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Small Data Scattering and Blow-up for a Wave Equation with a Cubic Convolution

By

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Dedicated to Professor Kiyoshi Mochizuki on his sixtieth birthday

1. Introduction

This paper is concerned with the small data scattering and blow-up for the wave equation with a cubic convolution

$$\partial_t^2 u - \Delta u + (V_\gamma * u^2)u = 0 \qquad \text{in } \boldsymbol{R}^{n+1}. \tag{1.1}$$

Here u is a real-valued unknown function, $V_\gamma = \lambda|x|^{-\gamma}$, $\lambda \in \boldsymbol{R}$, $0 < \gamma < n$ and $*$ denotes the convolution in the space variables.

In the nonlinear scattering theory the asymptotic behavior for large time is studied from the point of view of the existence of asymptotically free solutions. The existence of an asymptotically free solution means that the wave keeps on going to spatial infinity, its local energy decays to zero and the effect of a nonlinear term is negligible for sufficiently large time. Under some situations, however, it is possible that the effect of a nonlinear interaction term causes waves which blow up in a finite time (see [26]–[27]). Thus it is quite interesting to determine sharp conditions which guarantees that all solutions under consideration are asymptotically free as $t \to +\infty$. In this paper we study the small data scattering and blow-up problem for the equation (1.1). Our main theorems show that, in the case of $n = 3$, the scattering operator is defined in a neighborhood of zero in the Hilbert space $X \times Y$ (defined below) for $2 < \gamma < 2 + 1/2$, and there exist blow-up solutions even for small data for $0 < \gamma < 2$ and $\lambda < 0$.

Here we shall refer to the previous results on the scattering and blow-up for (1.1). Mochizuki [17] studied the low energy scattering and showed by employing the space-time mixed norm estimate due to Pecher [19] that the scattering operator can be defined as a mapping from a neighborhood of zero in $\dot{H}^1 \times L^2$ into $\dot{H}^1 \times L^2$ for the case $\gamma = 4 < n$. When the requirement that the data be arbitrary within a neighborhood of zero in $\dot{H}^1 \times L^2$ is dispensable, Mochizuki and Motai [18] proved by making use of the classical $L^p - L^q$ time decay estimate of Strichartz [29] that the allowed values of γ are extended to

$2+2/3(n-1)<\gamma<n$. One of the main purposes of this paper is to show that the allowed values of γ can be reduced to two (limit excluded) when $n=3$. As for the blow-up, when $\lambda<0$, the nonexistence of global solutions was shown by Perla Menzala and Strauss [16] for some large data. The key point of our blow-up theorem is that, in the case of $n=3$, $\lambda<0$ and $0<\gamma<2$, there exists a case where the nonexistence of global solutions occurs even for small data.

Our method of the proof of scattering is to solve the associated integral equation by the contraction mapping principle. For the purpose we make use of a variant of the classical L^p-L^q estimate of Strichartz (3.9), the generalized Strichartz estimates and the infinitesimal generators of the Lorentz group and the dilation operator. We also employ the Li–Zhou inequality (3.13) to estimate the convolution $V_\gamma * u^2 (= C\omega^{-(n-\gamma)}u^2, \omega=\sqrt{-\Delta})$. The Li–Zhou inequality has an advantage over the Hardy–Littlewood–Sobolev inequality in the respect that we can take into account the difference of behaviors of solutions near the characteristic cone and away from it.

As for the proof of our blow-up theorem, we analyze non-negative, radial solutions. After taking into account the basic fact that, for radial solution u, $V_\gamma * u^2$ is reduced to the special form (see (6.6)), we shall be on the same lines as in [8], [6], [22] and [21] to show the blow-up. As far as the author knows, this theorem is the first one showing the blow-up of solutions for small data to the equation (1.1).

Finally, we give three comments. First, in a similar (in fact, slightly easier) way, it is possible to prove that, in all higher space dimensions $n\geq 4$, the scattering operator is defined in a neighborhood of zero in a suitable Hilbert space when $2<\gamma\leq 3$. This is an improvement of the above-mentioned result of Mochizuki and Motai. However, such a result would be out of proportion to the present interest because our blow-up theorem is limited to the three space dimensional case. Thus, in the present paper we refrain from considering the scattering problem in higher space dimensions. Secondly, the low energy scattering for the Klein–Gordon equation with a cubic convolution was studied in [16], [17], [18], [25]. Finally, the three space dimensional nonlinear wave equation with a negative potential $\Box u+\lambda(1+|x|^2)^{-l/2}u=|u|^{p-1}u$ $(\lambda<0)$ has been studied in [28] in detail. The critical number of l dividing the global existence and blow-up for small data is two.

This paper is organized as follows: In the next section we give the notation used in this paper and then state our main results concerning scattering. In Section 3 some fundamental lemmas and key estimates are proved. In Section 4 we prove the result concerning the initial value problem with data given at $t=0$ and investigate the asymptotic behavior of solutions. In Section 5 we discuss the solvability of the final value problem with data given at $t=\pm\infty$. In the final section the blow-up theorem is proved.

2. Scattering

Following Klainerman [11]–[12], we introduce several partial differential operators as follows: $\partial_0 = \partial/\partial t$, $\partial_j = \partial/\partial x_j$ $L_j = t\partial_j + x_j\partial t$ $(j = 1, \dots, n)$, $\Omega_{kl} = x_k\partial_l - x_l\partial_k$ $(1 \le k < l \le n)$, $L_0 = t\partial t + x_1\partial_1 + \cdots + x_n\partial_n$. These operators $\partial_0, \dots, \partial_n$, $L_1, \dots, L_n$, $\Omega_{12}, \dots, \Omega_{n-1n}$ and L_0 are denoted by $\Gamma_0, \dots, \Gamma_\nu$ in this order, where $\nu = (n^2 + 3n + 2)/2$. For a multi-index $\alpha = (\alpha_0, \dots, \alpha_\nu)$ we denote $\Gamma_0^{\alpha_0} \dots \Gamma_\nu^{\alpha_\nu}$ by Γ^α. It is also necessary to define the norm for $1 \le p$, $q < \infty$

$$\|v(\cdot)\|_{L^{p,q}} = \|v(r\zeta)r^{(n-1)/p}\|_{L^p(\boldsymbol{R}^+; L^q(S^{n-1}))} = \left(\int_0^\infty \left(\int_{S^{n-1}} |v(r\zeta)|^q dS_\zeta\right)^{p/q} r^{n-1} dr\right)^{1/p}, \tag{2.1}$$

where $r = |x|$, $\zeta \in S^{n-1}$. It is clear that the $L^{p,q}$ norm coincides with the usual L^p norm when $p = q$. We also define

$$\begin{cases} \|v(\cdot)\|_{L^{p,\infty}} = \|v(r\zeta)r^{(n-1)/p}\|_{L^p(\boldsymbol{R}^+; L^\infty(S^{n-1}))} & \text{for } 1 \le p < \infty, \\ \|v(\cdot)\|_{L^{\infty,q}} = \sup_{r>0}\left(\int_{S^{n-1}} |v(r\zeta)|^q dS_\zeta\right)^{1/q} & \text{for } 1 \le q < \infty, \\ \|v(\cdot)\|_{L^{\infty,\infty}} = \|v(\cdot)\|_{L^\infty}. \end{cases} \tag{2.2}$$

In [13] Li and Yu utilized effectively these types of norms for the existence theory of solutions to fully nonlinear wave equations. Let N be a non-negative integer and Ψ be a characteristic function of a set of $\boldsymbol{R}^{n+1}$. We define the norm

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\Psi} := \sum_{|\alpha| \le N} \|\Psi(t,\cdot)\Gamma^\alpha u(t,\cdot)\|_{L^{p,q}} \quad (1 \le p, q \le \infty) \tag{2.3}$$

for any function $u(t,x)$ for which the above right-hand side is finite for every t. When $\Psi \equiv 1$ in (2.3), we omit the sub-index Ψ. When $p = q$, we omit q. When $N = 0$, we omit Γ and N. In sum, the notation of the norm in (2.3) is abbreviated to

$$\|u(t,\cdot)\|_{\Gamma,N,p,q,\Psi} = \begin{cases} \|u(t,\cdot)\|_{\Gamma,N,p,q}, & \text{if } \Psi \equiv 1, \\ \|u(t,\cdot)\|_{\Gamma,N,p,\Psi}, & \text{if } p = q, \\ \|u(t,\cdot)\|_{p,q,\Psi}, & \text{if } N = 0. \end{cases} \tag{2.4}$$

According to the rule, we denote by $\|u(t,\cdot)\|_p$ the usual L^p norm.

For a positive integer k, we mean by $H^k = H^k(\boldsymbol{R}^n)$ the usual Sobolev space of functions whose derivatives up to order k belong to L^2. We shall work with the homogeneous Sobolev space $\dot{H}_r^s = \dot{H}_r^s(\boldsymbol{R}^n)$ $(s \in \boldsymbol{R}, 1 < r < \infty)$. The space $\dot{H}_2^s$ will be simply denoted by $\dot{H}^s$. For a comprehensible description of the definition of the homogeneous Sobolev space and fundamental properties such as completeness and the embedding theorem, refer to the appendix of [1], [4]. For any interval I and any Banach space B we denote by $C(I;B)$ (resp. $BC(I;B)$) the space of continuous (resp. bounded continuous) functions from I to B. We denote by $\alpha(p)$ the variable defined by $\alpha(p) = 1/2 - 1/p$. The variables $\beta(p)$, $\gamma(p)$ and $\delta(p)$ are also defined by

$$\alpha(p) = \frac{2}{n+1}\beta(p) = \frac{1}{n-1}\gamma(p) = \frac{1}{n}\delta(p). \tag{2.5}$$

The Hölder conjugate exponent of p is denoted by p': $1/p + 1/p' = 1$. We denote by $\bar{I}$ the closure of an interval I in $\boldsymbol{R} \cup \{\pm\infty\}$.

Put $\omega := \sqrt{-\Delta}$. The initial data will be taken in the space $X \times Y$ defined by

$$X = \Big\{ f \in L^2(\boldsymbol{R}^3) \cap \dot{H}^1(\boldsymbol{R}^3) \mid \|f\|_X^2 = \sum_{|\alpha| \le 2} \|\Omega^\alpha f\|_2^2 + \sum_{|\alpha| \le 2} \|\langle x\rangle \omega \Omega^\alpha f\|_2^2 < \infty \Big\}, \tag{2.6}$$

$$Y = \Big\{ g \in \dot{H}^{-1}(\boldsymbol{R}^3) \cap L^2(\boldsymbol{R}^3) \mid \|g\|_Y^2 = \sum_{|\alpha| \le 2} \|\Omega^\alpha \omega^{-1} g\|_2^2 + \sum_{|\alpha| \le 2} \|\langle x\rangle \Omega^\alpha g\|_2^2 < \infty \Big\}. \tag{2.7}$$

Here $\Omega^\alpha = \Omega_{12}^{\alpha_{12}} \Omega_{23}^{\alpha_{23}} \Omega_{13}^{\alpha_{13}}$ for a multi-index $\alpha = (\alpha_{12}, \alpha_{23}, \alpha_{13})$ and $\langle x\rangle = (1+|x|^2)^{1/2}$. $X \times Y$ is the Hilbert space with the norm defined by $\|(f,g)\|_{X\times Y}^2 = \|f\|_X^2 + \|g\|_Y^2$. Now we can state our main theorems.

