{H^2}^2 + \|\rho\|_{H^3}^2) + C_\varepsilon p(\|u\|_{L^2} + \|\rho\|_{H^1})$$

holds with an arbitrary $\varepsilon > 0$. For example, by (2.3) and (2.4),

$$\int_I \chi'(\rho)^2\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial \rho}{\partial x}\right)^2 dx \leq C\|u\|_{H^1}^2 \left\|\frac{\partial \rho}{\partial x}\right\|_{L^\infty}^2 (\|\rho\|_{L^\infty} + 1)^{2r}$$

$$\leq C\|u\|_{H^1}^2 \|\rho\|_{H^2}(\|\rho\|_{H^1} + 1)^{2r+1}$$

$$\leq C\|u\|_{H^2}\|u\|_{L^2}\|\rho\|_{H^3}^{1/2}(\|\rho\|_{H^1} + 1)^{2r+3/2}$$

$$\leq \varepsilon(\|u\|_{H^2}^2 + \|\rho\|_{H^3}^2) + C_\varepsilon\|u\|_{L^2}^4(\|\rho\|_{H^1} + 1)^{8r+6}.$$

It is the same for other integrals.

On the other hand, we obtain from the second equation in (KS) the energy inequality

$$(4.9)\qquad \frac{1}{2}\frac{d}{dt}\int_I \left(\frac{\partial^2 \rho}{\partial x^2}\right)^2 dx + b\int_I \left(\frac{\partial^3 \rho}{\partial x^3}\right)^2 dx + d\int_I \left(\frac{\partial^2 \rho}{\partial x^2}\right)^2 dx$$

$$= -c\int_I \frac{\partial u}{\partial x}\frac{\partial^3 \rho}{\partial x^3} dx \le \frac{b}{2}\int_I \left(\frac{\partial^3 \rho}{\partial x^3}\right)^2 dx + \frac{c^2}{2b}\int_I \left(\frac{\partial u}{\partial x}\right)^2 dx.$$

Here,

$$\int_I \left(\frac{\partial u}{\partial x}\right)^2 dx \le \varepsilon \|u\|_{H^2}^2 + C_\varepsilon \|u\|_{L^2}^2.$$

Therefore, (4.9) jointed with (4.8) yields that

$$\frac{d}{dt}\int_I \left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial^2 \rho}{\partial x^2}\right)^2\right\} dx + \int_I \left\{\frac{a}{2}\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{b}{2}\left(\frac{\partial^3 \rho}{\partial x^3}\right)^2\right\} dx$$

$$+ \int_I \left\{\left(\frac{\partial u}{\partial x}\right)^2 + d\left(\frac{\partial^2 \rho}{\partial x^2}\right)^2\right\} dx \le p(\|u\|_{L^2} + \|\rho\|_{H^1}).$$

Solving this differential inequality and noting (4.3) and (4.4), we conclude that

$$(4.10)\qquad \|u(t)\|_{H^1}^2 + \|\rho(t)\|_{H^2}^2$$

$$\le C\Big[e^{-\eta t}(\|u_0\|_{H^1}^2 + \|\rho_0\|_{H^2}^2) + \|u_0\|_{L^1}^2$$

$$+ \int_0^t e^{-\eta(t-s)} p(\|u(s)\|_{L^2} + \|\rho(s)\|_{H^1}) ds\Big], \qquad 0 \le t < T_{u,\rho},$$

$$(4.11)\qquad \int_0^t (\|u(s)\|_{H^2}^2 + \|\rho(s)\|_{H^3}^2) ds$$

$$\le C\Big[\|u_0\|_{H^1}^2 + \|\rho_0\|_{H^2}^2 + \int_0^t p(\|u(s)\|_{L^2} + \|\rho(s)\|_{H^1}) ds\Big], \qquad 0 \le t < T_{u,\rho}.$$

From (4.3) and (4.6), we conclude also that

$$(4.12)\qquad \|u(t)\|_{H^1} + \|\rho(t)\|_{H^2} \le p(\|u_0\|_{H^1} + \|\rho_0\|_{H^2}), \qquad 0 \le t < T_{u,\rho},$$

$$(4.13)\qquad \int_0^t (\|u(s)\|_{H^2}^2 + \|\rho(s)\|_{H^3}^2)ds$$

$$\leq C[\|u_0\|_{H^1}^2 + \|\rho_0\|_{H^2}^2 + tp(\|u_0\|_{L^2} + \|\rho_0\|_{H^1})], \qquad 0 \leq t < T_{u,\rho}.$$

Step 5. We first notice from the assumption that the formula

$$\frac{d}{dt}\int_I \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx = -2\int_I \frac{\partial^2 u}{\partial t \partial x}\frac{\partial^3 u}{\partial x^3}dx, \qquad \text{a.e. } t \in (0, T_{u,\rho})$$

holds (cf. [19, Chap. 5, Lemma 5.1]). Take the product of the first equation of (KS) operated by $\partial/\partial x$ and $\partial^3 u/\partial x^3$, and integrate the product in x. Then, using the formula above, we verify that

$$(4.14)\qquad \frac{1}{2}\frac{d}{dt}\int_I \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx + a\int_I \left(\frac{\partial^3 u}{\partial x^3}\right)^2 dx$$

$$= \int_I \left\{\frac{\partial^2}{\partial x^2}\left(u\frac{\partial \chi}{\partial x}\right)\right\}\frac{\partial^3 u}{\partial x^3}dx$$

$$\leq \frac{a}{2}\int_I \left(\frac{\partial^3 u}{\partial x^3}\right)^2 dx + \frac{1}{2a}\int_I \left\{\frac{\partial^2}{\partial x^2}\left(u\frac{\partial \chi}{\partial x}\right)\right\}^2 dx.$$

