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# Stability for a Parabolic Variational Inequality Associated with Total Variation Functional

By

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## Introduction

In this paper we study an evolution equation of the form

$$v'(t) + \kappa \partial V(v(t)) \ni v(t) + \theta_0 \qquad \text{in } L^2(0, L), \quad t \geq 0, \tag{0.1}$$

where $0 < L < +\infty$, $\kappa$ is a (small) positive number, $\theta_0$ is a given constant, and $\partial V$ is the subdifferential of the proper l.s.c. and convex function $V$ on $L^2(0, L)$, which is the total variation functional with constraints $-1$ and $1$.

This problem is motivated by a one dimensional phase field model in a mesoscopic length scale, which was proposed and studied by Visintin (cf. [8]), of the form

$$\begin{cases} (\theta + v)_t - \theta_{xx} = 0 \ \text{ in } Q := (0, +\infty) \times (0, L), \\ v_t(t) + \kappa \partial V(v(t)) \ni v(t) + \theta(t) \qquad \text{in } L^2(0, L), \quad t > 0, \end{cases}$$

subject to suitable initial and boundary conditions. In such a context $\theta$ is the (relative) temperature field and $v$ is the order parameter which indicates the physical situation of the material.

Supposing $p \geq 2$ and

$$V(z) := \begin{cases} \dfrac{1}{p} \displaystyle\int_0^L |z_x|^p dx, & \text{if } z \in W^{1,p}(0, L), |z| \leq 1 \text{ on } (0, L), \\ +\infty & \text{otherwise}, \end{cases}$$

equation (0.1) is just of the form

$$\begin{cases} v_t - \kappa(|v_x|^{p-2} v_x)_x + \xi - v = \theta_0 \qquad \text{in } Q, \\ \xi \in \partial I_{[-1,1]}(v) \qquad \text{in } Q, \\ v_x(t, 0) = v_x(t, L) = 0 \qquad \text{for } t > 0, \\ v(0, \cdot) = v_0 \qquad \text{in } L^2(0, L), \end{cases}$$

which was studied by Chen and Elliott (cf. [1]), and Ito (cf. [3]).

The equilibrium equation for (0.1) is of the form

$$\kappa\partial V(w) \ni w+\theta_0 \quad \text{in } L^2(0,L), \tag{0.2}$$

and it was shown in [7] that any solution of (0.2) was piecewise constant with a finite number of discontinuities in $[0,L]$ (cf. Fig. 1). (0.2) is the Euler-Lagrange inclusion of the functional (free energy)

$$F_{\theta_0}(z) := \kappa V(z) - \frac{1}{2}\int_0^L |z+\theta_0|^2 dx, \qquad z \in L^2(0,L),$$

and it was shown in [7] that any local minimizer of $F_{\theta_0}$ takes only two values 1 and $-1$ in $[0,L]$ except for a finite number of discontinuous points (cf. Fig. 2).

In this paper, we are interested in the dynamics of the solid region $\{w=-1\}$, liquid region $\{w=1\}$ and mushy region $\{-1<w<1\}$, especially their stability. To this end we introduce a concept of stability for any pair $(J,c)$, $J$ being a subinterval of $[0,L]$ and $c$ being a constant in $[-1,1]$. Within this concept of stability we discuss the following items (a), (b) and (c) by making use of the structural result of stationary solutions.

(a) the stability of $(J,c)$ for a subinterval $J \subset [0,L]$ and a constant $c \in [-1,1]$;
(b) the stability of solutions of (0.2);
(c) the asymptotic convergence of solutions of (0.1) as $t \to +\infty$, when initial data are sufficiently close to stable stationary solutions.

## 1. Preliminaries

Throughout this paper, $L$ is a positive number, $|\cdot|_{L^p(0,L)}$ denotes the norm of $L^p(0,L)$ for any $1 \le p \le +\infty$, $(\cdot,\cdot)$ denotes the usual inner product in $L^2(0,L)$, and for a proper l.s.c. and convex function $\varphi$ on $L^2(0,L)$, we denote by $D(\varphi)$, $\partial\varphi$ and $D(\partial\varphi)$ the effective domain, the subdifferential of $\varphi$ in $L^2(0,L)$ and its domain, respectively.

For any $f_i \in L^1(0,L)$ $(i=1,2)$, we define

$$(f_1 \wedge f_2)(x) := \min\{f_1(x), f_2(x)\} \quad \text{and} \quad (f_1 \vee f_2)(x) := \max\{f_1(x), f_2(x)\},$$
$$\text{a.e. } x \in (0,L).$$

Especially, $f^+ := f \vee 0$ for any $f \in L^1(0,L)$. Also, given an interval $J \subset R$ we denote by $|J|$ the linear measure of $J$.

Let $V_0$ be the total variation functional defined on $L^1(0,L)$ by

$$V_0(z) := \sup\left\{ \int_0^L z\psi_x\, dx \;\middle|\; \begin{array}{l} \psi \in C^1[0,L] \text{ with compact support in } (0,L) \\ \text{and } |\psi| \le 1 \text{ on } [0,L] \end{array} \right\}$$

By [2, Chapter 5], $V_0$ is proper l.s.c. and convex on $L^1(0,L)$. We put

$$BV[0,L] := \{z \in L^1(0,L) \mid V_0(z) < +\infty\}.$$

Now, consider a functional $V : L^2(0,L) \to [0,+\infty]$ such that

$$V(z) := \begin{cases} V_0(z), & \text{if } z \in L^2(0,L) \text{ and } |z| \le 1 \text{ a.e. on } [0,L], \\ +\infty, & \text{otherwise.} \end{cases}$$

Clearly, $V$ is proper l.s.c. and convex on $L^2(0,L)$. Also, we observe that (cf. [2, Chapter 5]) for each $v \in D(V)$ there is a function $\tilde{v}$, a fine expression of $v$, such that $v = \tilde{v}$ a.e. on $(0,L)$ and $V_0(v) = V_1(\tilde{v})$, where $V_1(\tilde{v})$ is the total variation of $\tilde{v}$ in the usual sense, namely

$$V_1(\tilde{v}) := \sup_{\Delta \in \mathscr{D}} \sum_{k=1}^{n_\Delta} |\tilde{v}(x_k) - \tilde{v}(x_{k-1})|,$$

for the family $\mathscr{D}$ of all partitions $\Delta := \{0 = x_0 < x_1 < \cdots < x_{n_\Delta} = L\}$ $(n_\Delta \in N)$. Moreover, we see that $\tilde{v}$ is continuous at $x = 0, L$, and

$$(\tilde{v}(x+) - \tilde{v}(x))(\tilde{v}(x-) - \tilde{v}(x)) \le 0 \qquad \text{for all } x \in (0,L),$$

where $\tilde{v}(x+)$ and $\tilde{v}(x-)$ are respectively the right and left limits of $\tilde{v}$ at $x$. Such a function $\tilde{v}$ is determined uniquely by $v$, except for at most countable number of points in $[0,L]$. In what follows, for functions in $D(V)$, we use their fine expressions.