Theorem 1. *Suppose that $n = 3$ and $2 < \gamma < 2 + 1/2$. There exists a constant $\delta > 0$ depending only on γ and λ with the following property: For any $(f,g) \in X \times Y$ with $\|(f,g)\|_{X\times Y} < \delta$ the integral equation*

$$u(t) = u_0(t) - \int_0^t \frac{\sin \omega(t-\tau)}{\omega} (V_\gamma * u^2(\tau)) u(\tau) d\tau \tag{2.8}$$

($u_0(t) = (\cos \omega t) f + (\omega^{-1} \sin \omega t) g$) has a unique solution $u(t,x)$ satisfying

$$(\Gamma^\alpha \Omega^\beta u, \Omega^\beta \omega^{-1} \partial_t u) \in BC(\boldsymbol{R}; L^2 \times L^2), \qquad (|\alpha| \le 1, |\beta| \le 2). \tag{2.9}$$

Moreover, there exists a unique pair of functions (f^+, g^+), $(f^-, g^-) \in X \times Y$ satisfying

(2.10) $$\|(f^{\pm}, g^{\pm})\|_{X\times Y} \le C_1\|(f, g)\|_{X\times Y} \quad \textit{for some constant } C_1 > 0,$$

(2.11) $$\|(\Gamma^{\alpha}\Omega^{\beta}u(t,\cdot) - \Gamma^{\alpha}\Omega^{\beta}u^{\pm}(t,\cdot), \Omega^{\beta}\omega^{-1}\partial_t u(t,\cdot) - \Omega^{\beta}\omega^{-1}\partial_t u^{\pm}(t,\cdot)\|_{L^2\times L^2} \to 0 \ (t \to \pm\infty)$$

for any $|\alpha| \le 1$ *and* $|\beta| \le 2$. *Here* $u^{\pm}(t) = (\cos \omega t)f^{\pm} + (\omega^{-1}\sin \omega t)g^{\pm}$.

The next theorem is concerned with the final value problem with data given at $t = -\infty$.

Theorem 2. *Suppose that* $n = 3$ *and* $2 < \gamma < 2 + 1/2$. *There exists a constant* $\delta > 0$ *depending only on* γ *and* λ *with the following property: For any* $(f_-, g_-) \in X \times Y$ *with* $\|(f_-, g_-)\|_{X\times Y} < \delta$ *the integral equation*

(2.12) $$u(t) = u_-(t) - \int_{-\infty}^{t} \frac{\sin \omega(t-\tau)}{\omega}(V_\gamma * u^2(\tau))u(\tau)d\tau$$

$(u_-(t) = (\cos \omega t)f_- + (\omega^{-1}\sin \omega t)g_-)$ *has a unique solution* $u(t,x)$ *satisfying*

(2.13) $$(\Gamma^{\alpha}\Omega^{\beta}u, \Omega^{\beta}\omega^{-1}\partial_t u) \in BC((-\infty, 0]; L^2 \times L^2), \qquad (|\alpha| \le 1, |\beta| \le 2),$$

(2.14) $$\|(\Gamma^{\alpha}\Omega^{\beta}u(t,\cdot) - \Gamma^{\alpha}\Omega^{\beta}u_-(t,\cdot), \Omega^{\beta}\omega^{-1}\partial_t u(t,\cdot) - \Omega^{\beta}\omega^{-1}\partial_t u_-(t,\cdot))\|_{L^2\times L^2} \to 0 \ (t \to -\infty)$$

for any $|\alpha| \le 1$, $|\beta| \le 2$. *Moreover, this solution* u *satisfies*

(2.15) $$\|(u(0,\cdot), \partial_t u(0,\cdot))\|_{X\times Y} \le C_2\|(f_-, g_-)\|_{X\times Y}$$

for some constant $C_2 > 0$.

Remark. Let B_r mean an open ball of $X \times Y$ centered at the origin with radius $r > 0$. By Theorem 2 we know the existence of the wave operator for negative time as a mapping from B_δ into $B_{C_2\delta}$. Theorem 1 together with Theorem 2 implies the existence of the scattering operator as a mapping from B_δ into $B_{C_1C_2\delta}$ for sufficiently small $\delta > 0$.

3. Preliminaries

In this section we prove a number of lemmas and propositions which play central roles in the proof of our theorems. Let $\chi_1 = \chi_1(t,x)$ be the characteristic function of the set $\{(t,x) \in \boldsymbol{R}^{n+1} \mid |x| \le (1+|t|)/2\}$ and put $\chi_2 := 1 - \chi_1$. Moreover, let $\phi_a = \phi_a(x)$ be the characteristic function of the set $\{x \in \boldsymbol{R}^n \mid |x| \le a\}$ $(a > 0)$. Those three characteristic functions will be frequently used throughout this paper.

Lemma 3.1. *For any $\theta \in \boldsymbol{R}$ the following commutation relations hold:*

$$[L_j, \omega^\theta] = \theta\omega^{\theta-2}\partial_j\partial_t, \qquad j = 1, \dots, n, \tag{3.1}$$

$$[\Omega_{kl}, \omega^\theta] = 0, \qquad 1 \le k < l \le n, \tag{3.2}$$

$$[L_0, \omega^\theta] = -\theta\omega^\theta. \tag{3.3}$$

Proof. They all can be easily verified with the help of the Fourier transform. Thus we omit the details. □

Lemma 3.2. (1) *Let $n \ge 2$. The following inequalities hold for any function for which the norm on the right-hand side is finite:*

$$\|u(t,\cdot)\|_{p,\chi_1} \le C(1+|t|)^{-\delta(p)}\|u(t,\cdot)\|_{\Gamma,1,2}, \tag{3.4}$$

$$\|(1+|t|+|\cdot|)^{\gamma(p)}u(t,\cdot)\|_p \le C\|u(t,\cdot)\|_{\Gamma,1,2}, \tag{3.5}$$

where $2 \le p < \infty$ for $n = 2$, $2 \le p \le 2n/(n-2)$ for $n \ge 3$,

$$\|\langle|t|-|\cdot|\rangle\nabla u(t,\cdot)\|_2, \qquad \|\langle|t|-|\cdot|\rangle\partial_t u(t,\cdot)\|_2 \le C\|u(t,\cdot)\|_{\Gamma,1,2}, \tag{3.6}$$

where $\langle|t|-|x|\rangle = 1 + ||t|-|x||$.

(2) *Let $n \ge 3$. The inequality*

$$\|\langle x\rangle^{\delta(p)}v\|_p \le C(\|v\|_2 + \|\langle x\rangle\nabla v\|_2) \tag{3.7}$$

holds for $2 \le p \le 2n/(n-2)$.

Proof. The inequality (3.5) follows from Propositions 3.3 and 3.4 in Ginibre and Velo [2]. The inequality (3.4) is also an immediate consequence of the previous propositions in [2] (see also the comment given after the proof of proposition 3.4 in [2], especially the bottom line of page 247 in [2]). The inequality (3.4) actually can be derived from the standard Sobelev inequality by a simple scaling argument. See, e.g., [14] on page 1214–1216 for this matter. The inequality (3.6) is an elementary result and follows from the fact that the differential operators ∂_j, ∂_t can be represented as

$$\partial_j = \frac{tL_j + \sum_{k=1}^n x_k(x_j\partial_k - x_k\partial_j) - x_jL_0}{t^2-|x|^2}, \qquad \partial_t = \frac{tL_0 - \sum_{j=1}^n x_jL_j}{t^2-|x|^2}. \tag{3.8}$$

The inequality (3.7) follows directly from the Hölder inequality and the Sobolev embedding $\dot{H}^1(\boldsymbol{R}^n) \hookrightarrow L^{2n/(n-2)}(\boldsymbol{R}^n)$. □

The classical $L^p - L^q$ estimate of Strichartz [29] is successfully injected into our framework. This is done in the following lemma due to Ginibre and Velo

[2]. In particular, this lemma is very useful for the estimate for $\omega^{-1}\partial_t u(t)$ in the L^p $(p>2)$ norm (see (4.29) below).

Lemma 3.3. *Let I be any interval of $\boldsymbol{R}$. Let $n\geq 2$ and $2<p\leq 2(n+1)/(n-1)$. Suppose that $\Gamma^\alpha u\in C(I;L^2)$ for any $|\alpha|\leq 1$. Then $u(t),\omega^{-1}\partial_t u(t)\in L^p$ and they satisfy the estimate*

$$\begin{aligned}(3.9)\qquad &\|u(t,\cdot)\|_p,\|\omega^{-1}\partial_t u(t,\cdot)\|_p\\ &\quad\leq C|t|^{-\gamma(p)}E_0(u(t),\partial_t u(t))^{\alpha(p)/2}Q_0(t,u(t),\partial_t u(t))^{(1-\alpha(p))/2}.\end{aligned}$$

If in addition $\partial_t u\in C(I;\dot{H}^{-1})$, then

$$(3.10)\qquad \|\omega^{-1}\partial_t u(t,\cdot)\|_p\leq C(1+|t|)^{-\gamma(p)}(\|u(t,\cdot)\|_{\Gamma,1,2}+\|\partial_t u(t,\cdot)\|_{\dot{H}^{-1}}).$$

Here $E_0(u(t),\partial_t u(t))=\|\nabla u(t,\cdot)\|_2^2+\|\partial_t u(t,\cdot)\|_2^2$ and

$$\begin{aligned}&Q_0(t,u(t),\partial_t u(t))\\ &\quad=\sum_{j=1}^n\|L_j u(t,\cdot)\|_2^2+\sum_{1\leq k<l\leq n}\|\Omega_{kl}u(t,\cdot)\|_2^2+\|(L_0+(n-1))u(t,\cdot)\|_2^2.\end{aligned}$$

Proof. The inequality (3.9) is a direct consequence of [2, (3.17)]. The inequality (3.10) easily follows from (3.9) and the Sobolev embedding $H^1(\boldsymbol{R}^n)\hookrightarrow L^p(\boldsymbol{R}^n)$. □

The following estimate (3.13) represents a generalization of the Li–Yu inequality in [13]. The estimate (3.11) is a variant of (3.12) and indispensable to our proof (see (4.65) below).

Lemma 3.4. *Let $n\geq 2$ and assume $1/2<s<n/2$. Set $1/q=1/2+s/n$. Let δ be arbitrary with $0\leq\delta\leq n/2-s$. Then the following inequalities hold:*

$$(3.11)\qquad \|\,|x|^\delta v\|_{\dot{H}^{-s}}\leq Ca^\delta\|v\|_{q,\phi_a}+Ca^{-(n/2-s)+\delta}\|v\|_{1,2,1-\phi_a},$$

$$(3.12)\qquad \|v\|_{\dot{H}^{-s}}\leq C\|v\|_{q,\phi_a}+Ca^{-(n/2-s)}\|v\|_{1,2,1-\phi_a}.$$

Here constants C are independent of a and v. Moreover

$$\begin{aligned}(3.13)\qquad &\|\,|\cdot|^\delta h(t,\cdot)\|_{\dot{H}^{-s}}\\ &\quad\leq C(1+|t|)^\delta\|h(t,\cdot)\|_{q,\chi_1}+C(1+|t|)^{-(n/2-s)+\delta}\|h(t,\cdot)\|_{1,2,\chi_2}.\end{aligned}$$

Proof. The proof of (3.12) can be found in Li and Zhou [14]. Their proof is based on the Sobolev embedding $\dot{H}^s(\boldsymbol{R}^n)\hookrightarrow L^q(\boldsymbol{R}^n)$, the inequality

$$(3.14)\qquad \|v\|_{\infty,2,1-\phi_a}\leq Ca^{-(n/2-s)}\|v\|_{\dot{H}^s}$$

(see [14, (2.31)]) and the duality argument. Noting that a stronger inequality

$$\| \, |x|^{n/2-s} v \|_{\infty,2} \le C \|v\|_{\dot{H}^s} \tag{3.15}$$

actually follows from the slight modification of Li–Zhou's proof of (3.14), we easily obtain (3.11) by repeating the same argument as in the proof of (3.12). For completeness the proof of (3.15) will be given in Appendix. The inequality (3.13) is a direct consequence of (3.11). □

We shall need the generalized Strichartz estimates for inhomogeneous wave equations (see the proof of Proposition 4.7 below).