Here, by a direct calculation, we observe from (χ) that

$$\left|\frac{\partial^2}{\partial x^2}\left(u\frac{\partial \chi}{\partial x}\right)\right| \leq C\left(\rho + \frac{1}{\rho}\right)^{3r}\left[\left|\frac{\partial^2 u}{\partial x^2}\right|\left|\frac{\partial \rho}{\partial x}\right| + \left|\frac{\partial u}{\partial x}\right|\left(\left|\frac{\partial^2 \rho}{\partial x^2}\right| + \left|\frac{\partial \rho}{\partial x}\right|^2\right)\right.$$

$$\left. + u\left(\left|\frac{\partial^3 \rho}{\partial x^3}\right| + \left|\frac{\partial^2 \rho}{\partial x^2}\right|\left|\frac{\partial \rho}{\partial x}\right| + \left|\frac{\partial \rho}{\partial x}\right|^3\right)\right].$$

From (4.1) it is then easy to see that

$$\int_I \left\{\frac{\partial^2}{\partial x^2}\left(u\frac{\partial \chi}{\partial x}\right)\right\}^2 dx \leq \varepsilon(\|u\|_{H^3}^2 + \|\rho\|_{H^4}^2) + C_\varepsilon p(\|u\|_{H^1} + \|\rho\|_{H^2})$$

with an arbitrary $\varepsilon > 0$.

On the other hand, we introduce another energy inequality

$$(4.15)\qquad \frac{1}{2}\frac{d}{dt}\int_I \left(\frac{\partial^3 \rho}{\partial x^3}\right)^2 dx + b\int_I \left(\frac{\partial^4 \rho}{\partial x^4}\right)^2 dx + d\int_I \left(\frac{\partial^3 \rho}{\partial x^3}\right)^2 dx$$

$$= -c\int_I \frac{\partial^2 u}{\partial x^2}\frac{\partial^4 \rho}{\partial x^4}dx \leq \frac{b}{2}\int_I \left(\frac{\partial^4 \rho}{\partial x^4}\right)^2 dx + \frac{c^2}{2b}\int_I \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx.$$

It is clear that

$$\int_I \left(\frac{\partial^2 u}{\partial x^2}\right)^2 dx \le \varepsilon \|u\|_{H^3}^2 + C_\varepsilon \|u\|_{H^1}^2.$$

Therefore, (4.15) jointed with (4.14) yields that

$$\frac{d}{dt}\int_I \left\{\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \left(\frac{\partial^3 \rho}{\partial x^3}\right)^2\right\} dx + \int_I \left\{\frac{a}{2}\left(\frac{\partial^3 u}{\partial x^3}\right)^2 + \frac{b}{2}\left(\frac{\partial^4 \rho}{\partial x^4}\right)^2\right\} dx$$
$$+ \int_I \left\{\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + d\left(\frac{\partial^3 \rho}{\partial x^3}\right)^2\right\} dx \le p(\|u\|_{H^1} + \|\rho\|_{H^2}).$$

Hence, we conclude that

$$\begin{aligned} (4.16)\qquad & \|u(t)\|_{H^2}^2 + \|\rho(t)\|_{H^3}^2 \\ & \le C\Big[e^{-\eta t}(\|u_0\|_{H^2}^2 + \|\rho_0\|_{H^3}^2) + \|u_0\|_{L^1}^2 \\ & \quad + \int_0^t e^{-\eta(t-s)} p(\|u(s)\|_{H^1} + \|\rho(s)\|_{H^2}) ds\Big], \qquad 0 \le t < T_{u,\rho}, \end{aligned}$$

$$\begin{aligned} (4.17)\qquad & \int_0^t (\|u(s)\|_{H^3}^2 + \|\rho(s)\|_{H^4}^2) ds \\ & \le C\Big[\|u_0\|_{H^2}^2 + \|\rho_0\|_{H^3}^2 + \int_0^t p(\|u(s)\|_{H^1} + \|\rho(s)\|_{H^2}) ds\Big], \qquad 0 \le t < T_{u,\rho}. \end{aligned}$$

From (4.12) and (4.13), we conclude also that

$$(4.18)\qquad \|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \le p(\|u_0\|_{H^2} + \|\rho_0\|_{H^3}), \qquad 0 \le t < T_{u,\rho},$$

$$\begin{aligned} (4.19)\qquad & \int_0^t (\|u(s)\|_{H^3}^2 + \|\rho(s)\|_{H^4}^2) ds \\ & \le C[\|u_0\|_{H^2}^2 + \|\rho_0\|_{H^3}^2 + tp(\|u_0\|_{H^1} + \|\rho_0\|_{H^2})], \qquad 0 \le t < T_{u,\rho}. \end{aligned}$$

Thus we have accomplished the proof of Proposition 4.1.

Thanks to Proposition 4.1, we obtain the global existence of solutions. Let us first verify the following result.

