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Weak Hyperbolicity of Delay Differential Equations and the 3/2-Type Stability Conditions

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1. Introduction

In the present paper we study the asymptotic stability of the Carathéodory scalar linear delay differential equation

(1) $$x'(t) + a(t)x(t) + b(t)x(t-\tau) = 0.$$

Our goal is to establish sufficient and "easily verifiable" conditions for such a stability. Throughout the paper we assume that $x \in \boldsymbol{R}$, $a(t) \geq 0$ and $b(t) \geq 0$ are measurable functions and $\tau > 0$ is a constant delay.

Note that conditions for the stability in the case of constant coefficients $a(t) \equiv a, b(t) \equiv b > 0$ are well known [5]. In particular, if $a \equiv 0$ the inequality $b\tau < \pi/2$ is necessary and sufficient for the asymptotic stability of equation (1). In the nonautonomous case a close to the latter sufficient condition in the form $\sup_{t \in \boldsymbol{R}} \int_t^{t+\tau} b(s)ds < 3/2$ is known as the 3/2-condition of Myshkis-Yorke [16] (see also [3, 5, 6, 7, 14, 15]). The following sharpest form of this result (for linear equations) was derived in [10] as an immediate consequence of Yorke theorem [16] by using an appropriate change of variables:

Theorem. *Let $b(t) \geq 0$ be locally integrable and $\tau = \tau(t) \geq 0$ be Lebesgue measurable, $t \in R_+$. Assume that $\int_{R_+} b(u)du = +\infty$ and denote by $C(t,s)$ the Cauchy operator*

$$x_t(\varphi, a) = C(t,a)\varphi : [a, +\infty) \to C = C[-\tau, 0]$$

of the initial value problem

(2) $$x'(t) + b(t)x(t-\tau) = 0, \qquad x_a(\varphi, a)(u) = \varphi(u), \qquad u \in [-\tau, 0].$$

Then there exist constants $N > 0, \gamma > 0$ depending on $b(t)$ only such that

$$|C(t,s)| \leq N \exp\left\{-\gamma \int_s^t b(s)ds\right\} \qquad \text{for all } t \geq s \geq 0,$$

provided

(3) $$\limsup_{t\to+\infty}\int_{t-\tau(t)}^{t} b(s)ds < \frac{3}{2} \quad \textit{and} \quad \liminf_{t\to+\infty}\int_{t-\tau(t)}^{t} b(s)ds > 0.$$

Moreover, if sign $<$ *in the first inequality of* (3) *is replaced by* $\le$ *and* $b(t) > 0$ *then the Cauchy operator of equation* (2) *is bounded.*

Note that a large number of previously known results about the asymptotic stability of equations (1) and (2) are proved by constructing appropriate Lyapunov functionals (see, for example, [3, 5] and further references therein). Some other methods were proposed in [1, 7, 17].

A new approach to the stability problem of equation (1) proposed in [4] is based on the use of methods of topological dynamics and exponential dichotomies for linear skew-product systems by Sacker and Sell [11, 12, 13]. In [4] a problem was posed to obtain sufficient conditions for the stability of equation (1) by applying the mentioned techniques. The present paper establishes new results in this direction some of which generalize the 3/2-condition.

2. Preliminaries

2.1 *Weak hyperbolicity of linear evolutionary systems*

For the reader's convenience we present in this subsection some basic definitions and facts adapted from [9, 12].

Let $\mathscr{E} = H \times X$ be given where X is a fixed Banach space (the state space) and H is a compact Hausdorff space. Assume that $\theta^t h = h \cdot t$ is a flow on H; that is, the mapping $(h,t) \to h \cdot t$ is continuous, $h \cdot 0 = h$, and one has $h \cdot (s+t) = (h \cdot s) \cdot t$ for all $s, t \in R$. A *linear evolutionary system* $\pi = (\theta, \Phi)$ on $\mathscr{E}$ is a mapping $\pi^t(G,x) = (G \cdot t, \Phi(G,t)x)$, defined for $t \ge 0$, with the following properties:

(1) $\Phi(G,0) = I$, the identity operator, for all $G \in H$.

(2) $\lim_{t\to+0} \Phi(G,t)x = x$, and this limit is uniform in G. This means that for every $x \in X$ and every $\varepsilon > 0$ there is a $\delta = \delta(x,\varepsilon) > 0$ such that $|\Phi(G,t)x - x| \le \varepsilon$, for all $G \in H$ whenever $0 \le t \le \delta$.

(3) $\Phi(G,t)$ is a bounded linear mapping from X to X such that for all $G \in H$, $0 \le s, t$ one has $\Phi(G, s+t) = \Phi(G \cdot t, s)\Phi(G,t)$.