Given a (small) positive number $\kappa$ and a function $\theta \in L^2_{loc}([0,+\infty); L^2(0,L))$ such that

$$S_\theta := \sup_{t \ge 0} |\theta|_{L^2(t,t+1;L^2(0,L))} < +\infty,$$

we consider an evolution equation, denoted by $(P)_\theta$, of the form

$$(P)_\theta \qquad v'(t) + \kappa \partial V(v(t)) \ni v(t) + \theta(t) \text{ in } L^2(0,L), \qquad t \ge 0,$$

where $v' = \dfrac{dv}{dt}$ in $L^2(0,L)$.

A function $v : [0,+\infty) \to L^2(0,L)$ is called a solution of $(P)_\theta$, if $v \in C([0,+\infty);\ L^2(0,L)) \cap W^{1,2}_{loc}((0,+\infty); L^2(0,L))$, $V(v) \in L^1_{loc}([0,+\infty))$ and $\kappa \partial V(v(t)) \ni v(t) + \theta(t) - v'(t)$ for a.e. $t \in (0,+\infty)$.

According to the well-known results (cf. [4]), for any $v_0 \in \overline{D(V)}$, the Cauchy problem for $(P)_\theta$ with initial condition $v(0) = v_0$ has a unique solution

$v \in C([0,+\infty); L^2(0,L))$, and there is a positive number $K(v_0,\theta)$, depending only on $|v_0|_{L^2(0,L)}$ and $S_\theta$, such that

$$\sup_{t\geq 0}|v(t)|^2_{L^2(0,L)} + \sup_{t\geq 0}\int_t^{t+1} V(v(\tau))d\tau$$
$$+ \sup_{t\geq 1}\int_t^{t+1} |v_t(\tau)|^2_{L^2(0,L)}d\tau + \sup_{t\geq 1} V(v(t)) \leq K(v_0,\theta).$$

We now prove the so-called "$T$-monotonicity" of $\partial V$.

**Proposition 1.1.**

(1.1) $(z_1^* - z_2^*, (z_1 - z_2)^+) \geq 0$ *for all* $z_i \in D(\partial V)$ *and* $z_i^* \in \partial V(z_i)$, $\quad i = 1,2$.

*Proof.* Let $z_i \in D(\partial V)$, $i = 1,2$. Then, by [2, Chapter 5] there are two sequences $\{z_i^{(j)}\} \subset C^\infty[0,L] \cap D(V)$ $(i = 1,2)$ such that

$$z_i^{(j)} \to z_i \text{ in } L^2(0,L) \text{ and } V(z_i^{(j)}) \to V(z_i) \text{ as } j \to +\infty, \qquad i = 1,2.$$

We observe that

$$V(z_1^{(j)} \wedge z_2^{(j)}) + V(z_1^{(j)} \vee z_2^{(j)}) = \int_0^L |(z_1^{(j)} \wedge z_2^{(j)})_x(x)|dx + \int_0^L |(z_1^{(j)} \vee z_2^{(j)})_x(x)|dx$$
$$= \int_0^L |(z_1^{(j)})_x(x)|dx + \int_0^L |(z_2^{(j)})_x(x)|dx$$
$$= V(z_1^{(j)}) + V(z_2^{(j)}) \qquad \text{for all } j \in N.$$

Since $V$ is l.s.c. in $L^2(0,L)$, letting $j \to +\infty$ yields that

(1.2) $$V(z_1 \wedge z_2) + V(z_1 \vee z_2) \leq V(z_1) + V(z_2).$$

By a result in [6], (1.2) is equivalent to (1.1). ■

**Corollary 1.1.** *For* $0 \leq s < +\infty$, $0 < T \leq +\infty$, *let*

$$J_{s,T} := \begin{cases} [s, s+T], & \text{if } T < +\infty, \\ [s,+\infty), & \text{if } T = +\infty, \end{cases}$$

*and* $\theta_i \in L^2_{loc}(J_{s,T}; L^2(0,L))$ $(i = 1,2)$ *be given functions such that*

$$\theta_1(t,x) \leq \theta_2(t,x) \qquad \text{for a.e. } (t,x) \in J_{s,T} \times (0,L).$$

*Let* $v_i \in C(J_{s,T}; L^2(0,L))$ *be the solutions of* $(P)_{\theta_i}$ $(i = 1,2)$ *such that*

$$v_1(s,x) \leq v_2(s,x) \qquad \text{for a.e. } x \in (0,L).$$

*Then*

(1.3) $\qquad v_1(t,x) \le v_2(t,x) \qquad$ *for any* $t \in J_{s,T}$ *and a.e.* $x \in [0,L]$.

*Proof.* Taking the difference of the both sides of $(P)_{\theta_2}$ and $(P)_{\theta_1}$ and multiplying it by $(v_1(t) - v_2(t))^+$, we see that

$$\frac{1}{2}\frac{d}{dt}|(v_1(t) - v_2(t))^+|^2_{L^2(0,L)} + (v_1^*(t) - v_2^*(t), (v_1(t) - v_2(t))^+)$$
$$= |(v_1(t) - v_2(t))^+|^2_{L^2(0,L)} + (\theta_1(t) - \theta_2(t), (v_1(t) - v_2(t))^+) \text{ for a.e. } t \in J_{s,T}.$$

It follows from our assumption and Proposition 1.1 that

$$\frac{d}{dt}|(v_1(t) - v_2(t))^+|^2_{L^2(0,L)} \le 2|(v_1(t) - v_2(t))^+|^2_{L^2(0,L)} \qquad \text{for a.e. } t \in J_{s,T}.$$

Applying Gronwall's lemma, we obtain that

$$|(v_1(t) - v_2(t))^+|^2_{L^2(0,L)} \le e^{2t}|(v_1(s) - v_2(s))^+|^2_{L^2(0,L)} = 0 \qquad \text{for any } t \in J_{s,T},$$

which implies (1.3). ■

For any given constant $\theta_0$, we consider the steady state problem

$$(P_\infty)_{\theta_0} \qquad \kappa\partial V(w) \ni w + \theta_0 \qquad \text{in } L^2(0,L),$$

which is the Euler-Lagrange inclusion of the energy functional $F_{\theta_0} : L^2(0,L) \to (-\infty, +\infty]$ given by

$$F_{\theta_0}(z) := \begin{cases} \kappa V(z) - \dfrac{1}{2}\displaystyle\int_0^L |z + \theta_0|^2 dx, & \text{if } z \in D(V), \\ +\infty & \text{otherwise.} \end{cases}$$

Next, we recall some known results of the asymptotic behavior of solutions of $(P)_\theta$ with $\theta = \theta_0$ (constant).

**Definition 1.1** ($\omega$-limit set). For any solution $v$ of $(P)_{\theta_0}$, the $\omega$-limit set $\omega(v)$ is defined by putting

$$\omega(v) := \left\{ v_\infty \in L^2(0,L) \;\middle|\; \begin{array}{l} \text{there is a sequence } \{t_i\} \text{ such that } t_i \nearrow +\infty \\ \text{and } v(t_i) \to v_\infty \text{ in } L^2(0,L), \text{ as } i \to +\infty \end{array} \right\}.$$

Any element $v_\infty$ of $\omega(v)$ is called an $\omega$-limit point of $v$.