Theorem 4.2. *Let* $0 \le u_0 \in H_N^2(I)$ *and* $\rho_0 \in H_N^3(I)$ *with* $\inf_{x \in I} \rho_0 > 0$. *Then,* (KS) *possesses a unique global solution such that*

$$0 \le u \in \mathscr{C}^1([0,\infty); L^2(I)) \cap \mathscr{C}([0,\infty); H_N^2(I)),$$
$$0 < \rho \in \mathscr{C}^1([0,\infty); H^1(I)) \cap \mathscr{C}([0,\infty); H_N^3(I)).$$

Proof. Let $\delta = \inf \rho_0$, and let $T \in (0,\infty)$ be an arbitrary finite time. From Remark 3.6, if there exists a local solution u, ρ on an interval $[0, T_{u,\rho})$, then

$$\rho(t) \ge \delta e^{-dT} \qquad \text{for every } 0 \le t < \min\{T_{u,\rho}, T\}.$$

By Theorems 3.4 and 3.5, there exists a unique local solution u, ρ on an interval $[0, T_1]$. Assuming that $T_1 < T$, we will prove that u, ρ can be extended as a local solution at least on the interval $[0, T]$. Since $u \in H^1(0, T_1; H^1(I)) \cap L^2(0, T_1; H_N^3(I))$ and $\rho \in H^1(0, T_1; H_N^2(I)) \cap L^2(0, T_1; H_{N^2}^4(I))$, it is seen that the limits $\lim_{t \to T_1} u(t) = u_1$ and $\lim_{t \to T_1} \rho(t) = \rho_1$ exist in $H^2(I)$ and $H^3(I)$, respectively (cf. [19, Chap. 5, Lemma 5.1]). Moreover, Proposition 4.1 shows that $\|u_1\|_{H^2} + \|\rho_1\|_{H^3}$ is estimated by $\|u_0\|_{H^2} + \|\rho_0\|_{H^3}$. We can then use again Theorems 3.4 and 3.5 to conclude that u, ρ is extended as a local solution on an interval $[0, T_1 + \tau)$, here $\tau > 0$ is determined by $\|u_1\|_{H^2} + \|\rho_1\|_{H^3}$ and hence by $\|u_0\|_{H^2} + \|\rho_0\|_{H^3}$ alone.

Repeating this procedure, we can extend u, ρ over the interval $[0, T]$. Since T is arbitrary, the global existence is proved.

Theorem 4.2 jointed with Theorems 3.3 and 3.4 then yields the final existence result.

Theorem 4.3. *Let* $0 \le u_0 \in L^2(I)$ *and* $\rho_0 \in H^1(I)$ *with* $\inf_{x \in I} \rho_0 > 0$. *Then,* (KS) *possesses a unique global solution such that*

$$0 \le u \in \mathscr{C}([0,\infty); L^2(I)) \cap \mathscr{C}^1((0,\infty); L^2(I)) \cap \mathscr{C}((0,\infty); H_N^2(I)),$$
$$0 < \rho \in \mathscr{C}([0,\infty); H^1(I)) \cap \mathscr{C}^1((0,\infty); H^1(I)) \cap \mathscr{C}((0,\infty); H_N^3(I)).$$

Proof. By Theorem 3.2, there exists a local solution u, ρ on an interval $[0, T_{u_0,\rho_0}]$. In addition, there is a time $t_1 < T_{u_0,\rho_0}$ arbitrary small such that $u_1 = u(t_1) \in H^1(I)$ and $\rho_1 = \rho(t_1) \in H_N^2(I)$. Then, from Theorem 3.3, it follows that $u \in L^2(t_1, T_{u_0,\rho_0}; H_N^2(I))$ and $\rho \in L^2(t_1, T_{u_0,\rho_0}; H_N^3(I))$; so that, with some $t_2 < T_{u_0,\rho_0}$ arbitrary small, $u_2 = u(t_2) \in H_N^2(I)$ and $\rho_2 = \rho(t_2) \in H_N^3(I)$. As shown above, there exists a unique global solution for the initial functions u_2, ρ_2. Hence we have verified that the desired global solution exists for the initial functions u_0, ρ_0.

Finally we establish estimates of the norms $\|u(t)\|_{H^2}$ and $\|\rho(t)\|_{H^3}$ in terms of the initial functions u_0, ρ_0.

Theorem 4.4. *Let* $0 \le u_0 \in L^2(I)$ *and* $\rho_0 \in H^1(I)$ *with* $\inf_{x \in I} \rho_0 > 0$. *Let* u, ρ *be the global solution to* (KS) *as in Theorem* 4.3, *and assume that* ρ *satisfies*

(4.20) $\rho(t) \geq \delta$ *for every* $0 \leq t < \infty$

with some $\delta > 0$. *Then, with some continuous increasing function* $p_\delta(\cdot)$ *dependent on* δ *but independent of the norms of* u_0, ρ_0, *the estimate*

$$\|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \leq p_\delta\left(\frac{1}{t} + \|u_0\|_{L^2} + \|\rho_0\|_{H^1}\right), \qquad 0 < t < \infty$$

is true.