Lemma 3.5 ([3, Lemma 2.3]). *Let $n \ge 2$. For any (r_1, q_1) and (r_2, q_2) with $r_1, r_2 < \infty$, $0 \le 2/q_1 = \gamma(r_1) < 1$ and $0 \le 2/q_2 = \gamma(r_2) < 1$, for any $t_0 \in \bar{I}$ the operator*

$$h \mapsto \int_{t_0}^{t} e^{i(t-\tau)\omega} h(\tau) d\tau$$

is bounded from $L^{q_1'}(I; \dot{H}_{r_1'}^{\beta(r_1)})$ to $L^{q_2}(I; \dot{H}_{r_2}^{-\beta(r_2)})$ with norm uniformly bounded with respect to I and t_0.

Remark. It has been recently proved that a larger set of the pair (r, q) can be allowed for $n \ge 4$. See [15], [5] and [10].

Proposition 3.1. *Assume $n = 3$. For any $(f, g) \in X \times Y$, $u_0(t) := (\cos \omega t) f + (\omega^{-1} \sin \omega t) g$ satisfies*

$$\Gamma^\alpha \Omega^\beta u_0, \Omega^\beta \omega^{-1} \partial_t u_0 \in BC(\mathbf{R}; L^2(\mathbf{R}^3)), \qquad |\alpha| \le 1, |\beta| \le 2, \tag{3.16}$$

$$\sum_{\substack{|\alpha| \le 1 \\ |\beta| \le 2}} \|\Gamma^\alpha \Omega^\beta u_0(t, \cdot)\|_2 + \sum_{|\beta| \le 2} \|\Omega^\beta \omega^{-1} \partial_t u_0(t, \cdot)\|_2 \le C \|(f, g)\|_{X \times Y}. \tag{3.17}$$

Proof. We need the following three equalities:

$$L_j u_0(t) = -\frac{\sin \omega t}{\omega} \partial_j f - (\sin \omega t) \omega (x_j f) + (\cos \omega t)(x_j g), \qquad j = 1, 2, 3 \tag{3.18}$$

$$\Omega_{kl} u_0(t) = (\cos \omega t)(\Omega_{kl} f) + \frac{\sin \omega t}{\omega} \Omega_{kl} g, \qquad 1 \le k < l \le 3 \tag{3.19}$$

$$L_0 u_0(t) = \sum_{j=1}^{3} (\cos \omega t)(x_j \partial_j f) - 2 \frac{\sin \omega t}{\omega} g + \sum_{j=1}^{3} \frac{\sin \omega t}{\omega} \partial_j (x_j g). \tag{3.20}$$

These equalities can be easily checked with the help of the Fourier transform. We also need the commutation relations

(3.21) $$[\cos \omega t, \Omega^\alpha] = [\sin \omega t, \Omega^\alpha] = 0$$

for any α. Then Proposition 3.1 is a simple result. We omit the details. □

4. Proof of Theorem 1

For any $(f, g) \in X \times Y$ let us consider the integral equation (2.8). In what follows the space dimension is assumed to be three and we consider real-valued solutions only for simplicity. We introduce the set of functions Z_δ $(\delta > 0)$ as follows:

$$Z_\delta := \Bigg\{ u = u(t,x) \mid$$

$$\Gamma^\alpha \Omega^\beta u \in C(\mathbf{R}; L^2), (|\alpha| \le 1, |\beta| \le 2), \quad \Omega^\beta \partial_t u \in C(\mathbf{R}; \dot{H}^{-1}), (|\beta| \le 2),$$

$$u(0,x) = f(x), \partial_t u(0,x) = g(x),$$

$$\|u\|_Z = \sum_{\substack{|\alpha| \le 1 \\ |\beta| \le 2}} \sup_{t \in \mathbf{R}} \|\Gamma^\alpha \Omega^\beta u(t,\cdot)\|_2 + \sum_{|\beta| \le 2} \sup_{t \in \mathbf{R}} \|\Omega^\beta \omega^{-1} \partial_t u(t,\cdot)\|_2 \le \delta \Bigg\}.$$

Z_δ is nonempty, complete metric space with the metric $\|u - v\|_Z$ $(u, v \in Z_\delta)$ if $\|(f,g)\|_{X \times Y}$ is sufficiently small relative to δ. From now on we simply denote $\|(f,g)\|_{X \times Y}$ by Λ. It follows from Proposition 3.1 that $u_0 \in Z_{C_3 \Lambda}$ for a suitable constant $C_3 \ge 1$.

In order to prove Theorem 1 we define the mapping

(4.1) $$M : u \mapsto Mu = u_0(t) - I_0[u](t)$$

$$\equiv u_0(t) - \int_0^t \frac{\sin \omega(t-\tau)}{\omega} (V_\gamma * u^2(\tau)) u(\tau) d\tau, \qquad (u \in Z_{2C_3 \Lambda}).$$

We shall show the following proposition.

Proposition 4.1. *If Λ is sufficiently small, then M carries $Z_{2C_3\Lambda}$ into itself and satisfies*

(4.2) $$\|Mu - Mv\|_Z \le \tfrac{1}{2} \|u - v\|_Z, \qquad (u, v \in Z_{2C_3\Lambda}).$$

Then, as a consequence of the contraction mapping principle, M has a unique fixed point in $Z_{2C_3\Lambda}$ which is a solution to the integral equation (2.8). The proof of uniqueness is similar to that of Proposition 4.1. Hence, in the rest of this section, we shall devote ourself to the proof of Proposition 4.1. We begin with the following proposition which implies that there exist universal constants C, C' such that

$$C \sum_{\substack{|\alpha| \le 1 \\ |\beta| \le 2}} \sup_{t \in \mathbf{R}} \|\Omega^\beta \Gamma^\alpha u(t,\cdot)\|_2 \le \sum_{\substack{|\alpha| \le 1 \\ |\beta| \le 2}} \sup_{t \in \mathbf{R}} \|\Gamma^\alpha \Omega^\beta u(t,\cdot)\|_2 \le C' \sum_{\substack{|\alpha| \le 1 \\ |\beta| \le 2}} \sup_{t \in \mathbf{R}} \|\Omega^\beta \Gamma^\alpha u(t,\cdot)\|_2 \tag{4.3}$$

for all $u \in Z_{2C_3A}$.

Proposition 4.2. *Let δ_{jk} be the Kronecker delta. The following commutation relations hold:*

$$[\partial_t, \Omega_{kl}] = 0, \qquad [\partial_j, \Omega_{kl}] = \delta_{jk}\partial_l - \delta_{jl}\partial_k, \tag{4.4}$$

$$[L_j, \Omega_{kl}] = \delta_{jk} L_l - \delta_{jl} L_k, \tag{4.5}$$

$$\begin{aligned}[\Omega_{ab}, \Omega_{kl}] = {} & \delta_{bk}(x_a\partial_l - x_l\partial_a) - \delta_{bl}(x_a\partial_k - x_k\partial_a) \\ & - \delta_{ak}(x_b\partial_l - x_l\partial_b) + \delta_{al}(x_b\partial_k - x_k\partial_b),\end{aligned} \tag{4.6}$$

$$[L_0, \Omega_{kl}] = 0. \tag{4.7}$$

Proof. They can be checked by direct computations. □

Proposition 4.3. *Let $u \in Z_{2C_3A}$. There exists a constant C depending on γ such that the following equalities hold:*

$$\begin{aligned} & L_j[(V_\gamma * u^2)u] \\ & = C(\omega^{-(3-\gamma)} L_j u^2)u - (3-\gamma)C(\omega^{-(5-\gamma)}\partial_j\partial_t u^2)u + C(\omega^{-(3-\gamma)}u^2)L_j u,\end{aligned} \tag{4.8}$$

$$\Omega_{kl}[(V_\gamma * u^2)u] = C(\omega^{-(3-\gamma)}\Omega_{kl}u^2)u + C(\omega^{-(3-\gamma)}u^2)\Omega_{kl}u, \tag{4.9}$$

$$\begin{aligned} & L_0[(V_\gamma * u^2)u] \\ & = C(\omega^{-(3-\gamma)} L_0 u^2)u + (3-\gamma)C(\omega^{-(3-\gamma)}u^2)u + C(\omega^{-(3-\gamma)}u^2)L_0 u.\end{aligned} \tag{4.10}$$

Proof. Since $V_\gamma * u^2 = C\omega^{-(3-\gamma)}u^2$ with C depending only on γ (see [24, Chapter V]), these three equalities can be checked with the help of the Fourier transform and Lemma 3.1. We omit the details. □

For a nice function $h = h(t,x)$ we have

$$\begin{aligned} & L_j \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega} h(\tau)\,d\tau \\ & = \frac{\sin\omega(t-\sigma)}{\omega}[x_j h(\sigma)] + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}[L_j h(\tau)]\,d\tau,\end{aligned} \tag{4.11}$$

$$\Omega_{kl}\int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}h(\tau)d\tau = \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}\Omega_{kl}h(\tau)d\tau, \tag{4.12}$$

$$\begin{aligned} L_0 \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}h(\tau)d\tau \\ &= \sigma\frac{\sin\omega(t-\sigma)}{\omega}h(\sigma) + 2\int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}h(\tau)d\tau \\ &\quad + \int_\sigma^t \frac{\sin\omega(t-\tau)}{\omega}[L_0 h(\tau)]d\tau. \end{aligned} \tag{4.13}$$

Combining (4.11)–(4.13) with (4.8)–(4.10), we have the following proposition.