(4) For each $t \ge 0$ the mapping of $\mathscr{E}$ into X given by $(G,x) \to \Phi(G,t)x$ is continuous.

The measure $\alpha(A)$ of noncompactness of a bounded set A in a Banach space X is defined by

$$\alpha(A) = \inf\{d : A \text{ has a finite covering with open sets of diameter } d\}.$$

A linear evolutionary system π is said to be *uniformly α-contracting* if for every bounded set $B = \{x \in X : |x| \le M\}$ there is a function k with $k(t) \to 0$ as $t \to \infty$ and such that $\alpha(\Phi(G,t)B) \le k(t)\alpha(B), G \in H$.

The following definition of the negative continuation generalizes the concept of a full trajectory for semidynamical system.

A point (G,x) is said to have a negative continuation if there exists a continuous function $\phi : (-\infty, 0] \to \mathscr{E}$ such that

(i) $\phi(s) = (G \cdot s, \phi^x(s)) \in \{G \cdot s\} \times X$ for each $s \le 0$;

(ii) $\phi(0) = (G,x)$;

(iii) $\pi^t(\phi(s)) = \phi(s+t)$ for all $s \le 0$, and $0 \le t \le -s$.

The following sets play a basic role in the Sacker-Sell theory:

$\mathscr{U}(G) = \{(G,x) \in \mathscr{E}$: there is a negative continuation ϕ of (G,x) such that $|\phi^x(t)| \to 0$ as $t \to -\infty\}$;

$\mathscr{B}^- = \{(G,x) \in \mathscr{E}$: there is a negative continuation ϕ of (G,x) such that $\sup_{t \le 0} |\phi^x(t)| < \infty\}$;

$$\mathscr{B}^+ = \left\{(G,x) \in \mathscr{E} : \sup_{t \ge 0} |\Phi(G,t)x| < \infty\right\}.$$

Now we are in a position to define the weak hyperbolicity of π and to present theorems necessary for our further exposition.

Definition. The linear evolutionary system $\pi^t(G,x) = (G \cdot t, \Phi(G,t)x)$ is said to be *weakly hyperbolic* on $\mathscr{E} = H \times X$ if π is uniformly α-contracting and $\mathscr{B}^+ \cap \mathscr{B}^- = H \times \{0\}$.

The Compatibility Theorem [12]. *Let $\pi = (\theta, \Phi)$ be a weakly hyperbolic linear evolutionary system on $\mathscr{E}$. Then π admits an exponential dichotomy over all minimal sets M_γ in H and* $\dim \mathscr{U}(G) = c_\gamma$ *is constant over every M_γ. Moreover, if* $\dim \mathscr{U}(G)$ *assumes the same value on all minimal sets in H, then π admits an exponential dichotomy over H.*

The Roughness Theorem [9]. *Assume that linear evolutionary systems $\pi_1 = (\theta, \Phi_1), \pi_2 = (\theta, \Phi_2)$ are defined over the flow $G \cdot t$ and π_1 has exponential dichotomy. Then there exists* $\varepsilon > 0$ *such that the inequality*

$$\sup_{G \in H} |\Phi_1(G,1) - \Phi_2(G,1)|_{L(X)} < \varepsilon$$

implies the exponential dichotomy of π_2.

2.2 *Linear evolutionary system associated with equation* (1)

The Cauchy operator $\omega(t,s) = \exp(-\int_s^t a(u)du)$ of equation $x' = -a(t)x$ will be denoted by $\omega(t,s)$. The notation $\omega(t,s,a)$ will be used if it will be necessary to explicitly indicate the dependence on $a(t)$.

$\mathrm{Cl}_X A$ stands for the closure of subset A of the topological space X. We use the notations $C, \boldsymbol{R}_+$ in the standard way: $\boldsymbol{R}_+ = [0, +\infty), C([-\tau, 0], \boldsymbol{R}) = C$.

In the sequel, we shall consider the uniform spaces $\mathscr{L}_q = \mathscr{L}_q(\boldsymbol{R}, \boldsymbol{R}_+)$, $1 \le q \le +\infty$. Algebraically each $\mathscr{L}_q(\boldsymbol{R}, \boldsymbol{R}_+)$ coincides with $L_{qloc}(\boldsymbol{R})$. If $1 \le q < +\infty$ then $\mathscr{L}_q(\boldsymbol{R}, \boldsymbol{R}_+)$ is endowed with the topology of weak convergence in $L_q(I)$ on compact intervals $I \subset \boldsymbol{R}$ (in other words, the generalized sequence $x_\alpha \to x_0$ in $\mathscr{L}_q$ if and only if $\int_I x_\alpha(u)z(u)du \to \int_I x_0(u)z(u)du$ for every compact interval $I \subset \boldsymbol{R}$ and for each $z \in L_p(I)$, where $p^{-1} + q^{-1} = 1$). $\mathscr{L}_\infty(\boldsymbol{R}, \boldsymbol{R}_+)$ is endowed with the weak *-topology in $L_\infty(I) = L_1(I)^*$ for any compact interval I. That is $x_\alpha \to x_0$ in $\mathscr{L}_\infty$ if and only if $\int_I x_\alpha(u)z(u)du \to \int_I x_0(u)z(u)du$ for each compact interval $I \subset \boldsymbol{R}$ and for each $z \in L_1(I)$. Partly the choice of such spaces is justified by Lemma 1 below.