Proof. We use the estimates estabilshed in the proof of Proposition 4.1. Let $0 < s < t$ and consider (KS) on an interval $[s, t]$ with initial functions $u(s), \rho(s)$. Then, with the aid of (4.3) and (4.6), we observe from (4.10) that

$$\begin{aligned}&\|u(t)\|_{H^1}^2 + \|\rho(t)\|_{H^2}^2 \\ &\quad \leq C[\|u(s)\|_{H^1}^2 + \|\rho(s)\|_{H^2}^2 + p(\|u_0\|_{L^2} + \|\rho_0\|_{H^1})], \qquad 0 < s < t.\end{aligned}$$

Integrating in $s \in (0, t)$ yields that

$$t(\|u(t)\|_{H^1}^2 + \|\rho(t)\|_{H^2}^2) \leq C\int_0^t (\|u(s)\|_{H^1}^2 + \|\rho(s)\|_{H^2}^2) ds + tp(\|u_0\|_{L^2} + \|\rho_0\|_{H^1}).$$

Therefore it follows from (4.7) that

$$\|u(t)\|_{H^1}^2 + \|\rho(t)\|_{H^2}^2 \leq (1/t + 1)p(\|u_0\|_{L^2} + \|\rho_0\|_{H^1}), \qquad 0 < t < \infty.$$

Let now $\frac{t}{2} < s < t$. In a similar way, we can verify from (4.16) and (4.12) that

$$\begin{aligned}&\|u(t)\|_{H^2}^2 + \|\rho(t)\|_{H^3}^2 \\ &\quad \leq C\left[\|u(s)\|_{H^1}^2 + \|\rho(s)\|_{H^3}^2 + p\left(\left\|u\left(\frac{t}{2}\right)\right\|_{H^1} + \left\|\rho\left(\frac{t}{2}\right)\right\|_{H^2}\right)\right], \qquad t/2 < s < t.\end{aligned}$$

Integrating in $s \in \left(\frac{t}{2}, t\right)$ then yields that

$$\begin{aligned}&(t/2)(\|u(t)\|_{H^2}^2 + \|\rho(t)\|_{H^3}^2) \\ &\quad \leq C\int_{t/2}^t (\|u(s)\|_{H^2}^2 + \|\rho(s)\|_{H^3}^2) ds + tp(\|u(t/2)\|_{H^1} + \|\rho(t/2)\|_{H^2}).\end{aligned}$$

Therefore, from (4.13),

$$\begin{aligned}\|u(t)\|_{H^2}^2 + \|\rho(t)\|_{H^3}^2 &\leq (1/t + 1)p\left(\left\|u\left(\frac{t}{2}\right)\right\|_{H^1} + \left\|\rho\left(\frac{t}{2}\right)\right\|_{H^2}\right) \\ &\leq p(1/t + \|u_0\|_{L^2} + \|\rho_0\|_{H^1}), \qquad 0 < t < \infty.\end{aligned}$$

5. Estimates from below

In this section, we shall establish a uniform estimate of ρ from below. This will then show that (4.20) is actually fulfilled.

We first consider an auxiliary linear problem

$$(5.1)\qquad \begin{cases} \dfrac{\partial\rho}{\partial t} = b\dfrac{\partial^2\rho}{\partial x^2} - d\rho + f(x,t), & (x,t) \in I \times (0,\infty), \\ \dfrac{\partial\rho}{\partial x} = 0, \quad x = \alpha, \beta, \ 0 < t < \infty, \\ \rho(x,0) = \rho_0(x), \quad x \in I \end{cases}$$

in I. Here, $b, d > 0$ are two positive constants, $0 \le f \in \mathscr{C}([0,\infty); L^2(I))$ is a given function such that

$$\int_I f(x,t)dx \equiv \ell \qquad \text{(a constant)},$$

and $0 \le \rho_0 \in L^2(I)$ is an initial function.

Proposition 5.1. *Let ρ be a solution of* (5.1) *such that*

$$0 \le \rho \in \mathscr{C}([0,\infty); L^2(I)) \cap \mathscr{C}^1((0,\infty); L^2(I)) \cap \mathscr{C}((0,\infty); H_N^2(I)).$$

Then, there exist a time $t_0 > 0$ and a constant $\delta_0 > 0$ which are independent of $f(t)$ and ρ_0 such that

$$\rho(t) \ge \delta_0 \ell \qquad \textit{for every } t \ge t_0.$$

Proof. Denote by L the differential operator $L = -b(\partial^2/\partial x^2)$ in $L^2(I)$ equipped with the Neumann boundary conditions at $x = \alpha, \beta$, L being a non-negative self-adjoint operator of $L^2(I)$. Using L, it is written as

$$\rho(t) = e^{-t(L+d)}\rho_0 + \int_0^t e^{-(t-s)(L+d)} f(s)ds.$$

Let $t_0 \ge 4$ be arbitrarily fixed. For every $t \ge t_0$,

$$\rho(t) \ge \int_0^{t-t_0/2} e^{-(t-s)(L+d)}\{\bar{f}(s) + f_m(s)\}ds,$$

where $f = \bar{f} + f_m$ is the orthogonal decomposition of $f \in L^2(I)$ such that $\bar{f} = |I|^{-1}\int_I f\,dx$ and $\int_I f_m\,dx = 0$. Since $\bar{f}(t) \equiv \bar{f} = \ell|I|^{-1}$ and $e^{-tL}\bar{f} \equiv \bar{f}$, it follows that

$$\begin{aligned} \rho(t) &\ge \int_0^{t-t_0/2} e^{d(s-t)}\bar{f}\,ds + \int_0^{t-t_0/2} e^{-(t-s)L}e^{d(s-t)}f_m(s)ds \\ &\ge \frac{e^{-dt_0/2}}{d}\{1 - e^{-dt_0/2}\}\bar{f} + \int_0^{t-t_0/2} e^{-(t-s)L}e^{d(s-t)}f_m(s)ds. \end{aligned}$$