**Proposition 1.2.** ([5, Section 9]). *For any solution $v$ of $(P)_{\theta_0}$, we have*
(i) $\omega(v)$ *is nonempty, connected and compact in* $L^2(0,L)$,
(ii) *any* $v_\infty \in \omega(v)$ *is a solution of* $(P_\infty)_{\theta_0}$,
(iii) *for any* $v_\infty, w_\infty \in \omega(v)$,

$$F_{\theta_0}(v_\infty) = F_{\theta_0}(w_\infty) \qquad \textit{and} \qquad F_{\theta_0}(v(t)) \searrow F_{\theta_0}(v_\infty) \qquad \text{as } t \to +\infty.$$

## 2. Steady state problem $(P_\infty)_{\theta_0}$

Throughout the rest of this paper, let $\theta_0$ be a constant. In the case of $|\theta_0| \geq 1$, the structure of the solution set is quite clear; in fact, as was remarked in [7],
(i) if $\theta_0 > 1$ (resp. $\theta_0 < -1$), then the steady state problem $(P_\infty)_{\theta_0}$ has a unique constant solution $w \equiv 1$ (resp. $w \equiv -1$);
(ii) if $|\theta_0| = 1$, then $(P_\infty)_{\theta_0}$ has two constant solutions $w_1 \equiv 1$ and $w_2 \equiv -1$, and has no any other solutions.

Therefore, in the sequel, assume that $|\theta_0| < 1$.

We recall some results on $(P_\infty)_{\theta_0}$ obtained in [7].

**Definition 2.1.** For each $\theta_0 \in (-1,1)$ and $n \in N \cup \{0\}$, we define a subclass $S_n(\theta_0)$ of $BV[0,L]$ as follows.
(I) $S_0(\theta_0) := \{-1, -\theta_0, 1\}$.
(II) (cf. Fig. 1) For $n \in N$, $z \in S_n(\theta_0)$ if and only if

$$z(x) = \begin{cases} c_k & \text{if } x \in J_k,\ k = 0,1,\dots,n, \\ -\theta_0 & \text{if } x \in [x_k^L, x_k^R] \text{ with } x_k^L \leq x_k^R,\ k = 1,\dots,n, \end{cases}$$

where $\{x_k^L, x_k^R \,|\, k = 1,\dots,n\}$ is a partition of $[0,L]$ and $c_k \in [-1,1]\backslash\{-\theta_0\}$, $k = 1,2,\dots,n$, such that the following (i), (ii) and (iii) are satisfied:

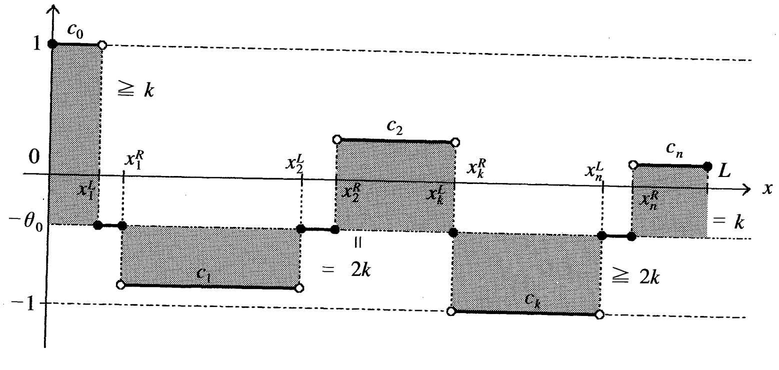


**Fig. 1**

(i) $0 < x_1^L \le x_1^R < \cdots < x_k^L \le x_k^R < \cdots < x_n^L \le x_n^R < L$,

$$J_k := \begin{cases} [0, x_1^L) & \text{for } k = 0, \\ (x_k^R, x_{k+1}^L) & \text{for } k = 1, \dots, n-1, \\ (x_n^R, L] & \text{for } k = n. \end{cases}$$

(ii) $(c_{k-1} + \theta_0)(c_k + \theta_0) < 0,\ k = 1, \dots, n.$

(iii) For $k = 0$ or $n$,

$$|c_k + \theta_0|\,|J_k| \ge \kappa$$

and if $c_k + \theta_0 > 0$ (resp. $c_k + \theta_0 < 0$), then

$$c_k = 1 \text{ or } \frac{\kappa}{|J_k|} - \theta_0 \left(\text{resp. } c_k = -1 \text{ or } -\frac{\kappa}{|J_k|} - \theta_0\right).$$

For $n \ge 2$ and $k = 1, \dots, n-1$,

$$|c_k + \theta_0|\,|J_k| \ge 2\kappa$$

and if $c_k + \theta_0 > 0$ (resp. $c_k + \theta_0 < 0$), then

$$c_k = 1 \text{ or } \frac{2\kappa}{|J_k|} - \theta_0 \left(\text{resp. } c_k = -1 \text{ or } -\frac{2\kappa}{|J_k|} - \theta_0\right).$$

Now, we define a class $S(\theta_0)$ in $BV[0, L]$ by

$$S(\theta_0) := \sum_{n=0}^{N_{\theta_0}} S_n(\theta_0),$$

where $N_{\theta_0} := \sup\{n \in N \mid S_n(\theta_0) \neq \varnothing\}$; notice that

$$1 \le N_{\theta_0} \le \frac{L}{2\kappa}(1 + |\theta_0|) < +\infty.$$

Throughout this paper, we assume that $\kappa$ is sufficiently small so that

$$0 < \kappa < \frac{L}{2}(1 + |\theta_0|),$$

which implies $N_{\theta_0} > 0$.

**Theorem 2.1** ([7]). *Assume that $|\theta_0| < 1$. Then a function $w$ is a solution of $(P_\infty)_{\theta_0}$ if and only if there is a function $w^\circ \in S(\theta_0)$ such that*

(2.1) *$w = w^\circ$ except for discontinuous points of $w^\circ$ in $[0, L]$.*

Next, we define a subclass $M_{loc}(\theta_0)$ of $S(\theta_0)$ by putting

$$M_{loc}(\theta_0) := \begin{cases} \{1,-1\}, & \text{if } 0<\theta_0<1, \\ \left\{ z \in S(0) \;\middle|\; \begin{array}{l} |z| = 1 \text{ a.e. on } [0,L], \text{ and} \\ \text{if } z \in S_n(0)\ (1 \le n \le N_{\theta_0}) \text{ with the} \\ \quad \text{same notation } c_k,\ x_k^L,\ x_k^R \text{ and } J_k \\ \quad \text{as in (II) of Definition 2.1, then} \\ x_k^L = x_k^R\ (k=1,\dots,n), \text{ and} \\ |J_0|, |J_n| > \kappa,\ |J_k| > 2\kappa\ (k=1,\dots,n-1). \end{array} \right\}, & \text{if } \theta_0 = 0, \\ \{-1,1\}, & \text{if } -1<\theta_0<0, \end{cases}$$