Proposition 4.4. *Let* $u \in Z_{2C_3 A}$ *and let* C *be the same constant as in Proposition* 4.3. *The following equalities hold:*

$$\begin{aligned} L_j I_0[u](t) \\ &= C\frac{\sin\omega t}{\omega}[x_j(\omega^{-(3-\gamma)}f^2)f] \\ &\quad + C\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}L_j u^2(\tau))u(\tau)]d\tau \\ &\quad + C(3-\gamma)\frac{\sin\omega t}{\omega}[(\omega^{-(5-\gamma)}\partial_j f^2)f] \\ &\quad - C(3-\gamma)\int_0^t \cos\omega(t-\tau)[(\omega^{-(5-\gamma)}\partial_j u^2(\tau))u(\tau)]d\tau \\ &\quad + C(3-\gamma)\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(5-\gamma)}\partial_j u^2(\tau))\partial_\tau u(\tau)]d\tau \\ &\quad + C\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}u^2(\tau))L_j u(\tau)]d\tau, \end{aligned} \tag{4.14}$$

$$\begin{aligned} \Omega_{kl} I_0[u](t) &= C\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega_{kl}u^2(\tau))u(\tau)]d\tau \\ &\quad + C\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}u^2(\tau))\Omega_{kl}u(\tau)]d\tau, \end{aligned} \tag{4.15}$$

$$\begin{aligned} L_0 I_0[u](t) &= C\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}L_0 u^2(\tau))u(\tau)]d\tau \\ &\quad + C(5-\gamma)\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}u^2(\tau))u(\tau)]d\tau \\ &\quad + C\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}u^2(\tau))L_0 u(\tau)]d\tau. \end{aligned} \tag{4.16}$$

Proof. It is enough to prove (4.14) because the others are easier to show. In view of (4.8) and (4.11) we notice that it suffices to show how to deal with the resulting term involving $\omega^{-(5-\gamma)}\partial_j\partial_\tau u^2$. Carrying out integration by parts with respect to the time variable, we see that

$$(4.17)\quad \int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(5-\gamma)}\partial_j\partial_\tau u^2(\tau))u(\tau)]d\tau$$
$$= -\frac{\sin\omega t}{\omega}[(\omega^{-(5-\gamma)}\partial_j f^2)f] + \int_0^t \cos\omega(t-\tau)[(\omega^{-(5-\gamma)}\partial_j u^2(\tau))u(\tau)]d\tau$$
$$- \int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(5-\gamma)}\partial_j u^2(\tau))\partial_\tau u(\tau)]d\tau.$$

Combining (4.8), (4.11) with (4.17), we have (4.14). □

Acting Ω^β ($|\beta| \le 2$) on (4.14)–(4.16), we finally have the following three equalities by (3.2), (3.21):

$$(4.18)\quad \Omega^\beta L_j I_0[u](t)$$
$$= C\frac{\sin\omega t}{\omega}\Omega^\beta[x_j(\omega^{-(3-\gamma)}f^2)f]$$
$$+ C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}L_j u^2(\tau))\Omega^{\beta-\beta'}u(\tau)]d\tau$$
$$+ C(3-\gamma)\frac{\sin\omega t}{\omega}\sum_{\beta'\le\beta}\binom{\beta}{\beta'}[(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j f^2)\Omega^{\beta-\beta'}f]$$
$$- C(3-\gamma)\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \cos\omega(t-\tau)[(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\Omega^{\beta-\beta'}u(\tau)]d\tau$$
$$+ C(3-\gamma)\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\Omega^{\beta-\beta'}\partial_\tau u(\tau)]d\tau$$
$$+ C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau))\Omega^{\beta-\beta'}L_j u(\tau)]d\tau,$$

$$(4.19)\quad \Omega^\beta \Omega_{kl} I_0[u](t)$$
$$= C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}\Omega_{kl}u^2(\tau))\Omega^{\beta-\beta'}u(\tau)]d\tau$$
$$+ C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau))\Omega^{\beta-\beta'}\Omega_{kl}u(\tau)]d\tau,$$

(4.20) $\Omega^\beta L_0 I_0[u](t)$

$$= C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}L_0u^2(\tau))\Omega^{\beta-\beta'}u(\tau)]d\tau$$

$$+ C(5-\gamma)\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau))\Omega^{\beta-\beta'}u(\tau)]d\tau$$

$$+ C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_0^t \frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau))\Omega^{\beta-\beta'}L_0u(\tau)]d\tau,$$

where

$$\binom{\beta}{\beta'} := \binom{\beta_1}{\beta_1'}\binom{\beta_2}{\beta_2'}\binom{\beta_3}{\beta_3'}$$

and the summation is taken over all the multi-indices $\beta' = (\beta_1', \beta_2', \beta_3')$ such that $\beta_j' \le \beta_j$ for $j = 1, 2, 3$. Now we are in a position to estimate $\Gamma^\alpha\Omega^\beta Mu$ ($|\alpha| \le 1, |\beta| \le 2$). Recalling (4.3), we start with the estimate for $\Omega^\beta L_j I_0[u]$. First, we estimate the fifth term on the right-hand side of (4.18) in the $L^\infty(\boldsymbol{R}; L^2)$ norm. We find that our task is to show

$$\|(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\Omega^{\beta-\beta'}\partial_\tau u(\tau)\|_{\dot H^{-1}} \in L^1(\boldsymbol{R}). \tag{4.21}$$

By duality we see that

(4.22) $\sup|((\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\Omega^{\beta-\beta'}\partial_\tau u(\tau), \phi)|$

$$= \sup\left|\left((\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\phi, \frac{\omega^2}{\omega}\Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\right)\right|$$

$$\le \sum_{k=1}^3 \sup|(\omega^{-1}\partial_k[(\omega^{-(5-\gamma)}\partial_k\Omega^{\beta'}\partial_j u^2(\tau))\phi], \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau))|$$

$$+ \sum_{k=1}^3 \sup|(\omega^{-1}\partial_k[(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\partial_k\phi], \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau))|$$

$$= \sum_{k=1}^3 \sup|((\omega^{-(5-\gamma)}\partial_k\Omega^{\beta'}\partial_j u^2(\tau))\omega^{-1}\partial_k\Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau), \phi)|$$

$$+ \sum_{k=1}^3 \sup|((\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\omega^{-1}\partial_k\Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau), \partial_k\phi)|$$

$$\leq \sum_{k=1}^{3} \|(\omega^{-(5-\gamma)}\partial_k \Omega^{\beta'}\partial_j u^2(\tau))\omega^{-1}\partial_k \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\|_{\dot{H}^{-1}}$$

$$+\sum_{k=1}^{3} \|(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\omega^{-1}\partial_k \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\|_2,$$

where by $(\cdot,\cdot)$ we mean the L^2 inner product and the supremum is taken over all $\phi \in \dot{H}^1$ with $\|\phi; \dot{H}^1\| = 1$. Further, thanks to (4.4), we find that the proof of (4.21) is reduced to that of

$$\|(\omega^{-(5-\gamma)}\partial_k \partial_l \Omega^{\beta'} u^2(\tau))\omega^{-1}\partial_k \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\|_{\dot{H}^{-1}} \in L^1(\boldsymbol{R}), \tag{4.23}$$

$$\|(\omega^{-(5-\gamma)}\partial_l \Omega^{\beta'} u^2(\tau))\omega^{-1}\partial_k \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\|_2 \in L^1(\boldsymbol{R}) \tag{4.24}$$

for $1 \leq k,\ l \leq 3$. First we show (4.23).

Proposition 4.5. *Assume that* $2 < \gamma < 5/2$. (1) *It is possible to choose* $s > 0$ *satisfying*

$$-3\gamma + \frac{13}{2} < s < -\gamma + \frac{5}{2}. \tag{4.25}$$

(2) *The norm in* (4.23) *is estimated from above as*

$$\cdots \leq C(1+|\tau|)^{-(\gamma-3/2+(s+1)/3)} \tag{4.26}$$

$$\times \left(\sum_{|\beta|\leq 2} \|\Omega^\beta u(\tau,\cdot)\|_{\Gamma,1,2}^2\right) \sum_{|\beta|\leq 2} (\|\Omega^\beta u(\tau,\cdot)\|_{\Gamma,1,2} + \|\Omega^\beta \omega^{-1}\partial_\tau u(\tau,\cdot)\|_2)$$

for s *satisfying* (4.25).

Remark. The condition $-3\gamma + 13/2 < s$ is equivalent to $\gamma - 3/2 + (s+1)/3 > 1$.

Proof. The proof of (1) is easy. Thus we omit it. Let us choose s satisfying (4.25) and put $1/p_1 = 1/2 - s/3$, $1/p_1 + 1/p_2 = 1/2 + 1/3$, $1/2 = 1/p_3 - (3-\gamma-s)/3$. Employing the Hardy–Littlewood–Sobolev inequality first and then the Hölder inequality, we see that the norm in (4.23) is estimated as

$$\cdots \leq C\|(\omega^{-(5-\gamma)}\partial_k \partial_l \Omega^{\beta'} u^2(\tau))\omega^{-1}\partial_k \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\|_{6/5} \tag{4.27}$$

$$\leq C\|\omega^{-(5-\gamma)}\partial_k \partial_l \Omega^{\beta'} u^2(\tau)\|_{p_1}\|\omega^{-1}\partial_k \Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\|_{p_2}.$$

Using the embedding $\dot{H}^s \hookrightarrow L^{p_1}$ first and then (3.13) with $\delta = 0$, we get

$$(4.28)\qquad \|\omega^{-(5-\gamma)}\partial_k\partial_l\Omega^{\beta'}u^2(\tau)\|_{p_1} \le C\|\omega^{-(3-\gamma-s)}\Omega^{\beta'}u^2(\tau)\|_2$$
$$\le C\|\Omega^{\beta'}u^2(\tau)\|_{p_3,\chi_1} + C(1+|\tau|)^{-3/2+(3-\gamma-s)}\|\Omega^{\beta'}u^2(\tau)\|_{1,2}$$
$$\le C(1+|\tau|)^{-3/2+(3-\gamma-s)}\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\tau)\|^2_{\Gamma,1,2}.$$

At the second inequality, the inequality (3.13) was applicable because $1/2 < 3-\gamma-s < 3/2$ holds now. At the last inequality we were able to employ the inequality (3.4) with $p = 2p_3$. We also used the fact that the Sobolev space on the sphere $W^{2,2}(S^2)$ (L^2 type of order 2) forms an algebra.

Let us turn our attention to the estimate for the second norm on the last line in (4.27). Since $1/p_2 = (1+s)/3$ and $0 < s < -\gamma + 5/2 < 1/2$, we find $2 < p_2 < 3$. Hence, employing the well-known multiplier theorem (see, e.g., [23] on page 15) first and then (3.10), we obtain

$$(4.29)\qquad \|\omega^{-1}\partial_k\Omega^{\beta-\beta'}\omega^{-1}\partial_\tau u(\tau)\|_{p_2} \le C\|\omega^{-1}\partial_\tau\Omega^{\beta-\beta'}u(\tau)\|_{p_2}$$
$$\le C(1+|\tau|)^{-\gamma(p_2)}(\|\Omega^{\beta-\beta'}u(\tau)\|_{\Gamma,1,2} + \|\Omega^{\beta-\beta'}\partial_\tau u(\tau)\|_{\dot H^{-1}}).$$

Since $-3/2 + (3-\gamma-s) - \gamma(p_2) = -\gamma + 3/2 - (s+1)/3$, we obtain (4.26) by (4.27)–(4.29). □

The estimate for the norm in (4.24) is given in the next proposition.

Proposition 4.6. *Assume* $2 < \gamma < 11/4$. (1) *It is possible to choose* $s \ge 3/4$ *satisfying*

$$(4.30)\qquad -3\gamma + \frac{15}{2} < s < -\gamma + \frac{7}{2}.$$

(2) *The norm in* (4.24) *is estimated from above as*

$$(4.31)\qquad \cdots \le C(1+|\tau|)^{-(\gamma-3/2+s/3)}$$
$$\times\left(\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\tau)\|^2_{\Gamma,1,2}\right)\sum_{|\beta|\le 2}(\|\Omega^{\beta}u(\tau)\|_{\Gamma,1,2} + \|\Omega^{\beta}\omega^{-1}\partial_\tau u(\tau)\|_2)$$

for s satisfying (4.30).

Remark. The condition $-3\gamma + 15/2 < s$ is equivalent to $\gamma - 3/2 + s/3 > 1$.