Let $L^2_m(I) = \{f \in L^2(I); \int_I f\,dx = 0\}$, and let L_m be the part of L in $L^2_m(I)$. Since $L^2_m(I)$ is the orthogonal complement of the eigen space of the eigen value 0, there exists some $\lambda_0 > 0$ such that $L_m \geq \lambda_0$. Then, using the fact that $e^{-L} \in \mathscr{L}(L^2(I), \mathscr{C}(I)) \cap \mathscr{L}(L^1(I), L^2(I))$, we can estimate the integral term as

$$\left\| e^{-L} \int_0^{t-t_0/2} e^{-(t-s-2)L_m} e^{d(s-t)} e^{-L} f_m(s)\,ds \right\|_{\mathscr{C}}$$

$$\leq \|e^{-L}\|_{\mathscr{L}(L^2,\mathscr{C})} \int_0^{t-t_0/2} e^{2\lambda_0} e^{(s-t)(\lambda_0+d)} \|e^{-L} f_m(s)\|_{L^2} ds$$

$$\leq \frac{e^{2\lambda_0} e^{-t_0(\lambda_0+d)/2}}{\lambda_0 + d} \|e^{-L}\|_{\mathscr{L}(L^2,\mathscr{C})} \|e^{-L}\|_{\mathscr{L}(L^1,L^2)} 2\ell.$$

Therefore, we obtain that

$$\rho(t) \geq \frac{e^{-dt_0/2}\ell}{d|I|} \{1 - e^{-dt_0/2} - 2d|I|\, \|e^{-L}\|_{\mathscr{L}(L^2,\mathscr{C})}$$
$$\times \|e^{-L}\|_{\mathscr{L}(L^1,L^2)} e^{(2-t_0/2)\lambda_0}\}, \qquad t \geq t_0.$$

This shows that, if t_0 is sufficiently large, then the desired estimate holds certainly.

Using this proposition we verify a uniform lower bound of ρ.

Theorem 5.2. *Let $0 \leq u_0 \in L^2(I)$ and $\rho_0 \in H^1(I)$ with $\inf_{x \in I} \rho_0 > 0$, and let u, ρ be the global solution to* (KS) *constructed in Theorem* 4.3. *Then, there exist a time t_0 and a constant $\delta_0 > 0$ independent of u_0, ρ_0 such that*

(5.2) $$\rho(t) \geq \delta_0 \|u_0\|_{L^1} \qquad \textit{for every } t \geq t_0.$$

Proof. Since $\|u(t)\|_{L^1} = \|u_0\|_{L^1}$ identically, it suffices to apply Proposition 5.1 with $f(t) = cu(t)$.

6. Construction of attractor set

Let $H = L^2(I) \times H^1(I)$. In H we consider the Cauchy problem

(6.1) $$\begin{cases} \dfrac{dU}{dt} + AU = F(U), & 0 < t < \infty, \\ U(0) = U_0. \end{cases}$$

Here, A is a positive definite self-adjoint operator of H given by

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad \mathscr{D}(A) = H^2_N(I) \times H^3_N(I),$$

where $A_1 = -a(\partial^2/\partial x^2) + 1$ and $A_2 = -b(\partial^2/\partial x^2) + d$.

$F(U)$ is defined for $U \in \mathscr{D}(A) \cap K_\ell$ by

$$F(U) = \begin{pmatrix} -(\partial/\partial x)(u(\partial/\partial x)(\chi(\rho))) + u \\ cu \end{pmatrix}, \qquad U = \begin{pmatrix} u \\ \rho \end{pmatrix} \in \mathscr{D}(A) \cap K_\ell,$$

where $K_\ell \subset H$ denotes a set of initial functions.

In fact, taking a constant $\ell > 0$, we set K_ℓ as

$$K_\ell = \left\{ {}^t(u, \rho) \in H; u \geq 0, \int_I u\, dx = \ell, \inf_{x \in I} \rho > 0 \right\}.$$

U_0 is then taken from K_ℓ.

By Theorems 3.5 and 4.3 we already know that (6.1) is well posed in the set K_ℓ, that is, for each $U_0 \in K_\ell$, there exists a unique global solution

$$U \in \mathscr{C}([0, \infty); K_\ell) \cap \mathscr{C}^1((0, \infty); H) \cap \mathscr{C}((0, \infty); \mathscr{D}(A))$$

(see also Remark 3.6, (4.2) and (5.2)). Therefore, (6.1) defines a continuous semigroup $\{S(t)\}_{t \geq 0}$ acting on K_ℓ.