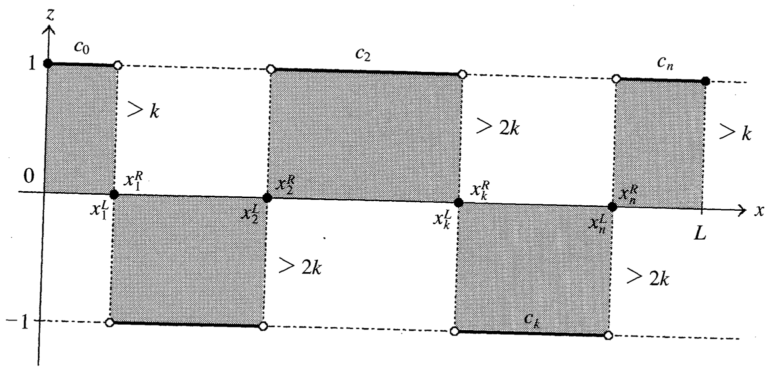


**Fig. 2**

**Theorem 2.2** (cf. [7]). *Assume that* $|\theta_0| < 1$. *Let* $w$ *be a solution of* $(P_\infty)_{\theta_0}$. *Then,* $w$ *is a local minimizer of* $F_{\theta_0}$ *if and only if* $w^\circ \in M_{loc}(\theta_0)$, *where* $w^\circ$ *is the same function as in Theorem* 2.1.

## 3. Statements of main results

In the case of $\theta_0 > 1$ (resp. $\theta_0 < -1$), the constant solution $w \equiv 1$ (resp. $w \equiv -1$) is stable and any solution $v$ of the evolution equation

$$(P)_{\theta_0} \qquad v'(t) + \kappa \partial V(v(t)) \ni v(t) + \theta_0 \qquad \text{in } L^2(0,L),\ t>0,$$

converges to $w \equiv 1$ (resp. $w \equiv -1$) uniformly on $[0,L]$ as $t \to +\infty$. As is easily checked, in the case of $\theta_0 = 1$ (resp. $\theta_0 = -1$), $w_1 \equiv 1$ (resp. $w_2 \equiv -1$) is stable, but $w_2$ (resp. $w_1$) is unstable.

Therefore, in the sequel, assuming again that $|\theta_0| < 1$, we investigate the stability for evolution problem $(P)_{\theta_0}$.

First, we introduce a concept of stability for any pair $(J, c)$ of a relatively open interval $J$ in $[0, L]$ and a constant $c \in [-1, 1]$.

**Definition 3.1.** Let $J$ be a relatively open interval in $[0, L]$ and $c$ be a constant with $-1 \leq c \leq 1$.

(i) (Stability) A pair $(J, c)$ is called stable if for any positive number $\varepsilon$, there exist a positive number $\delta_\varepsilon$ and a time $t_\varepsilon \geq 0$ such that

$$|v(t) - c|_{L^\infty(J)} < \varepsilon$$

for any $t \geq t_\varepsilon$ and any solution $v$ of $(P)_{\theta_0}$ with $|v(0) - c|_{L^\infty(J)} < \delta_\varepsilon$.

(ii) (Instability) A pair $(J, c)$ is called unstable if $(J, c)$ is not stable. More precisely, there exists a positive number $\varepsilon_0$ such that for any positive number $\delta$ and time $T \geq 0$ we can find a time $t_0 \geq T$ and a solution $v$ of $(P)_{\theta_0}$ satisfying

$$|v(0) - c|_{L^\infty(J)} < \delta \qquad \text{and} \qquad |v(t_0) - c|_{L^\infty(J)} \geq \varepsilon_0.$$

Our first theorem is concerned with a characterization of the stability of a pair $(J, c)$.

**Theorem 3.1.** *Assume that $|\theta_0| < 1$. Let $c$ be a constant with $-1 \leq c \leq 1$ and $J$ be a relatively open interval such that*

(3.1) $J = [0, L]$ *or* $[0, b)$ *or* $(a, L]$ *or* $(a, b)$ *with* $\kappa \leq a < L$ *and* $0 < b \leq L - \kappa$.

*Then the pair $(J, c)$ is stable if and only if $|c| = 1$ and*

$$|c + \theta_0|\,|J| > \begin{cases} \kappa, & \text{if } \bar{J} \cap \{0, L\} \neq \varnothing, \\ 2\kappa, & \text{if } \bar{J} \cap \{0, L\} = \varnothing. \end{cases} \tag{3.2}$$

**Corollary 3.1.** *Assume that $|\theta_0| < 1$. Let $v$ be a solution of $(P)_{\theta_0}$ and $v_\infty$ be an $\omega$-limit point of $v$. Assume that $v_\infty$ is constant, say $c$, on a relatively open interval $J$ in $[0, L]$ such that* (3.2) *holds and $(J, c)$ is a stable pair. Then, for each compact subset $J_1$ of $J$ there exists a finite time $t_1 \geq 0$ such that*

$$v(t, \cdot) = c \qquad \text{on } J_1 \text{ for all } t \geq t_1.$$

Next, based on the result of Theorem 3.1, we shall discuss the stability of solutions of the steady state problem

$$(P_\infty)_{\theta_0} \qquad \kappa \partial V(w) \ni w + \theta_0 \qquad \text{in } L^2(0, L).$$

For each solution $w$ of $(P_\infty)_{\theta_0}$ and each (small) positive number $\mu$, we define a relatively open set $J_\mu(w)$ by putting

(3.3) $J_\mu(w) := \{x \in [0, L] \mid |x - y| > \mu$ for any discontinuous point $y$ of $w\}$.

**Definition 3.2.** Let $w$ be a solution of $(P_\infty)_{\theta_0}$ and let $S_n(\theta_0)$ $(n \in N \cup \{0\})$, $x_k^L$, $x_k^R$, $J_k$, $c_k$ and $w^\circ$ be the same as in Definition 2.1 and Theorem 2.1 corresponding to $w$. Then:

(i) (Stability) A solution $w$ of $(P_\infty)_{\theta_0}$ is called stable if the following statement holds:
- if $w^\circ \in S_0(\theta_0)$, then $w \equiv 1$ or $w \equiv -1$ on $[0, L]$;
- if $w^\circ \in S_n(\theta_0)$ with $n \in N$, then $x_k^L = x_k^R$ for $k = 1, \ldots, n$, and any pair $(J_k, c_k)$ is stable for $k = 0, 1, \ldots, n$.

(ii) (Instability) A solution $w$ of $(P_\infty)_{\theta_0}$ is called unstable if the following statement holds:
- if $w^\circ \in S_0(\theta_0)$, then $w \equiv -\theta_0$ on $[0, L]$;
- if $w^\circ \in S_n(\theta_0)$ with $n \in N$, then there is a number $k_0 \in \{0, 1, \ldots, n\}$ such that $x_{k_0}^L < x_{k_0}^R$ or $(J_{k_0}, c_{k_0})$ is unstable.

Our second result is concerned with the stability of stationary solutions.