Proof. (1) is obvious. Let us take $s \ge 3/4$ satisfying (4.30) and put $1/p_1 = 1/2 - s/3$, $1/2 = 1/p_1 + 1/p_2$, $1/2 = 1/p_3 - (4-\gamma-s)/3$. Employing the Hölder inequality first and then the embedding $\dot H^s \hookrightarrow L^{p_1}$, we see that the norm in (4.24) is estimated as

$$\cdots \le \|\omega^{-(5-\gamma)}\partial_l \Omega^{\beta'} u^2(\tau)\|_{p_1} \|\omega^{-1}\partial_k \Omega^{\beta-\beta'} \omega^{-1}\partial_\tau u(\tau)\|_{p_2} \tag{4.32}$$

$$\le C\|\Omega^{\beta'} u^2(\tau)\|_{\dot H^{-(4-\gamma-s)}} \|\omega^{-1}\partial_k \Omega^{\beta-\beta'} \omega^{-1}\partial_\tau u(\tau)\|_{p_2}.$$

Since $1/2 < 4-\gamma-s < 3/2$, we can employ (3.13) with $\delta = 0$ in showing

$$\|\Omega^{\beta'} u^2(\tau)\|_{\dot H^{-(4-\gamma-s)}} \tag{4.33}$$

$$\le C\|\Omega^{\beta'} u^2(\tau)\|_{p_3,\chi_1} + C(1+|\tau|)^{-(3/2-(4-\gamma-s))}\|\Omega^{\beta'} u^2(\tau)\|_{1,2}$$

$$\le C(1+|\tau|)^{-(3/2-(4-\gamma-s))} \sum_{|\beta|\le 2} \|\Omega^{\beta} u(\tau)\|^2_{\Gamma,1,2}$$

as we did in (4.28). On the other hand, since $p_2 > 2$ by the conditions $\gamma > 2$ and $s < -\gamma + 7/2$ and the condition $s \ge 3/4$ implies $p_2 \le 4$, we can use (3.10) to show

$$\|\omega^{-1}\partial_k \Omega^{\beta-\beta'} \omega^{-1}\partial_\tau u(\tau)\|_{p_2} \le C\|\omega^{-1}\partial_\tau \Omega^{\beta-\beta'} u(\tau)\|_{p_2} \tag{4.34}$$

$$\le C(1+|\tau|)^{-\gamma(p_2)}(\|\Omega^{\beta-\beta'} u(\tau)\|_{\Gamma,1,2} + \|\Omega^{\beta-\beta'}\partial_\tau u(\tau)\|_{\dot H^{-1}}).$$

Since $-3/2 + (4-\gamma-s) - \gamma(p_2) = -\gamma + 3/2 - s/3$, we have shown (4.31). □

Combining Propositions 4.5 and 4.6, we have shown (4.21). Let us turn our attention to the last term on the right-hand side of (4.18). In order to estimate it the generalized Strichartz estimate plays a crucial role.

Proposition 4.7. *Assume that* $2 < \gamma < 5/2$. *The* $L^\infty(\boldsymbol{R}; L^2)$ *norm of the last term on the right-hand side of* (4.18) *is estimated as*

$$\cdots \le C\left(\sum_{|\beta|\le 2} \sup_{t\in\boldsymbol{R}} \|\Omega^\beta u(t)\|^2_{\Gamma,1,2}\right) \sum_{\substack{1\le j\le 3\\ |\beta|\le 2}} \sup_{t\in\boldsymbol{R}} \|L_j \Omega^\beta u(t)\|_2. \tag{4.35}$$

Proof. For any γ with $2 < \gamma < 5/2$ choose and fix a small number ε such that $0 < \varepsilon < 5/2 - \gamma$. Then we choose and fix (r, q) as follows: $1/r = 5/2 - \gamma - \varepsilon$, $2/q = \gamma(r)$. Observe that $0 < 2/q = \gamma(r) < 1$ for $2 < \gamma < 5/2$. Thus we may employ Lemma 3.5 with $(r_1, q_1) = (r, q)$, $(r_2, q_2) = (2, \infty)$ and we find that our task is to estimate

$$\|(\omega^{-(3-\gamma)}\Omega^{\beta'} u^2)\Omega^{\beta-\beta'} L_j u; L^{q'}(\boldsymbol{R}; \dot H^{\beta(r)-1}_{r'})\|. \tag{4.36}$$

Define p_1, p_2 and s as follows: $1/r' = 1/p_1 - (1-\beta(r))/3$, $1/p_1 = 1/2 + 1/p_2$ and $s = 1/r$. Observe that $p_1 > 1$ by the conditions $1/r = 5/2 - \gamma - \varepsilon$, $\varepsilon < 5/2 - \gamma$ and that the equality $1/p_2 = 1/2 - s/3$ holds. By the embeddings $\dot H^0_{p_1} \hookrightarrow \dot H^{\beta(r)-1}_{r'}$ and $\dot H^s \hookrightarrow L^{p_2}$ we see that

$$\begin{aligned}(4.37)\qquad &\|(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau))\Omega^{\beta-\beta'}L_ju(\tau);\dot{H}_{r'}^{\beta(r)-1}\|\\ &\le C\|(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau))\Omega^{\beta-\beta'}L_ju(\tau)\|_{p_1}\\ &\le C\|\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau)\|_{p_2}\|\Omega^{\beta-\beta'}L_ju(\tau)\|_2\\ &\le C\|\omega^{-(3-\gamma-s)}\Omega^{\beta'}u^2(\tau)\|_2\|\Omega^{\beta-\beta'}L_ju(\tau)\|_2.\end{aligned}$$

Taking account of the condition $1/r = 5/2-\gamma-\varepsilon$, we can easily verify that $1/2 < 3-\gamma-s < 3/2$. Thus we may employ (3.13) with $\delta = 0$ to show

$$\begin{aligned}(4.38)\qquad &\|\omega^{-(3-\gamma-s)}\Omega^{\beta'}u^2(\tau)\|_2\\ &\le C\|\Omega^{\beta'}u^2(\tau)\|_{p_3,\chi_1}\qquad\left(\frac{1}{2}=\frac{1}{p_3}-\frac{3-\gamma-s}{3}\right)\\ &\quad + C(1+|\tau|)^{-3/2+(3-\gamma-s)}\|\Omega^{\beta'}u^2(\tau)\|_{1,2}\\ &\le C(1+|\tau|)^{3/2-\gamma-s}\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\tau)\|_{\Gamma,1,2}^2.\end{aligned}$$

At the last inequality we have used the inequality (3.4) with $p = 2p_3$. Finally it is easy to check that $(1+|\tau|)^{-\gamma-s+3/2}\in L^{q'}(\boldsymbol{R})$ thanks to the condition $\gamma > 2$. Thus we have completed the estimate for the norm in (4.36). □

Next we shall estimate the second term on the right-hand side of (4.18). Our task is to show

$$(4.39)\qquad \|(\omega^{-(3-\gamma)}\Omega^{\beta'}L_ju^2(\tau))\Omega^{\beta-\beta'}u(\tau)\|_{\dot{H}^{-1}}\in L^1(\boldsymbol{R}).$$

Proposition 4.8. *Assume that* $2<\gamma<5/2$. (1) *It is possible to choose* $s>0$ *satisfying*

$$(4.40)\qquad -3\gamma+\frac{13}{2}<s<-\gamma+\frac{5}{2}.$$

(2) *The norm in* (4.39) *is estimated from above as*

$$\begin{aligned}(4.41)\qquad \cdots &\le C(1+|\tau|)^{-(s/3+\gamma-7/6)}\left(\sum_{|\beta|\le 2}\sup_{t\in\boldsymbol{R}}\|\Omega^{\beta}u(\tau)\|_{\Gamma,1,2}^2\right)\\ &\times\sum_{\substack{1\le k\le 3\\ |\beta|\le 2}}\sup_{t\in\boldsymbol{R}}\|L_k\Omega^{\beta}u(\tau)\|_2\end{aligned}$$

for s *satisfying* (4.40).

Remark. It follows from the condition $-3\gamma+13/2<s$ that $s/3+\gamma-7/6>1$.

Proof. The proof of (1) is easy. To show (2) we choose p_1, p_2 and p_3 as follows: $1/p_1=1/2-s/3$, $1/p_1+1/p_2=1/2+1/3$, $1/2=1/p_3-(3-\gamma-s)/3$. Employing the Sobolev embedding and the Hölder inequality, we see that the norm in (4.39) is estimated as

$$\begin{aligned}(4.42)\qquad \cdots &\le C\|(\omega^{-(3-\gamma)}\Omega^{\beta'}L_j u^2(\tau))\Omega^{\beta-\beta'}u(\tau)\|_{6/5}\\ &\le C\|\omega^{-(3-\gamma)}\Omega^{\beta'}L_j u^2(\tau)\|_{p_1}\|\Omega^{\beta-\beta'}u(\tau)\|_{p_2}.\end{aligned}$$

Moreover, making use of the embedding $\dot{H}^s \hookrightarrow L^{p_1}$ first and then (3.13) with $\delta=0$, we get

$$\begin{aligned}(4.43)\qquad \|\omega^{-(3-\gamma)}\Omega^{\beta'}L_j u^2(\tau)\|_{p_1} &\le C\|\omega^{-(3-\gamma-s)}\Omega^{\beta'}L_j u^2(\tau)\|_2\\ &\le C\|\Omega^{\beta'}L_j u^2(\tau)\|_{p_3,\chi_1}+C(1+|\tau|)^{-3/2+(3-\gamma-s)}\|\Omega^{\beta'}L_j u^2(\tau)\|_{1,2}\\ &\le C\left(\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\tau)\|_{3/(3-\gamma-s),\chi_1}\right)\left(\sum_{\substack{1\le k\le 3\\ |\beta|\le 2}}\|L_k\Omega^{\beta}u(\tau)\|_2\right)\\ &\quad + C(1+|\tau|)^{-(\gamma+s-3/2)}\left(\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\tau)\|_2\right)\left(\sum_{\substack{1\le k\le 3\\ |\beta|\le 2}}\|L_k\Omega^{\beta}u(\tau)\|_2\right)\\ &\le C(1+|\tau|)^{-(\gamma+s-3/2)}\left(\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\tau)\|_{\Gamma,1,2}\right)\left(\sum_{\substack{1\le k\le 3\\ |\beta|\le 2}}\|L_k\Omega^{\beta}u(\tau)\|_2\right).\end{aligned}$$

At the last inequality we have used (3.4). Finally we employ (3.5) to show

$$(4.44)\qquad \|\Omega^{\beta-\beta'}u(\tau)\|_{p_2}\le C(1+|\tau|)^{-\gamma(p_2)}\|\Omega^{\beta-\beta'}u(\tau)\|_{\Gamma,1,2}.$$

Since $(\gamma+s-3/2)+\gamma(p_2)=s/3+\gamma-7/6$, (4.41) follows from (4.42)–(4.44). □

As for the fourth term on the right-hand side of (4.18), we can prove for $2<\gamma<3$

$$(4.45)\qquad \|(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2)\Omega^{\beta-\beta'}u; L^1(\boldsymbol{R};L^2)\|\le C\|u\|_Z^3$$

by repeating essentially the same argument as in the proof of Proposition

4.6. The only difference is in the use of (3.5) instead of (3.10). We omit the proof of (4.45).

It is also easy to prove by the standard argument that the $L^\infty(\boldsymbol{R}; L^2)$ norm of the third term on the right-hand side of (4.18) is estimated from above by $C\|f\|_X^3$.