Let ${}^t(u(t), \rho(t)) = S(t)U_0$ with $U_0 \in K_\ell$. As proved in Theorem 5.2, there exists a time t_0 and a constant $\delta_0 > 0$ such that

$$\rho(t) \geq \delta_0 \ell \qquad \text{for every } t \geq t_0,$$

t_0 and δ_0 being independent of $U_0 \in K_\ell$. Using this time t_0, we will reset the subset K_ℓ as

$$\tilde{K}_\ell = \overline{S(t_0)K_\ell} \subset K_\ell.$$

Obviously, $S(t)$ maps $\tilde{K}_\ell$ into itself. Hence, $(\{S(t)\}_{t \geq 0}, \tilde{K}_\ell)$ defines a dynamical system in H. Furthermore, by definition, ${}^t(u(t), \rho(t)) = S(t)U_0$, $U_0 \in \tilde{K}_\ell$, satisfies

$$\rho(t) \geq \delta_0 \ell \qquad \text{for every } t \geq 0.$$

Our goal is then to construct an exponential attractor for $(\{S(t)\}_{t \geq 0}, \tilde{K}_\ell)$.

We begin with verifying the following proposition.

Proposition 6.1. *With some universal constant C_ℓ which may depend on ℓ, the following statement is true. For each bounded ball $B_r = \{U_0 \in \tilde{K}_\ell; \|U_0\|_H \leq r\}$, there exists a time t_r depending on B_r such that*

$$\sup_{t \geq t_r} \sup_{U_0 \in B_r} \|S(t)U_0\|_{\mathscr{D}(A)} \leq C_\ell. \tag{6.2}$$

Proof. Let $U_0 = {}^t(u_0, \rho_0) \in B_r$ and $S(t)U_0 = {}^t(u(t), \rho(t))$. By Theorem 4.4, $S(t)U_0 \in \mathscr{D}(A) = H_N^2(I) \times H_N^3(I)$ for every $t > 0$ with the estimate

$$(6.3) \qquad \|U(t)\|_{\mathscr{D}(A)} = \|u(t)\|_{H^2} + \|\rho(t)\|_{H^3} \leq p_\ell(1/t + r), \qquad 0 < t < \infty.$$

The desired estimates will then be established step by step as in the proof of Proposition 4.1. Throughout the proof, C_r and t_r denote some constant and some time, respectively, which may depend on r but is uniform for $U_0 \in B_r$.

From (4.3), we have:

$$\|\rho(t)\|_{H^1} \leq C[e^{-dt}\|\rho_0\|_{H^1} + \ell] \leq C[re^{-dt} + \ell], \qquad 0 < t < \infty.$$

Therefore there exist a universal constant C_ℓ and a time $t_r > 0$ such that

$$\|\rho(t)\|_{H^1} \leq C_\ell \qquad \text{for every } t \geq t_r.$$

Next, let us apply (4.4) to u, ρ on the interval $[t_r, \infty)$. Then, from (6.3),

$$\begin{aligned}\|u(t)\|_{L^2}^2 &\leq C\left[e^{-\eta(t-t_r)}(\|u(t_r)\|_{L^2}^2 + \|\rho(t_r)\|_{H^1}^2) + \int_{t_r}^t e^{-\eta(t-s)} p_\ell(\|\rho(s)\|_{H^1} + \ell)ds\right] \\ &\leq C\left[C_r e^{-\eta t} + p_\ell(C_\ell)\int_{t_r}^t e^{-\eta(t-s)}\,ds\right], \qquad t_r \leq t < \infty.\end{aligned}$$

This then shows that there exists a universal constant C_ℓ and a time $t_r > 0$ such that

$$\|u(t)\|_{L^2} \leq C_\ell \qquad \text{for every } t \geq t_r.$$

We repeat the same argument using (4.10) and (4.16) with the aid of (6.3). Then the desired result is obtained.

Proposition 6.1 then shows that the set $\mathscr{B} = \{U_0 \in \tilde{K}_\ell; \|U_0\|_{\mathscr{D}(A)} \leq C_\ell\}$, where C_ℓ denotes the constant in (6.2), is an absorbing set of $S(t)$ which is obviously a compact set of H. Hence, by virtue of [21, Chap. I, Theorem 1.1], there exists a global attractor $\mathscr{A} \subset \tilde{K}_\ell$, $\mathscr{A}$ being a compact and connected subset of H.

We set $\mathscr{X} = \overline{\bigcup_{t \geq t_{\mathscr{B}}} S(t)\mathscr{B}}$, where $t_{\mathscr{B}}$ denotes a time such that $S(t)\mathscr{B} \subset \mathscr{B}$ for every $t \geq t_{\mathscr{B}}$. Then, $\mathscr{X}$ is a compact subset of H with $\mathscr{A} \subset \mathscr{X} \subset \mathscr{B}$, and is an absorbing and invariant set for $(\{S(t)\}_{t\geq 0}, \tilde{K}_\ell)$. In order to construct an exponential attractor for $S(t)$ it is now sufficient to apply the Theorem 2.1.

From (3.7), we have:

$$\begin{aligned}\|F(U) - F(V)\|_H &\leq p_l(\|A^{1/2}U\|_H + \|A^{1/2}V\|_H)\|A^{1/2}(U - V)\|_H \\ &\leq C_{\mathscr{X}}\|A^{1/2}(U - V)\|_H, \qquad U, V \in \mathscr{X}.\end{aligned}$$

Therefore, the condition (F) is fulfilled.