**Theorem 3.2.** *Assume that* $|\theta_0| < 1$. *Then a solution* $w$ *of* $(P_\infty)_{\theta_0}$ *is stable if and only if there exists a positive number* $\mu_0$ *such that for any number* $\mu \in (0, \mu_0]$ *and any positive number* $\varepsilon$ *we can find a positive number* $\delta_{\mu,\varepsilon}$ *and a time* $t_{\mu,\varepsilon} \geq 0$ *satisfying*

$$|v(t) - w|_{L^\infty(J_\mu(w))} < \varepsilon$$

*for any* $t \geq t_{\mu,\varepsilon}$ *and any solution* $v$ *of* $(P)_{\theta_0}$ *with* $|v(0) - w|_{L^\infty(J_\mu(w))} < \delta_{\mu,\varepsilon}$.

Our third result is concerned with the asymptotic convergence of solutions of $(P)_{\theta_0}$.

**Theorem 3.3.** *Assume* $|\theta_0| < 1$. *Let* $w$ *be a stable solution of* $(P_\infty)_{\theta_0}$. *Then there are (small) positive constants* $\mu$ *and* $\delta$ *such that if* $v$ *is a solution of* $(P)_{\theta_0}$ *and* $|v(0) - w|_{L^\infty(J_\mu(w))} < \delta$, *then* $v(t)$ *converges to a solution* $v_\infty$ *of* $(P_\infty)_{\theta_0}$ *in* $L^2(0, L)$ *as* $t \to +\infty$. *Moreover, for such a solution* $v$ *of* $(P)_{\theta_0}$ *and the limit* $v_\infty = \lim_{t \to +\infty} v(t)$ *it holds that* $v(t) \to v_\infty$ *as* $t \to +\infty$ *uniformly on each compact subset of the complement of the set consisting of all discontinuous points of* $v_\infty$ *in* $[0, L]$.

We have still an interesting question "Does any solution of $(P)_{\theta_0}$ converge to a solution of $(P_\infty)_{\theta_0}$?". The above Theorem 3.3 gives a partial answer to this question, namely a solution of $(P)_{\theta_0}$ asymptotically converges in $L^2(0, L)$ as $t \to +\infty$, if the initial value is sufficiently close to a stable solution of $(P_\infty)_{\theta_0}$.

We shall prove our theorems in sections 4 and 5.

## 4. Proof of Theorem 3.1.

In this section, we give the proof of Theorem 3.1.

*Proof of Theorem* 3.1. We first show the sufficiency. Assume that $|c| = 1$ and (3.2) holds. We show it below only in case $c = 1$ and $\bar{J} \cap \{0, L\} = \emptyset$, since the other cases are similarly proved.

In such a case, consider a positive number

(4.1) $$\delta_0 := \frac{1}{2|J|}(|1 + \theta_0||J| - 2\kappa),$$

and define a function $v_J : [0, 1] \to L^2(0, L)$ (see Fig. 3) by

$$v_J(t, x) := \begin{cases} 1 + \delta_0(t - 1), & \text{if } x \in J, \\ -1, & \text{otherwise,} \end{cases}$$

for any $t \in [0, 1]$.

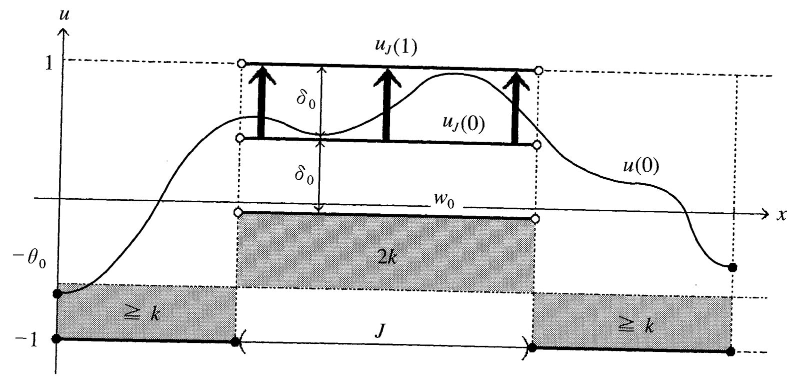


**Fig. 3**

Then, the function $v_J$ satisfies the evolution inclusion

(4.2) $v_J'(t) + \kappa\partial V(v_J(t)) \ni v_J(t) + \theta_0 - \delta_0 t \chi_J$ in $L^2(0, L)$, for any $t \in [0, L]$,

where $\chi_J$ is the characteristic function of $J$. In fact, the function

$$w_0(x) := \begin{cases} 1 - 2\delta_0 & \text{for } x \in J, \\ -1 & \text{otherwise} \end{cases}$$

is a solution of $(P_\infty)_{\theta_0}$ (see Fig. 3), so that

(4.3) $$\kappa V(w_0) - \int_0^L (w_0 + \theta_0) w_0 \, dx \leq \kappa V(z) - \int_0^L (w_0 + \theta_0) z \, dx$$

for any $z \in D(V)$.

Since $v_J(t) = w_0 + \delta_0(t+1)\chi_J$, it follows from (4.1) and (4.3) that

$$\kappa V(v_J(t)) - \int_0^L (w_0 + \theta_0) v_J(t) dx$$
$$= (\kappa V(w_0) + 2\kappa\delta_0(t+1)) - \left(\int_0^L (w_0 + \theta_0) w_0 \, dx + \delta_0(t+1)((1+\theta_0) - 2\delta_0)|J|\right)$$
$$= \kappa V(w_0) - \int_0^L (w_0 + \theta_0) w_0 \, dx \le \kappa V(z) - \int_0^L (w_0 + \theta_0) z \, dx$$

for any $t \in [0,1]$ and any $z \in D(V)$. This implies that

$$\kappa\partial V(v_J(t)) \ni w_0 + \theta_0 \qquad \text{in } L^2(0,L) \text{ for any } t \in [0,1].$$

Since $v_J'(t) = \delta_0 \chi_J$ and $v_J(t) = w_0 + \delta_0(t+1)\chi_J$, we obtain (4.2).

Now, let $v$ be any solution of $(P)_{\theta_0}$ so that

$$|v(0) - 1|_{L^\infty(J)} < \delta_0.$$

Since

$$v(0) \ge v_J(0) \qquad \text{a.e. on } [0,L],$$

by Corollary 1.1 with $s = 0$, $T = 1$, $\theta_1 = \theta_0 - \delta_0 t \chi_J$, $\theta_2 = \theta_0$, $v_1 = v_J$ and $v_2 = v$, we have

$$v_J(t,x) \le v(t,x) \qquad \text{for any } t \in [0,1] \text{ and a.e. } x \in [0,L],$$

in particular,

$$(4.4) \qquad v_J(1,x) \le v(1,x) \qquad \text{for a.e. } x \in [0,L]$$

Next, note from assumption (3.1) that $v_J(1)$ is a stationary solution of $(P)_{\theta_0}$. By Corollary 1.1 with $s = 1$, $T = +\infty$, $\theta_1 = \theta_2 = \theta_0$, $v_1 = v_J$ and $v_2 = v$, (4.4) implies that

$$v_J(1,x) \le v(t,x) \qquad \text{for all } t \ge 1 \text{ and a.e. } x \in [0,L].$$

Hence,

$$v(t,x) = 1 \qquad \text{for any } t \ge 1 \text{ and a.e. } x \in J.$$

Thus $(J,c)$ is stable. In fact, for any positive number $\varepsilon$, we can take $\delta_0$ and 1 as $\delta_\varepsilon$ and $t_\varepsilon$, respectively, in (i) of Definition 3.1; namely

$$|v(t) - 1|_{L^\infty(J)} = 0 < \varepsilon$$

for any $t \ge 1$ and any solution $v$ of $(P)_{\theta_0}$ with $|v(0) - 1|_{L^\infty(J)} < \delta_0$.