As far as (4.18) is concerned, only the first term on the right remains to be estimated. Observing

$$\Omega^\beta(x_j(\omega^{-(3-\gamma)}f^2)f) = \sum_{\beta' \le \beta} \binom{\beta}{\beta'} (\omega^{-(3-\gamma)}\Omega^{\beta'}f^2)\Omega^{\beta-\beta'}(x_j f), \tag{4.46}$$

we need to bound the $\dot{H}^{-1}$ norm of $(\omega^{-(3-\gamma)}\Omega^{\beta'}f^2)\Omega^{\beta-\beta'}(x_j f)$. Let p_1 be the number defined by $2/3 = 1/p_1 - (3-\gamma)/3$. Then we see that $p_1 > 1$ for $\gamma > 2$ and $1/2p_1 \ge 1/6$ for $\gamma < 3$. Hence we get by using the Sobolev embedding and the Hölder inequality

$$\begin{aligned} (4.47)\quad & \|(\omega^{-(3-\gamma)}\Omega^{\beta'}f^2)\Omega^{\beta-\beta'}(x_j f)\|_{\dot{H}^{-1}} \\ & \le C\|(\omega^{-(3-\gamma)}\Omega^{\beta'}f^2)\Omega^{\beta-\beta'}(x_j f)\|_{6/5} \\ & \le C\|\omega^{-(3-\gamma)}\Omega^{\beta'}f^2\|_{3/2}\|\Omega^{\beta-\beta'}(x_j f)\|_6 \\ & \le C\sum_{\beta'' \le \beta'} \binom{\beta'}{\beta''} \|\Omega^{\beta''}f\|_{2p_1}\|\Omega^{\beta'-\beta''}f\|_{2p_1}\|\Omega^{\beta-\beta'}(x_j f)\|_6 \le C\|f\|_X^3. \end{aligned}$$

At the last inequality we have used (3.7) to estimate $\|\Omega^{\beta-\beta'}(x_j f)\|_6$. Hence the estimate for the first term on the right-hand side of (4.18) is complete and we have shown

$$\sum_{\substack{1 \le j \le 3 \\ |\beta| \le 2}} \sup_{t \in \boldsymbol{R}} \|\Omega^\beta L_j I_0[u](t)\|_2 \le C\|f\|_X^3 + C\|u\|_Z^3 \tag{4.48}$$

for $u \in Z_{2C_3 A}$, providing that $2 < \gamma < 5/2$.

Let us turn our attention to the estimate for $\Omega^\beta \Omega_{kl} I_0[u]$ and $\Omega^\beta L_0 I_0[u]$. In view of (4.19)–(4.20) we get for $2 < \gamma < 5/2$

$$\sum_{\substack{1 \le k \le l \le 3 \\ |\beta| \le 2}} \sup_{t \in \boldsymbol{R}} \|\Omega^\beta \Omega_{kl} I_0[u](t)\|_2 + \sum_{|\beta| \le 2} \sup_{t \in \boldsymbol{R}} \|\Omega^\beta L_0 I_0[u](t)\|_2 \le C\|u\|_Z^3 \tag{4.49}$$

by repeating the same argument as in the proof of Propositions 4.7–4.8. We omit the proof of (4.49).

It remains to show

$$\sum_{\substack{0\le a\le 3\\ |\beta|\le 2}} \sup_{t\in\boldsymbol{R}} \|\Omega^\beta \partial_a I_0[u](t)\|_2 \le C\|u\|_Z^3, \tag{4.50}$$

$$\sum_{|\beta|\le 2} \sup_{t\in\boldsymbol{R}} \|\Omega^\beta \omega^{-1}\partial_t I_0[u](t)\|_2 \le C\|u\|_Z^3. \tag{4.51}$$

Following the proof of Proposition 4.8, we can easily show (4.51). The proof of (4.50) is easy. Thus we omit it. Hence it has been shown that

$$\|Mu\|_Z \le \|u_0\|_Z + \|I_0[u]\|_Z \le C_3\Lambda + C\|f\|_X^3 + C\|u\|_Z^3 \tag{4.52}$$

for $u \in Z_{2C_3\Lambda}$. It follows from (4.52) that M carries $Z_{2C_3\Lambda}$ into itself, provided that Λ is sufficiently small. Repeating the same argument as above, we can show that M satisfies (4.2) and hence is the contraction mapping. Thus we have completed the proof of Proposition 4.1.

Our next task is to prove (2.10)–(2.11). Define $u^\pm(t)$ by

$$u^\pm(t) = u(t) - \int_t^{\pm\infty} \frac{\sin\omega(t-\tau)}{\omega}(V_\gamma * u^2(\tau))u(\tau)d\tau \tag{4.53}$$

for the solution u to (2.8). The second term on the right-hand side of (4.53) is well-defined in $BC(\boldsymbol{R}; H^1)$. We prove the following proposition.

Proposition 4.9. *Let C be the same constant as in Proposition* 4.3. *Then $u^\pm$ defined in* (4.53) *satisfy*

$$(\Gamma^\alpha\Omega^\beta u^\pm, \Omega^\beta\omega^{-1}\partial_t u^\pm) \in BC(\boldsymbol{R}; L^2\times L^2), \qquad (|\alpha|\le 1, |\beta|\le 2). \tag{4.54}$$

Moreover, the following equalities hold in $BC(\boldsymbol{R}; L^2)$:

(4.55) $\Omega^\beta L_j u^\pm(t)$

$$= \Omega^\beta L_j u(t)$$

$$- C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_t^{\pm\infty}\frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}L_j u^2(\tau))\Omega^{\beta-\beta'}u(\tau)]d\tau$$

$$+ C(3-\gamma)\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_t^{\pm\infty}\cos\omega(t-\tau)[(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\Omega^{\beta-\beta'}u(\tau)]d\tau$$

$$- C(3-\gamma)\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_t^{\pm\infty}\frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(5-\gamma)}\Omega^{\beta'}\partial_j u^2(\tau))\Omega^{\beta-\beta'}\partial_\tau u(\tau)]d\tau$$

$$- C\sum_{\beta'\le\beta}\binom{\beta}{\beta'}\int_t^{\pm\infty}\frac{\sin\omega(t-\tau)}{\omega}[(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\tau))\Omega^{\beta-\beta'}L_j u(\tau)]d\tau,$$

(4.56) $\Omega^\beta \Omega_{kl} u^\pm(t)$

$$= \Omega^\beta \Omega_{kl} u(t)$$

$$- C \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \int_t^{\pm\infty} \frac{\sin \omega(t-\tau)}{\omega} [(\omega^{-(3-\gamma)} \Omega^{\beta'} \Omega_{kl} u^2(\tau)) \Omega^{\beta-\beta'} u(\tau)] d\tau$$

$$- C \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \int_t^{\pm\infty} \frac{\sin \omega(t-\tau)}{\omega} [(\omega^{-(3-\gamma)} \Omega^{\beta'} u^2(\tau)) \Omega^{\beta-\beta'} \Omega_{kl} u(\tau)] d\tau,$$

(4.57) $\Omega^\beta L_0 u(t)$

$$= \Omega^\beta L_0 u^\pm(t)$$

$$- C \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \int_t^{\pm\infty} \frac{\sin \omega(t-\tau)}{\omega} [(\omega^{-(3-\gamma)} \Omega^{\beta'} L_0 u^2(\tau)) \Omega^{\beta-\beta'} u(\tau)] d\tau$$

$$- C(5-\gamma) \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \int_t^{\pm\infty} \frac{\sin \omega(t-\tau)}{\omega} [(\omega^{-(3-\gamma)} \Omega^{\beta'} u^2(\tau)) \Omega^{\beta-\beta'} u(\tau)] d\tau$$

$$- C \sum_{\beta' \le \beta} \binom{\beta}{\beta'} \int_t^{\pm\infty} \frac{\sin \omega(t-\tau)}{\omega} [(\omega^{-(3-\gamma)} \Omega^{\beta'} u^2(\tau)) \Omega^{\beta-\beta'} L_0 u(\tau)] d\tau.$$

Proof. We begin with the proof of (4.55). In view of (4.11), (4.14) and (4.18) we have only to show

$$\left\| \frac{\sin \omega(t-\sigma)}{\omega} \Omega^\beta [x_j (\omega^{-(3-\gamma)} u^2(\sigma)) u(\sigma)] \right\|_2 \to 0, \tag{4.58}$$

$$\left\| \frac{\sin \omega(t-\sigma)}{\omega} [(\omega^{-(5-\gamma)} \Omega^{\beta'} \partial_j u^2(\sigma)) \Omega^{\beta-\beta'} u(\sigma)] \right\|_2 \to 0 \tag{4.59}$$

as $\sigma \to \pm\infty$. First we prove (4.58). Since $[\Omega_{kl}, x_j] = x_k \delta_{jl} - x_l \delta_{jk}$, it suffices to show for $1 \le k \le 3$, $\beta' + \beta'' \le \beta$

$$\| x_k (\omega^{-(3-\gamma)} \Omega^{\beta'} u^2(\sigma)) \Omega^{\beta''} u(\sigma) \|_{\dot{H}^{-1}} \to 0, \qquad (\sigma \to \pm\infty). \tag{4.60}$$

Since $2 < \gamma < 5/2$, we can pick η_1 $(0 < \eta_1 < 1)$ sufficiently close to 1 such that $\eta_1 \ge \gamma - 3/2$, $\eta_1 > 5 - 2\gamma$ and $0 < 1 - \eta_1 < 2(\gamma - 2)$. Put $\eta_2 := 1 - \eta_1$. Recalling $V_\gamma * (\Omega^{\beta'} u^2) = C\omega^{-(3-\gamma)} \Omega^{\beta'} u^2$, we easily see that the norm in (4.60) is estimated as

$$\cdots \le C\| \, |x|(\omega^{-(3-\gamma)}\Omega^{\beta'}u^2(\sigma))\Omega^{\beta''}u(\sigma)\|_{6/5} \tag{4.61}$$
$$\le C\|(\omega^{-(3-\gamma+\eta_1)}|\Omega^{\beta'}u^2(\sigma)|)|x|^{\eta_2}\Omega^{\beta''}u(\sigma)\|_{6/5}$$
$$+ C\|(\omega^{-(3-\gamma)}(|x|^{\eta_1}|\Omega^{\beta'}u^2(\sigma)|))|x|^{\eta_2}\Omega^{\beta''}u(\sigma)\|_{6/5}.$$

To estimate the first term on the last line in (4.61) we put $s := 1/2 - 3\eta_2/2$ and pick p_1, p_2 such that $1/p_1 = 1/2 - s/3$, $1/p_1 + 1/p_2 = 5/6$. Since the equality $\eta_2 = 2(1/2 - 1/p_2)$ holds, we have

$$\|(\omega^{-(3-\gamma+\eta_1)}|\Omega^{\beta'}u^2(\sigma)|)|x|^{\eta_2}\Omega^{\beta''}u(\sigma)\|_{6/5} \tag{4.62}$$
$$\le \|\omega^{-(3-\gamma+\eta_1)}|\Omega^{\beta'}u^2(\sigma)|\,\|_{p_1}\| \, |x|^{2(1/2-1/p_2)}\Omega^{\beta''}u(\sigma)\|_{p_2}$$
$$\le C\|\omega^{-(3-\gamma+\eta_1-s)}|\Omega^{\beta'}u^2(\sigma)|\,\|_2\|\Omega^{\beta''}u(\sigma)\|_{\Gamma,1,2}.$$

At the last inequality we were able to use (3.5). Moreover, since the inequality $1/2 < 3 - \gamma + \eta_1 - s < 3/2$ is true, we apply the inequality (3.13) with $\delta = 0$ to get

$$\|\omega^{-(3-\gamma+\eta_1-s)}|\Omega^{\beta'}u^2(\sigma)|\,\|_2 \tag{4.63}$$
$$\le C\|\Omega^{\beta'}u^2(\sigma)\|_{p_3,\chi_1} \qquad \left(\frac{1}{2} = \frac{1}{p_3} - \frac{3-\gamma+\eta_1-s}{3}\right)$$
$$+ C(1+|\sigma|)^{-3/2+(3-\gamma+\eta_1-s)}\|\Omega^{\beta'}u^2(\sigma)\|_{1,2}$$
$$\le C(1+|\sigma|)^{-(2(\gamma-2)-\eta_2)/2}\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\sigma)\|^2_{\Gamma,1,2}.$$

At the last inequality we were able to use (3.4). Since $2(\gamma-2)-\eta_2 > 0$, it follows from (4.62)–(4.63) that the first term on the last line in (4.61) converges to zero as $\sigma \to \pm\infty$.