For $U_0, V_0 \in \mathscr{X}$, let $W(t) = S(t)U_0 - S(t)V_0$, $0 \leq t < \infty$. Obviously $W(t)$ is a solution to the problem

$$(6.4)\qquad \begin{cases} \dfrac{dW}{dt} + AW = F(S(t)U_0) - F(S(t)V_0), & 0 < t < \infty, \\ W(0) = U_0 - V_0. \end{cases}$$

Therefore,

$$\frac{1}{2}\frac{d}{dt}\|W\|_H^2 + \|A^{1/2}W\|_H^2 = (F(S(t)U_0) - F(S(t)V_0), W)_H \le C_{\mathscr{X}}\|A^{1/2}W\|_H\|W\|_H.$$

Hence, $\|W(t)\|_H \le e^{C_{\mathscr{X}}t}\|W_0\|_H$; this shows that $S(t)U_0$ is Lipschitz continuous in $U_0 \in \mathscr{X}$.

Let $U_0 \in \mathscr{X}$; since $\mathscr{X} \subset \mathscr{B}$, $\|AU_0\|_H \le C_\ell$; then, by Proposition 4.1, $\|AS(t)U_0\|_H \le C_{\mathscr{X}}$ for every $t \ge 0$. On the other hand, from (3.6), $\|F(S(t)U_0)\|_H \le C_{\mathscr{X}}$ for every $t \ge 0$. Therefore, we observe that

$$\|S(t)U_0 - S(s)U_0\|_H \le \int_s^t \left\|\frac{dU}{dt}(\tau)\right\|_H d\tau \le C_{\mathscr{X}}(t-s), \qquad 0 \le s \le t < \infty.$$

Thus the condition (G) is also fulfilled.

Hence, we have derived the main result of this paper.

Theorem 6.2. *The dynamical system* $(\{S(t)\}_{t\ge 0}, \mathscr{X})$ *determined by* (KS) *admits an exponential attractor* $\mathscr{M}$ *in* $K_\ell \subset H$.

7. Global Lyapunov function

Throughtout this section we assume that

$$(7.1)\qquad \chi(\rho) = k\rho, \qquad \rho > 0$$

with some constant $k > 0$. In this case a global Lyapunov functional will be constructed in the same manner as in Nagai, Senba and Yoshida [13] (cf. also Biler, Hebisch and Nadzieja [1], Gajewski and Zacharias [6]). Thanks to the functional, it can be shown that every ω-limit set $\omega(U_0)$ of the solution to (6.1) contains a stationary solution.

Let $U_0 = {}^t(u_0, \rho_0) \in H^1(I) \times H^1(I)$ be the initial data with $\inf_{x\in I} u_0(x) > 0$ and $\inf_{x\in I} \rho_0(x) > 0$. Let $S(t)U_0 = {}^t(u(t), \rho(t))$ be the global solution to (6.1). If we continue obtaining the existence results as in Section 3 in Sobolev spaces of higher exponents, it is possible to prove that

$$u \in \mathscr{C}^1((0,\infty); H^1(I)) \cap \mathscr{C}([0,\infty); H^1(I)) \cap \mathscr{C}((0,\infty); H_N^3(I)),$$

$$\rho \in \mathscr{C}^1((0,\infty); H_N^2(I)) \cap \mathscr{C}([0,\infty); H^1(I)) \cap \mathscr{C}((0,\infty); H_{N^2}^4(I)).$$

From (2.3), it follows that

$$u \in \mathscr{C}^1((0,\infty);\mathscr{C}(\bar{I})) \cap \mathscr{C}([0,\infty);\mathscr{C}(\bar{I})) \cap \mathscr{C}((0,\infty);\mathscr{C}^2_N(\bar{I})),$$

$$\rho \in \mathscr{C}^1((0,\infty);\mathscr{C}^1(\bar{I})) \cap \mathscr{C}([0,\infty);\mathscr{C}(\bar{I})) \cap \mathscr{C}((0,\infty);\mathscr{C}^3_{N^2}(\bar{I})),$$

where

$$\begin{cases} \mathscr{C}^2_N(\bar{I}) = \{u \in \mathscr{C}^2(\bar{I}); (du/dx)(\alpha) = (du/dx)(\beta) = 0\}, \\ \mathscr{C}^3_{N^2}(\bar{I}) = \{u \in \mathscr{C}^3(\bar{I}); (du/dx)(\alpha) = (du/dx)(\beta) = 0, \\ \qquad (d^3u/dx^3)(\alpha) = (d^3u/dx^3)(\beta) = 0\}. \end{cases}$$

Hence u, ρ is a classical solution. We can then apply the maximal principle of parabolic equations (cf. Protter and Weinberger [15]) to obtain that $\inf_{x\in I} u(t) > 0$ for every $t \geq 0$.