In the rest of our proof, we show the necessity. Under the assumption that $|c| < 1$ or a pair $(J, c)$ satisfying

$$(4.5) \qquad |c + \theta_0|\,|J| \le \begin{cases} \kappa, & \text{if } \bar{J} \cap \{0, L\} \neq \emptyset, \\ 2\kappa, & \text{if } \bar{J} \cap \{0, L\} = \emptyset, \end{cases}$$

it is enough to prove that $(J, c)$ is unstable. This is done in the following two cases.

(Case 1) The case of $|c| < 1$.

We prove the unstability of $(J, c)$ only in the case of $-\theta_0 \le c < 1$, since the other cases are similarly proved. In this case, for any (small) positive number $\delta$, put

$$T_\delta := \frac{1 - c}{\delta} - 1,$$

and define a function $v_\delta : [0, T_\delta] \to L^2(0, L)$ (see Fig. 4) by

$$v_\delta(t) := c + \delta(t + 1) \qquad \text{on } [0, L] \text{ for any } t \in [0, T_\delta].$$

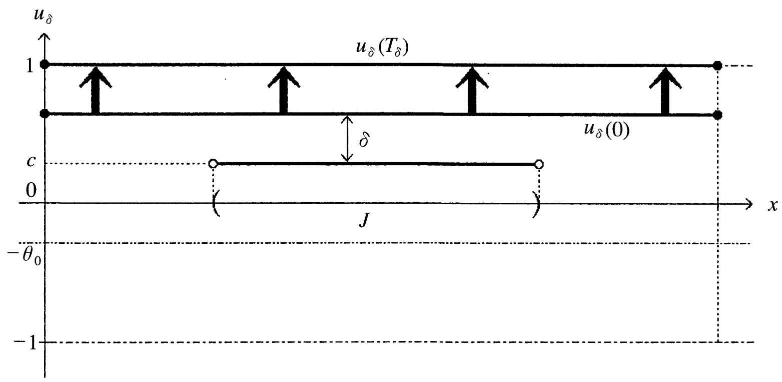


**Fig. 4**

Then the function $v_\delta$ satisfies the following evolution inclusion

$$(4.6) \qquad v'_\delta(t) + \kappa \partial V(v_\delta(t)) \ni v_\delta(t) - c - \delta t \qquad \text{in } L^2(0, L) \text{ for any } t \in [0, T_\delta].$$

In fact, noting that $v_\delta$ satisfies

$$\kappa \partial V(v_\delta(t)) \ni v_\delta(t) - c - \delta(t+1) (= 0) \text{ and } v'_\delta(t) = \delta \text{ in } L^2(0, L) \text{ for any } t \in [0, T_\delta],$$

we immediately see (4.6).

Let $v$ be the solution of the Cauchy problem for $(P)_{\theta_0}$ with initial condition $v(0) = v_\delta(0)$ in $L^2(0, L)$. Then, since

$$-c - \delta t \le -c \le \theta_0 \qquad \text{for any } t \in [0, T_\delta],$$

we can apply Corollary 1.1 with $s = 0$, $T = T_\delta$, $\theta_1 = -c - \delta t$, $\theta_2 = \theta_0$, $v_1 = v_\delta$ and $v_2 = v$ to obtain that

$$v_\delta(t, x) \le v(t, x) \qquad \text{for any } t \in [0, T_\delta] \text{ and a.e. } x \in [0, L],$$

in particular,

$$v_\delta(T_\delta, x) \le v(T_\delta, x) \text{ for a.e. } x \in [0, L].$$

Next, since $v_\delta(T_\delta)(\equiv 1)$ is a stationary solution for $(P)_{\theta_0}$, it follows from Corollary 1.1 with $s = T_\delta$, $T = +\infty$, $\theta_1 = \theta_2 = \theta_0$, $v_1 = v_\delta$ and $v_2 = v$ that

$$v_\delta(T_\delta, x) \le v(t, x) \qquad \text{for any } t \ge T_\delta \text{ and a.e. } x \in [0, L]$$

and

$$v(t, x) = 1 \qquad \text{for any } t \ge T_\delta \text{ and a.e. } x \in J.$$

Thus $(J, c)$ is unstable. In fact, for the positive number $\varepsilon_0 := 1 - c$ and for any positive number $\delta$ the solution $v$ of the Cauchy problem for $(P)_{\theta_0}$ with initial condition $v(0) = c + \delta$ satisfies that

$$|v(0) - c|_{L^\infty(J)} < 2\delta$$

and

$$|v(t) - c|_{L^\infty(J)} = 1 - c = \varepsilon_0 \qquad \text{for all } t \ge T_\delta.$$

(Case 2) The case that (4.5) holds for $(J, c)$.

By the result of (Case 1), it is enough to give a proof only in case of $|c| = 1$. We prove the instability of $(J, c)$ only in case $c = 1$ and $\bar{J} \cap \{0, L\} = \varnothing$, since the other cases are similarly proved. Let $\tilde{\delta}_0$ be a nonnegative number given by

$$\tilde{\delta}_0 := \frac{1}{|J|}(2\kappa - |1 + \theta_0||J|). \tag{4.7}$$

Here for any (small) positive number $\delta$, put

$$\tilde{T}_\delta := \frac{2}{\delta} - 1,$$

and define a function $\tilde{v}_\delta : [0, \tilde{T}_\delta] \to L^2(0, L)$ (see Fig. 5) by

$$\tilde{v}_\delta(t, x) := \begin{cases} 1 - \delta(t+1) & \text{for } x \in J, \\ -1 & \text{otherwise,} \end{cases}$$

for any $t \in [0, \tilde{T}_\delta]$.