We turn our attention to the second term on the last line in (4.61). Recall that η_1 has been picked sufficiently close to one such that $\eta_1 \ge \gamma - 3/2$, $\eta_1 > 5 - 2\gamma$ and $0 < 1 - \eta_1 < 2(\gamma - 2)$. Moreover, we have set $\eta_2 := 1 - \eta_1$. Now we take s as $s := -\gamma + 3/2 + \eta_1$ and pick p_1, p_2 such that $1/p_1 = 1/2 - s/3$, $1/p_1 + 1/p_2 = 5/6$. Since the inequalities $2\alpha(p_2) - \eta_2 > 0$ and $1/6 \le 1/p_2 \le 1/2$ are satisfied, it follows from the inequality (3.5)

$$\|(\omega^{-(3-\gamma)}(|x|^{\eta_1}|\Omega^{\beta'}u^2(\sigma)|))|x|^{\eta_2}\Omega^{\beta''}u(\sigma)\|_{6/5} \tag{4.64}$$
$$\le \|\omega^{-(3-\gamma)}(|x|^{\eta_1}|\Omega^{\beta'}u^2(\sigma)|)\|_{p_1}\| \, |x|^{\eta_2}\Omega^{\beta''}u(\sigma)\|_{p_2}$$
$$\le C(1+|\sigma|)^{-2\alpha(p_2)+\eta_2}\|\omega^{-(3-\gamma)}(|x|^{\eta_1}|\Omega^{\beta'}u^2(\sigma)|)\|_{p_1}\|\Omega^{\beta''}u(\sigma)\|_{\Gamma,1,2}.$$

Observe that the inequality $1/2 < 3 - \gamma - s < 3/2$ and the equality $\eta_1 = 3/2 - (3 - \gamma - s)$ hold. Employing the inequality (3.13) with $\delta = \eta_1$, we continue the estimate as

$$
\begin{aligned}
(4.65) \quad & \|\omega^{-(3-\gamma)}(|x|^{\eta_1}|\Omega^{\beta'}u^2(\sigma)|)\|_{p_1} \\
& \le C\|\omega^{-(3-\gamma-s)}(|x|^{\eta_1}|\Omega^{\beta'}u^2(\sigma)|)\|_2 \\
& \le C(1+|\sigma|)^{\eta_1}\|\Omega^{\beta'}u^2(\sigma)\|_{p_3,\chi_1} + C\|\Omega^{\beta'}u^2(\sigma)\|_{1,2} \quad \left(\frac{1}{2} = \frac{1}{p_3} - \frac{3-\gamma-s}{3}\right) \\
& \le C\sum_{|\beta|\le 2}\|\Omega^{\beta}u(\sigma)\|_{\Gamma,1,2}^2.
\end{aligned}
$$

At the last inequality we have employed (3.4) with $p = 2p_3$. Combining (4.64)–(4.65), we have shown that the second term on the last line in (4.61) also converges to zero as $\sigma \to \pm\infty$. Therefore the proof of (4.58) is complete. Next we need to show (4.59). However the proof of (4.59) is much easier than that of (4.58). Indeed we have only to repeat essentially the same argument as in the proof of Propositions 4.5–4.8. Thus the proof of (4.59) is omitted and we have finished the proof of (4.55). The proof (4.56)–(4.57) is much easier, therefore we omit it. Further the proof of the fact $\partial_a\Omega^\beta u^\pm$, $\Omega^\beta\omega^{-1}\partial_t u \in BC(\boldsymbol{R}; L^2)$ $(a = 0,\dots,3, |\beta| \le 2)$ is trivial. Hence it is also omitted. We have finished the proof of Proposition 4.9. □

Now we are ready to show (2.10). It follows from the definition of (4.53) that for $t \in \boldsymbol{R}$

$$
(4.66) \qquad \|(\Omega^\beta u^\pm(t), \Omega^\beta\partial_t u^\pm(t))\|_{L^2\times\dot{H}^{-1}} \le C\Lambda, \qquad (|\beta| \le 2).
$$

Moreover, it also follows from (3.6) and Proposition 4.9 that for $t \in \boldsymbol{R}$, $|\beta| \le 2$

$$
\begin{aligned}
(4.67) \qquad & \|\langle|t| - |\cdot|\rangle\nabla\Omega^\beta u^\pm(t)\|_2, \ \|\langle|t| - |\cdot|\rangle\Omega^\beta\partial_t u^\pm(t)\|_2 \\
& \le C\|\Omega^\beta u^\pm(t)\|_{\Gamma,1,2} \le C\Lambda.
\end{aligned}
$$

In particular, taking $t = 0$, we get (2.10). (2.11) is an immediate consequence of (4.53), (4.55)–(4.57). Therefore the proof of Theorem 1 is complete. □

5. Proof of Theorem 2

The proof of Theorem 2 is quite similar to that of Theorem 1. Thus we only sketch the proof. For any $(f_-, g_-) \in X \times Y$ let us consider the integral equation (2.12). The set W_δ $(\delta > 0)$ of functions defined in $(-\infty, 0] \times \boldsymbol{R}^3$ is introduced as follows:

$$W_\delta := \left\{ u = u(t,x) \mid \Gamma^\alpha \Omega^\beta u \in C((-\infty,0];L^2), (|\alpha| \le 1, |\beta| \le 2), \right.$$

$$\Omega^\beta \partial_t u \in C((-\infty,0];\dot{H}^{-1}), (|\beta| \le 2),$$

$$\left. \|u\|_W = \sum_{\substack{|\alpha|\le 1\\ |\beta|\le 2}} \sup_{t\in \boldsymbol{R}^-} \|\Gamma^\alpha \Omega^\beta u(t,\cdot)\|_2 + \sum_{|\beta|\le 2} \sup_{t\in \boldsymbol{R}^-} \|\Omega^\beta \omega^{-1}\partial_t u(t,\cdot)\|_2 \le \delta \right\}.$$

W_δ is nonempty, complete metric space with the metric $\|u - v\|_W$ $(u, v \in W_\delta)$. Denoting $\|(f_-, g_-)\|_{X\times Y}$ by Λ, we easily see from Proposition 3.1 that $u_- \in W_{C_3\Lambda}$ for a suitable constant C_3. For the proof of Theorem 2 we define the mapping

$$(5.1)\quad M : u \mapsto Mu = u_-(t) - I_{-\infty}[u](t)$$

$$\equiv u_-(t) - \int_{-\infty}^t \frac{\sin\omega(t-\tau)}{\omega}(V_\gamma * u^2(\tau))u(\tau)d\tau, \qquad (u \in W_{2C_3\Lambda}).$$

As in the proof of Theorem 1, it can be shown that the mapping M carries $W_{2C_3\Lambda}$ into itself and is the contraction mapping. Therefore it has a unique fixed point u in $W_{2C_3\Lambda}$ which is a solution to the integral equation (2.12). Following the argument in (4.66)–(4.67), we can easily prove (2.15).

It remains to prove the required uniqueness. We shall be on the same lines as in Reed [20] (see [20] on page 75). Let u and v be the solutions to the integral equation (2.12) verifying (2.13)–(2.14). By modifying the contraction argument slightly, we can take $T < 0$ with $|T|$ large such that

$$(5.2)\qquad \sup_{t\in(-\infty,T)} \|(\Gamma^\alpha\Omega^\beta u(t) - \Gamma^\alpha\Omega^\beta v(t), \Omega^\beta\omega^{-1}\partial_t u(t) - \Omega^\beta\omega^{-1}\partial_t v(t)\|_{L^2\times L^2} = 0.$$

Therefore $u = v$ on $(-\infty, T]$. Next we need to extend this uniqueness to the whole interval $(-\infty, 0]$. Note that u is a solution to the integral equation

$$(5.3)\qquad u(t) = (\cos\omega(t-T))v(T) + \frac{\sin\omega(t-T)}{\omega}\partial_t v(T)$$

$$- \int_T^t \frac{\sin\omega(t-\tau)}{\omega}(V_\gamma * u^2(\tau))u(\tau)d\tau \qquad \text{in } (T,0)\times \boldsymbol{R}^3.$$

Since v is also a solution to (5.3), it is possible to extend the uniqueness to the whole interval $(-\infty, 0]$ by the standard argument. Hence we have finished the proof of Theorem 2. □

6. Blow up

In this section we use the following result concerning the local solutions to the Cauchy problem

$$\partial_t^2 u - \Delta u + (V_\gamma * u^2)u = 0, \qquad t \in (0, T), \qquad x \in \boldsymbol{R}^3, \tag{6.1}$$

$$u(0, x) = f(x), \qquad \partial_t u(0, x) = g(x), \tag{6.2}$$

where $V_\gamma(x) = \lambda|x|^{-\gamma}$ as before.

Proposition 6.1. *Let* $0 < \gamma < 3$ *and* $(f, g) \in H^1 \times L^2$.

(1) *There exists a maximal interval* $I = (0, T_+)$ *and a unique solution* $(u, \partial_t u) \in C(I; H^1 \times L^2)$ *to the problem* (6.1)–(6.2). *If* $T_+ < \infty$, *then*

$$\|u(t)\|_{H^1} + \|\partial_t u(t)\|_2 \to \infty \qquad (t \to T_+). \tag{6.3}$$

(2) *If initial data* (f, g) *is spherically symmetric, then* $u(t, \cdot)$ *is also spherically symmetric as long as it exists.*

(3) *If* $\operatorname{supp} f \cup \operatorname{supp} g \subset \{|x| < R\}$ *for some* $R > 0$, *then* $\operatorname{supp} u(t, \cdot) \subset \{|x| < R + t\}$ *for any* $t < T_+$.

(4) *Let* k *be a non-negative integer. If* $(f, g) \in H^{k+1} \times H^k$, *then the solution given by* (1) *satisfies*

$$(u, \partial_t u, \partial_t^2 u) \in C(I; H^{k+1} \times H^k \times H^{k-1}). \tag{6.4}$$

(5) *Let* $\lambda < 0$ *and* $f \equiv 0$. *If* $g \in H^4$ *and* $g \geq 0$, *then* $u(t, x) \geq 0$ *for all* $(t, x) \in I \times \boldsymbol{R}^3$.

Proof. The results of (1), (3)–(4) are proved in Menzala–Strauss [16] (see Theorems 1, 3 and 4 in [16]). The statement (5) follows from (6.4) and Keller's comparison theorem for classical solutions [9]. Finally, the statement (2) is an immediate consequence of (1) and the invariance of the equation (6.1) under rotation. □

We need the following lemma.

Lemma 6.1 ([22, Lemma 4]). *Suppose that* $F(t) \in C^2[a, b)$ *and that*

$$F(t) \geq C_1(t+1)^r, \qquad F''(t) \geq C_2(t+1)^{-q} F(t)^p$$

for $t \in [a, b)$, *where* C_1, C_2 *are positive constants. If* $p > 1$, $r \geq 1$ *and* $(p-1)r > q - 2$, *then* b *must be finite.*

The second purpose of this paper is to prove the following theorem.