We are ready to introduce the Lyapunov functional. Multplying $a \log u - k\rho$ to the first equation of (KS) and integrating the product in x, we obtain that

$$\int_I \frac{\partial u}{\partial t}(a\log u - k\rho)dx = -\int_I u\left\{\frac{\partial}{\partial x}(a\log u - k\rho)\right\}^2 dx.$$

On the other hand, from the second equation of (KS),

$$\int_I \frac{\partial \rho}{\partial t}\left(-b\frac{\partial^2\rho}{\partial x^2} + d\rho - cu\right)dx = -\int_I \left(\frac{\partial\rho}{\partial t}\right)^2 dx.$$

Therefore,

$$\frac{d}{dt}\int_I \left\{ac(u\log u - u) + \frac{bk}{2}\left(\frac{\partial\rho}{\partial x}\right)^2 + \frac{dk}{2}\rho^2 - cku\rho\right\}dx \tag{7.2}$$

$$= -c\int_I u\left\{\frac{\partial}{\partial x}(a\log u - k\rho)\right\}^2 dx - k\int_I \left(\frac{\partial\rho}{\partial t}\right)^2 dx, \qquad 0 < t < \infty.$$

Hence, the functional

$$\Phi(U) = \int_I \left\{ac(u\log u - u) + \frac{bk}{2}\left(\frac{\partial\rho}{\partial x}\right)^2 + \frac{dk}{2}\rho^2 - cku\rho\right\}dx$$

is a global Lyapunov functional of the problem (6.1). But, since

$$\int_I u\rho\, dx \leq \varepsilon\|\rho\|^2_{H^1} + C_\varepsilon\|u\|^2_{L^1}$$

with an arbitrary $\varepsilon > 0$, there exists some universal constant C such that $\Phi(S(t)U_0) \geq -C$ for every $t > 0$.

Let $\omega(U_0) = \bigcap_{t\geq 0}\overline{\{S(\tau)U_0; \tau \geq t\}}$ denote the ω-limit set of the solution $S(t)U_0$. Then we verify the following result.

Theorem 7.1. *Let* $U_0 \in H^1(I) \times H^1(I)$ *with* $\inf_{x\in I} u_0(x) > 0$ *and* $\inf_{x\in I} \rho_0(x) > 0$. *In the case of* (7.1), *the ω-limit set of $S(t)U_0$ contains at least one stationary solution. In particular, $S(t)U_0$ is never a periodic solution.*

Proof. As verified above, $\Phi(S(t)U_0)$ is a decreasing function bounded from below. Therefore, there must exist some increasing sequence $\{t_n\}$, $t_n \to \infty$, for which $(d/dt)\Phi(S(t)U_0)|_{t=t_n}$ is convergent to 0. In other words, as $n \to \infty$,

$$\sqrt{u(t_n)}\left\{\frac{\partial}{\partial x}(a \log u(t_n) - k\rho(t_n))\right\} \to 0 \qquad \text{in } L^2(I),$$

$$b\frac{\partial^2 \rho}{\partial x^2}(t_n) + cu(t_n) - d\rho(t_n) \to 0 \qquad \text{in } L^2(I),$$

where ${}^t(u(t_n), \rho(t_n)) = S(t_n)U_0$.

On the other hand, from Proposition 6.1, we can assume that $S(t_n)U_0$ converges to a limit $\bar{U} = {}^t(\bar{u}, \bar{\rho})$ strongly in $H^1(I) \times H^2(I)$ and weakly in $H^2(I) \times H^3(I)$. In view of the formula

$$\begin{aligned} \text{(7.3)} \qquad & a\frac{\partial^2 u}{\partial x^2}(t_n) - k\frac{\partial}{\partial x}\left\{u(t_n)\frac{\partial \rho}{\partial x}(t_n)\right\} \\ &= \frac{\partial}{\partial x}\sqrt{u(t_n)}\left\{\sqrt{u(t_n)}\frac{\partial}{\partial x}(a \log u(t_n) - k\rho(t_n))\right\}, \end{aligned}$$

we conclude that $\bar{U}$ is a stationary solution to (6.1). By definition, $\bar{U} \in \omega(U_0)$.

The second assertion then follows from the fact that $\omega(U_0) = \{S(t)U_0; 0 \leq t \leq p\}$ if $S(t)U_0$ is a periodic solution, where $p > 0$ denotes its period.

Remark 7.2. If $S(t)U_0 = {}^t(u(t), \rho(t))$ satisfies a uniform estimate $\inf_{x\in I} u(t) \geq \delta$ for every $t \geq 0$ with some constant $\delta > 0$, then we shall verify a stronger result. In this case it is seen from (7.2) that

$$\left|\frac{d^2}{dt^2}\Phi(S(t)U_0)\right| \leq p_\delta(\|u(t)\|_{H^3} + \|\rho(t)\|_{H^4}), \qquad 0 < t < \infty$$

with some continuous increasing function $p_\delta(\cdot)$. On the other hand, if we continue estimating higher norms of the solution $S(t)U_0$ as in the proof of Proposition 4.1, it is possible to prove that $\|u(t)\|_{H^3} + \|\rho(t)\|_{H^4}$ remains bounded as $t \to \infty$. Hence, $(d^2/dt^2)\Phi(S(t)U_0)$ is a bounded function. As a consequence, $\lim_{t\to\infty}(d/dt)\Phi(S(t)U_0) = 0$. Let a sequence $S(t_n)U_0$ converge to

$\overline{U}$ in $L^2(I) \times H^1(I)$. Then, by the same argument as above, we conclude that $\overline{U}$ must be a stationary solution to (6.1). This means that the ω-limit set $\omega(U_0)$ consists of stationary solutions only.

Furthermore, if the set of stationary solutions of (6.1) is observed to be discrete, then every ω-limit set must be a singleton.

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