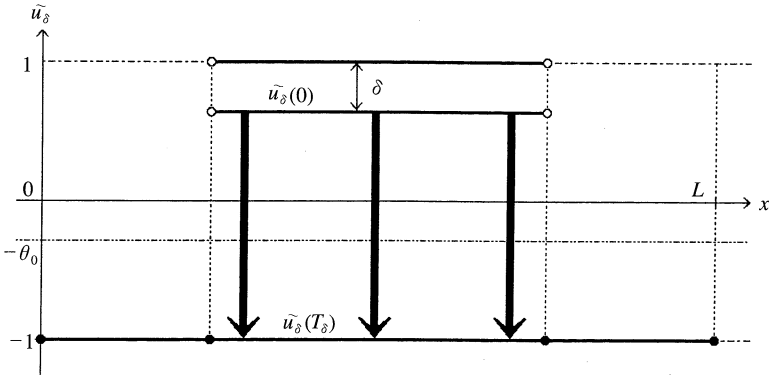


**Fig. 5**

Then the function $\tilde{v}_\delta$ satisfies the following evolution inclusion

$\tilde{v}_\delta'(t) + \kappa \partial V(\tilde{v}_\delta(t)) \ni \tilde{v}_\delta(t) + \theta_0 + (\delta(t+1) + \tilde{\delta}_0)\chi_J$ in $L^2(0, L)$ for any $t \in [0, \tilde{T}_\delta]$.

In fact, just as [7, section 3], we can show that a function $\tilde{w}_0 : [0, L] \to R$, defined by

$$\tilde{w}_0(x) := \begin{cases} 1, & \text{for } x \in J, \\ -1, & \text{otherwise,} \end{cases}$$

satisfies the following variational inequality

$$\kappa V(\tilde{w}_0) - \int_0^L (\tilde{w}_0 + \theta_0 + \tilde{\delta}_0 \chi_J)\tilde{w}_0 \, dx \le \kappa V(z) - \int_0^L (\tilde{w}_0 + \theta_0 + \tilde{\delta}_0 \chi_J) z \, dx \tag{4.8}$$

for any $z \in D(V)$. By (4.7) and (4.8), we see that

$$\begin{aligned} &\kappa V(\tilde{v}_\delta(t)) - \int_0^L (\tilde{w}_0 + \theta_0 + \tilde{\delta}_0 \chi_J)\tilde{v}_\delta(t) dx \\ &= (\kappa V(\tilde{w}_0) - 2\kappa\delta) - \left( \int_0^L (\tilde{w}_0 + \theta_0 + \tilde{\delta}_0 \chi_J)\tilde{w}_0 \, dx - \delta(1 + \theta_0 + \tilde{\delta}_0)|J| \right) \\ &\le \kappa V(\tilde{w}_0) - \int_0^L (\tilde{w}_0 + \theta_0 + \tilde{\delta}_0 \chi_J)\tilde{w}_0 \, dx \le \kappa V(z) - \int_0^L (\tilde{w}_0 + \theta_0 + \tilde{\delta}_0 \chi_J) z \, dx \end{aligned}$$

for any $t \in [0, \tilde{T}_\delta]$ and any $z \in D(V)$. This implies that

$$\kappa V(\tilde{v}_\delta(t)) \ni \tilde{w}_0 + \theta_0 + \tilde{\delta}_0 \chi_J \text{ in } L^2(0, L) \text{ for any } t \in [0, \tilde{T}_\delta].$$

Since

$$\tilde{v}_\delta'(t) = \delta \chi_J \quad \text{and} \quad \tilde{v}_\delta(t) = \tilde{w}_0 - \delta(t+1)\chi_J,$$

we obtain that

$$\tilde{v}_\delta'(t) + \kappa \partial V(\tilde{v}_\delta(t)) \ni \tilde{v}_\delta(t) + \theta_0 + (\delta(t+1) + \delta_0)\chi_J \quad \text{in } L^2(0, L),$$

for any $t \in [0, \tilde{T}_\delta]$.

Now, let $v$ be the solution of the Cauchy problem for $(P)_{\theta_0}$ with initial condition $v(0) = \tilde{v}_\delta(0)$ in $L^2(0, L)$. Applying Corollary 1.1 with $s = 0$, $T = 1$, $\theta_1 = \theta_0$, $\theta_2 = \theta_0 + (\delta(t+1) + \delta_0)\chi_J$, $v_1 = v$ and $v_2 = \tilde{v}_\delta$, we have

$$v(t, x) \leq \tilde{v}_\delta(t, x) \quad \text{for any } t \in [0, \tilde{T}_\delta] \text{ and a.e. } x \in [0, L],$$

in particular

$$v(\tilde{T}_\delta, x) \leq \tilde{v}_\delta(\tilde{T}_\delta, x) \quad \text{for a.e. } x \in [0, L]. \tag{4.9}$$

Next, since $\tilde{v}_\delta(\tilde{T}_\delta) (\equiv -1)$ is a stationary solution for $(P)_{\theta_0}$, it follows from Corollary 1.1 that

$$v(t, x) \leq \tilde{v}_\delta(\tilde{T}_\delta, x) \quad \text{for any } t \geq \tilde{T}_\delta \text{ and a.e. } x \in [0, L],$$

so that

$$v(t, x) = -1 \quad \text{for any } t \geq \tilde{T}_\delta \text{ and a.e. } x \in J.$$

Thus $(J, c)$ is unstable. In fact, for $\varepsilon_0 := 2$, we see that for any positive number $\delta$ the solution $v$ of the Cauchy problem for $(P)_{\theta_0}$ with initial condition $v(0) = \tilde{w}_0 + \tilde{\delta}_0 \chi_J$ satisfies that

$$|v(0) - 1|_{L^\infty(J)} < 2\delta$$

and

$$|v(t) - 1|_{L^\infty(J)} = 2 = \varepsilon_0 \quad \text{for all } t \geq \tilde{T}_\delta.$$

Now the proof of Theorem 3.1 is complete. ■

## 5. Proofs of the other results

In our proof of Corollary 3.1, we need the following fundamental result on functions of bounded variation on a compact interval.

**Lemma 5.1.** *Let $I$ be a compact interval in $R$ and $\{z_i\}$ be a sequence of functions of bounded variation on $I$ such that*

$$z_i \to c \ (= const.) \qquad in\ L^2(I)$$

*and*

$$V_0(z_i; I) \to 0 \qquad as\ i \to +\infty,$$

*where $V_0(z_i; I)$ denotes the total variation of $z_i$ on $I$. Then $z_i \to c$ uniformly on $I$ as $i \to +\infty$.*

The proof of Lemma 5.1 is quite standard, so we omit it.

*Proof of Corollary* 3.1. We use the same notation for $v$, $v_\infty$, $c$ and $J$ as in the statement of Corollary 3.1. By assumption, $|c| = 1$ and $J$ is of the form $[0, L]$ or $[0, a)$ or $(b, L]$ or $(a, b)$ for $\kappa \le a < L$ and $0 < b \le L - \kappa$. We prove below the corollary in case $c = 1$ and $J = (a, b)$, since any other case is similarly proved.