Theorem 6.1. *Suppose that* $0 < \gamma < 2$, $\lambda < 0$ *and* $f \equiv 0$. *Let* $g \in H^4$ *and* $g \geq 0$ $(\not\equiv 0)$. *Moreover, assume that* g *is spherically symmetric and* $\operatorname{supp} g \subset$

$\{|x| < R\}$ *for some* $R > 0$. *Then the maximal existence time* T_+ *given by* (1) *of Proposition* 6.1 *must be finite.*

Proof. Without loss of generality we may take $R = 1$ and $\lambda = -1$. Note that the corresponding solution is non-negative and $\operatorname{supp} u(t, \cdot) \subset \{|x| < t+1\}$ for any $t < T_+$. Assume $T_+ = \infty$. Then the function $\int_{\boldsymbol{R}^3} u(t,x)dx$ must be in the C^2-class for all time. However we shall get the contradiction.

Our proof is divided into two cases: the cases $0 < \gamma < 1$ and $1 \leq \gamma < 2$. We begin with the former. Since u vanishes outside $\{|x| < t+1\}$, we get by integrating (6.1) over $\boldsymbol{R}^3$

$$\frac{d^2}{dt^2}\int_{\boldsymbol{R}^3} u(t,x)dx = -\int_{\boldsymbol{R}^3} (V_\gamma * u^2)(t,x)u(t,x)dx. \tag{6.5}$$

Since $u(t,\cdot)$ is spherically symmetric, we may write $u(t,x)$ as $v(t,r)$ $(r = |x|)$. Then the right-hand side of (6.5) can be written as

$$\begin{aligned}
\cdots &= \int_{\boldsymbol{R}^3} \frac{2\pi}{r}\int_0^\infty \int_{|r-\eta|}^{r+\eta} \rho^{1-\gamma}\eta v^2(t,\eta)d\rho d\eta v(t,r)dx \\
&= 8\pi^2 \int_0^{t+1}\int_r^{t+1} \eta v^2(t,\eta)\left(\frac{1}{r}\int_{\eta-r}^{\eta+r} \rho^{1-\gamma}d\rho\right)d\eta v(t,r)r^2dr \\
&\quad + 8\pi^2 \int_0^{t+1}\int_0^{r} \eta v^2(t,\eta)\left(\frac{1}{r}\int_{r-\eta}^{r+\eta} \rho^{1-\gamma}d\rho\right)d\eta v(t,r)r^2dr.
\end{aligned} \tag{6.6}$$

Since $\gamma < 1$, we easily get

$$\frac{1}{r}\int_{\eta-r}^{\eta+r} \rho^{1-\gamma}d\rho \geq 2(\eta-r)^{1-\gamma}, \qquad \frac{1}{r}\int_{r-\eta}^{r+\eta} \rho^{1-\gamma}d\rho \geq \frac{2\eta}{r}(r-\eta)^{1-\gamma}. \tag{6.7}$$

We therefore obtain from (6.5)–(6.7)

$$\frac{d^2}{dt^2}\int_{\boldsymbol{R}^3} u(t,x)dx \geq \frac{C}{t+1}\int_0^{t+1}\int_0^{t+1} \eta^2 v^2(t,\eta)|r-\eta|^{1-\gamma}d\eta v(t,r)r^2dr. \tag{6.8}$$

By an elementary computation we see

$$\begin{aligned}
&\int_0^{t+1} |r-\eta|^{-(1-\gamma)}\eta^2 d\eta \\
&\quad = \int_0^{r/2} (r-\eta)^{-(1-\gamma)}\eta^2 d\eta + \int_{r/2}^{r} (r-\eta)^{-(1-\gamma)}\eta^2 d\eta + \int_r^{t+1} (\eta-r)^{-(1-\gamma)}\eta^2 d\eta \\
&\quad \leq C(t+1)^{2+\gamma}.
\end{aligned} \tag{6.9}$$

Hence we get by the Hölder inequality and (6.9)

$$\int_0^{t+1} \eta^2 v^2(t,\eta)|r-\eta|^{1-\gamma} d\eta \geq C(t+1)^{-(2+\gamma)} \left(\int_0^{t+1} \eta^2 v(t,\eta) d\eta \right)^2. \tag{6.10}$$

Inserting (6.10) into (6.8), we finally obtain for all $t > 0$

$$\frac{d^2}{dt^2} \int_{\boldsymbol{R}^3} u(t,x)dx \geq C(t+1)^{-(3+\gamma)} \left(\int_{\boldsymbol{R}^3} u(t,x)dx \right)^3. \tag{6.11}$$

Set $F(t) = \int_{\boldsymbol{R}^3} u(t,x)dx$. Since $F''(t) \geq 0$ for all $t > 0$, we get $F(t) \geq t\|g\|_{L^1}$ by integrating $F''(t)$ twice. Thus these exists $T_1 > 0$ such that the inequalities

$$F(t) \geq C(t+1), \qquad F''(t) \geq C(t+1)^{-(3+\gamma)} F(t)^3$$

hold for all $t \in [T_1, \infty)$. However this contradicts Lemma 6.1 because of $\gamma < 1$ and hence the maximal existence time T_+ is finite. We have completed the proof of the blow-up for the case $0 < \gamma < 1$.

Next we prove the blow-up for the case $1 \leq \gamma < 2$. Our first task is to modify the inequalities in (6.7)–(6.8). Since $1 - \gamma \leq 0$, we get

$$\frac{1}{r} \int_{\eta-r}^{\eta+r} \rho^{1-\gamma} d\rho \geq \frac{2}{(\eta+r)^{\gamma-1}}, \qquad \frac{1}{r} \int_{r-\eta}^{r+\eta} \rho^{1-\gamma} d\rho \geq \frac{2\eta}{r(r+\eta)^{\gamma-1}}. \tag{6.12}$$

Inserting (6.12) into (6.6) and noting $r, \eta \leq t+1$, we have

$$\frac{d^2}{dt^2} \int_{\boldsymbol{R}^3} u(t,x)dx \geq \frac{C}{(t+1)^\gamma} \int_{\boldsymbol{R}^3} u^2(t,x)dx \int_{\boldsymbol{R}^3} u(t,x)dx. \tag{6.13}$$

Next it is necessary to show $\|u(t,\cdot)\|_2 \geq C > 0$ for all $t \geq 1$. We follow the argument in [6], [22] (see [22] on page 383). Since u is non-negative and the fundamental solution is positive, the inequality $u(t,x) \geq u_0(t,x) (= (\omega^{-1} \sin \omega t) g)$ is true for all $(t,x) \in I \times \boldsymbol{R}^3$. Hence we have by the Hölder inequality for $t \geq 1$

$$\int_{t-1 \leq |x| \leq t+1} u_0(t,x)dx \leq \int_{t-1 \leq |x| \leq t+1} u(t,x)dx \leq C(t+1)\|u(t,\cdot)\|_2. \tag{6.14}$$

On the other hand, we get

$$\frac{d^2}{dt^2} \int_{\boldsymbol{R}^3} u_0(t,x)dx = 0 \tag{6.15}$$

by integrating $\Box u_0 = 0$ over $\boldsymbol{R}^3$. Integrating (6.15) twice, we have

$$\int_{\boldsymbol{R}^3} u_0(t,x)dx = t \int_{\boldsymbol{R}^3} g(x)dx. \tag{6.16}$$

Since $\operatorname{supp} u_0(t,\cdot) \subset \{t-1 \le |x| \le t+1\}$ by the Huygens principle, it follows from (6.14) and (6.16) that

$$\|u(t,\cdot)\|_2 \ge \frac{Ct}{t+1} \ge \frac{C}{2} \tag{6.17}$$

for $t \ge 1$.

Therefore we have shown by (6.13) and (6.17) that for $t \ge 1$

$$F''(t) \ge C(t+1)^{-\gamma}F(t), \tag{6.18}$$

where $F(t) = \int_{R^3} u(t,x)dx$ as before. Since $F'(0) = \int g(x)dx > 0$ and $F''(t) \ge 0$, we see $F'(t) \ge 0$ for all $t \ge 0$. Thus, multiplying (6.18) by $F'(t)$, we see $(F'(t)^2)' \ge C((t+1)^{-\gamma}F(t)^2)'$ for $t \ge 1$. Integrating this inequality from 1 to t, we get

$$F'(t)^2 \ge C(t+1)^{-\gamma}F(t)^2 + F'(1)^2 - C2^{-\gamma}F(1)^2. \tag{6.19}$$

Since $\gamma < 2$ and

$$F(t) \ge \int_{R^3} u_0(t,x)dx = t\int_{R^3} g(x)dx,$$

we see that the right-hand side of (6.19) is estimated from below by $(C/2)(t+1)^{-\gamma}F(t)^2$ for large t. Hence the inequality

$$F'(t) \ge C(t+1)^{-\gamma/2}F(t) \tag{6.20}$$

holds for large t. This implies

$$F(t) \ge \exp[C(t+1)^{1-\gamma/2}] \tag{6.21}$$

for large t. Inserting (6.21) into (6.13), we get

$$\begin{aligned} \frac{d^2}{dt^2}\int_{R^3} u(t,x)dx &\ge C(t+1)^{-\gamma}\exp[C(t+1)^{1-\gamma/2}]\int_{|x|\le t+1} u^2(t,x)dx \\ &\ge C(t+1)^{-\gamma-3}\exp[C(t+1)^{1-\gamma/2}]F(t)^2 \end{aligned} \tag{6.22}$$

for large t. At the last inequality we have used the Hölder inequality. Since $1-\gamma/2 > 0$ for $\gamma < 2$, we have $F''(t) \ge CF(t)^2$ for large t. As before, this implies $F'(t) \ge CF(t)^{3/2}$ for large t. However, $F(t)$ must become infinity in a finite time. This is a contradiction. Thus the maximal interval $(0, T_+)$ must be finite. Therefore we have completed the proof of the blow-up. □

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Appendix

Here we prove the inequality (3.15). It is convenient to use the spherically symmetric Paley–Littlewood dyadic decompositions which are defined in the following way. Let $\psi \in C_0^\infty(\boldsymbol{R}^n)$ be spherically symmetric with $0 \le \psi \le 1$, $\psi(x) = 1$ for $|x| \le 1$ and $\psi(x) = 0$ for $|x| \ge 2$. Define radial functions $\varphi_0(x) = \psi(x) - \psi(2x)$ and $\varphi_j(x) = \varphi_0(2^{-j}x)$ for $j = \pm 1, \pm 2, \dots$. Then it is easily seen that $\operatorname{supp} \varphi_j \subset \{x \in \boldsymbol{R}^n \mid 2^{j-1} \le |x| \le 2^{j+1}\}$, $\varphi_j \le 1$ and

$$(*) \qquad \sum_{j=-\infty}^{\infty} \varphi_j(x) = 1 \qquad \text{for } |x| > 0.$$

In this sum there are at most two nonvanishing terms for every $|x|$. It is easy to observe that we get for $j = \pm 1, \pm 2, \dots$

$$\|2^{(n/2-s)j} \varphi_j v\|_{L^{\infty,2}} \le C\|v\|_{\dot{H}^s}, \qquad \frac{1}{2} < s < \frac{n}{2}$$

by proceeding in the same way that the inequality (2.45) in Li–Zhou [14] is obtained. In this inequality the constant C is completely independent of j. Recall that in the sum of $(*)$ there are at most two nonvanishing terms. Thus for every $r > 0$ we can choose j such that $r\zeta \in \operatorname{supp} \varphi_j$ and $\varphi_j(r\zeta) \ge 1/2$ for all $\zeta \in S^{n-1}$. Since $2^{j-1} \le |x| \le 2^{j+1}$ on $\operatorname{supp} \varphi_j$, the inequality $r^{n/2-s}\|v(r\cdot)\|_{L^2(S^{n-1})} \le C\|v\|_{\dot{H}^s}$ holds with the constant C independent of r. This immediately implies (3.15). □

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