Given a compact subset $J_1$ of $J$, choose a (small) positive number $\mu$ so that

$$J_1 \subset (a + \mu, b - \mu) =: J_\mu$$

and the pair $(J_\mu, 1)$ is stable; this is possible by Theorem 3.1. Now, let $\{t_i\}$ be an increasing sequence in $R$ such that $t_i \nearrow +\infty$ and $v(t_i) \to v_\infty$ in $L^2(0, L)$ as $i \to +\infty$. Then it follows from Proposition 1.2 that $V_0(v(t_i, \cdot)) \to V_0(v_\infty)$ as $i \to +\infty$. Putting $z_i = v(t_i)|_I$ for $I := \overline{J_\mu}$, we see that $z_i \to 1$ in $L^2(I)$ and $V_0(z_i; I) \to 0$, since $v_\infty = 1$ on $I$. Therefore, by Lemma 5.1, $z_i = v(t_i) \to 1$ uniformly on $\overline{J_\mu}$ as $i \to +\infty$. This together with the stability of $(J_\mu, 1)$ implies that $v(t) \to 1$ uniformly on $\overline{J_\mu}$ as $t \to +\infty$. Moreover, as was easily checked in the proof of the sufficiency of Theorem 3.1, $v(t, \cdot)$ reaches the value 1 on $\overline{J_\mu}$ in a finite time $t_1$. ■

*Proof of Theorem* 3.2. If $w$ is constant on $[0, L]$, then the proof of Theorem 3.2 is the same as in the proof of Theorem 3.1. Therefore we suppose that $w^\circ \in S_n(\theta_0)$ for a certain $n \in N$, where $w^\circ$ is as in Theorem 2.1.

By Theorem 3.1, there exists a (small) positive number $\mu_0 \in (0, \kappa)$ such that

$$\text{(5.1)} \qquad \begin{cases} |c_0 + \theta_0|(|J_0| - \mu_0) > \kappa, & |c_n + \theta_0|(|J_n| - \mu_0) > \kappa, \\ |c_k + \theta_0|(|J_k| - 2\mu_0) > 2\kappa, & k = 1, \dots, n-1, \end{cases}$$

where $c_k$, $x_k^L = x_k^R$, $J_k$ are as in (II) of Definition 2.1 corresponding to $w^\circ$. For any $\mu \in (0, \mu_0]$, put

$$J_{k,\mu} := \begin{cases} [0, x_1^R - \mu), & k = 0, \\ (x_k^R + \mu, x_{k+1}^R - \mu), & k = 1, \dots, n-1, \\ (x_n^R + \mu, L], & k = n. \end{cases}$$

Then, by Theorem 3.1 again, any pair $(J_{k,\mu}, c_k)$ is stable and hence for any $k \in \{0, 1, \dots, n\}$, $\mu \in (0, \mu_0]$ and $\varepsilon > 0$ there exist a positive number $\delta_{k,\mu,\varepsilon}$ and a time $t_{k,\mu,\varepsilon} \geq 0$ such that

$$|v(t) - c_k|_{L^\infty(J_{k,\mu})} < \varepsilon$$

for any $t \geq t_{k,\mu,\varepsilon}$ and any solution $v$ of $(P)_{\theta_0}$ with $|v(0) - c_k|_{L^\infty(J_{k,\mu})} < \delta_{k,\mu,\varepsilon}$. Since $J_\mu(w) = \sum_{k=0}^n J_{k,\mu}$, it is easy to see that $\delta_{\mu,\varepsilon} := \min_{0 \leq k \leq n} \delta_{k,\mu,\varepsilon}$ and $t_{\mu,\varepsilon} := \max_{0 \leq k \leq n} t_{k,\mu,\varepsilon}$ satisfy the required property. The converse is also proved by using Theorem 3.1. ■

*Proof of Theorem* 3.3. First take a (small) positive number $\mu_0$ for which the property in Theorem 3.2 is satisfied; more precisely, (5.1) is satisfied; we use the same notation $c_k$, $x_k^L = x_k^R$, $J_k$ $(0 \leq k \leq n)$ and $J_{k,\mu}$ $(0 < \mu \leq \mu_0)$, for the stable solution $w$ of $(P_\infty)_{\theta_0}$, as in the proof of Theorem 3.2.

Now, fix a number $\mu$ with $0 < \mu < \min\{\mu_0, \kappa\}$ and let $\varepsilon$ be a (small) positive number such that

$$\text{(5.2)} \qquad \begin{cases} |c_0 \pm \varepsilon + \theta_0|(|J_0| - \mu_0) > \kappa, & |c_n \pm \varepsilon + \theta_0|(|J_n| - \mu_0) > \kappa, \\ |c_k \pm \varepsilon + \theta_0|(|J_k| - \mu_0) > 2\kappa, & k = 1, \dots, n-1; \end{cases}$$

note that $c_k = 1$ or $-1$, $k = 0, 1, \dots, n$. To these $\varepsilon$ and $\mu$, by using Theorem 3.2, choose a positive number $\delta_{\mu,\varepsilon}$ and a time $t_{\mu,\varepsilon}$ such that

$$|v(t) - w|_{L^\infty(J_\mu(w))} < \varepsilon \qquad \text{for all } t \geq t_{\mu,\varepsilon},$$

whenever $v$ is a solution of $(P)_{\theta_0}$ with

$$\text{(5.3)} \qquad |v(0) - w|_{L^\infty(J_\mu(w))} < \delta_{\mu,\varepsilon}.$$

We now show that any solution $v$ satisfying (5.3) converges in $L^2(0, L)$ as $t \to +\infty$. In fact, let $v_\infty$ be any $\omega$-limit point of $v$. Then, since $v_\infty$ is a solution of $(P_\infty)_{\theta_0}$, our choice of $\varepsilon$ (cf. (5.2)) implies that

$$v_\infty = c_k \qquad \text{on } J_{k,\mu},\ k = 0, 1, \dots, n,$$

and $v_\infty$ has at most two discontinuous points in $[x_k^R - \mu, x_k^R + \mu]$ for each $k = 1, \dots, n$. Therefore, the quantities $c_k^\infty, x_k^{R,\infty}, x_k^{L,\infty}, J_k^\infty$ $(0 \leq k \leq n^\infty)$ for the expression of $v_\infty$ such as in (II) of Definition 2.1 satisfy that $n^\infty = n$,

$$c_k = c_k^\infty, \qquad J_{k,\mu} \subset J_k^\infty, \qquad k = 0, 1, \dots, n,$$

$$x_k^R - \mu \leq x_k^{R,\infty} \leq x_k^{L,\infty} \leq x_k^R + \mu, \qquad k = 1, \dots, n,$$

and $(J_k^\infty, c_k^\infty)$ is stable for $k = 0, 1, \dots, n$. Here, applying Corollary 3.1 and Lebesgue's convergence theorem, we see that

$$v(t) \to v_\infty \qquad \text{in } L^2\left(\sum_{k=0}^n J_k^\infty\right) \text{ as } t \to +\infty.$$

This fact holds for any $\omega$-limit point of $v$, which implies that the set of intervals $J_k^\infty$, $k = 0, 1, \ldots, n$, is independent of the choice of an $\omega$-limit point of $v$; in the other word the $\omega$-limit set of $v$ is a singleton, and by the way we have

$$v(t) \to v_\infty \qquad \text{in } L^2(0, L) \text{ as } t \to +\infty.$$

Moreover, it is a direct application of Corollary 3.1 with Lemma 5.1 that $v(t) \to v_\infty$ as $t \to +\infty$ uniformly on each compact subset of the complement of the set of all discontinuous points of $v_\infty$ in $[0, L]$. ■

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