The following basic "continuity" hypotheses on the coefficients $a(t), b(t)$: $\boldsymbol{R} \to \boldsymbol{R}_+ = [0, +\infty)$ of equation (1) will be assumed throughout the paper:

(H1) $a(t), b(t) \in L_{qloc}(R), 1 < q < +\infty$ (that is $a(t), b(t) \in L_q(I)$ for any compact interval $I \subset \boldsymbol{R}$) and there is a positive number c such that

$$\int_t^{t+1} (a^q(u) + b^q(u))du \le c \qquad \text{for all } t \in \boldsymbol{R}.$$

(H2) $a(t), b(t) \in L_\infty(\boldsymbol{R})$ with $\operatorname{ess\,sup}_{\boldsymbol{R}}(a(u) + b(u)) \le c$.

Note that under these conditions a unique Carathéodory solution $x_t(\varphi, a) : [a, +\infty) \to C$ of the initial value problem

$$x_a(\varphi, a)(s) = \varphi(s), \qquad s \in [-\tau, 0], \qquad \varphi \in C([-\tau, 0], R) = C$$

for equation (1) exists for all $t \ge a$. To prove this it is sufficient to use step by step continuation method and the existence and uniqueness theorem for the Carathéodory ordinary differential equations.

Every solution $x_t(\varphi, a)$ is associated with an ordinary solution (which is absolutely continuous function) $x(t, a, \varphi) : [a, +\infty) \to R_+$ by the relation $x(t + s, a, \varphi) = x_t(\varphi, a)(s), s \in [-\tau, 0]$.

Lemma 1. *Let* $\{a(t), b(t)\}$ *satisfy one of the assumptions* **H1** *or* **H2**. *Then the set of shifts* $\{a(t + h), b(t + h), \ h \in R\}$ *is precompact in* $\mathscr{L}_q$.

Proof. For the case **H1 H2**, see propositions IV.8.4, V.6.1 (or Corollary V.4.3, respectively) of [2]. □

With equation (1), or formally with the pair $F = (a(t), b(t)) \in \mathscr{L}_q \times \mathscr{L}_q$, we can associate the so-called hull [11, 12, 13] $H(F)$ consisting of all systems $G = (\mu(t), \nu(t)) \in \mathscr{L}_q \times \mathscr{L}_q$ of the form

$$x'(t) + \mu(t)x(t) + \nu(t)x(t - \tau) = 0, \tag{4}$$

where, in $\mathscr{L}_q$,

$$\mu(t) = \lim_{k\to+\infty} a(t+s_k), \qquad \nu(t) = \lim_{k\to+\infty} b(t+s_k)$$

for some sequence s_k of real numbers. The exact expression for the hull is

$$H(F) = \mathrm{Cl}_{\mathscr{L}_q\times\mathscr{L}_q}\{\theta^s F = (a(t+s), b(t+s)), s \in \boldsymbol{R}\}.$$

As the following Lemma shows the shift operator θ generates a continuous flow on $H = H(F)$.

Lemma 2. *Let* $a(t), b(t) \geq 0$ *satisfy one of the assumptions* **H1** *or* **H2**. *Then* $H(F) \subset \mathscr{L}_q$ *is a compact set and the map* $\theta^s F : \boldsymbol{R} \times H(F) \to H(F)$ *is continuous.*

Proof. The compactness of $H(F)$ follows from Lemma 1. Proof of the continuity of $\theta^s F$ in the case $H(F) \subset \mathscr{L}_\infty$ can be found in [13], section III.M. Here we consider in detail the case $H(F) \subset \mathscr{L}_q$, $q < \infty$.

To prove the continuity of $\theta^s F$ it is sufficient to show that on every compact interval I one has

$$\text{(5)} \qquad \sup_n \int_I a_n^p(u+h_n)du < \infty \qquad \text{and} \qquad \int_I a_n(u+h_n)du \to \int_I a_0(u)du$$

where $h_n \to 0$ and a_n is an arbitrary sequence weakly converging in $L_q(J)$ to a_0 (see [2], chapter IV, exercise 13.24). We shall prove the second part of (5) (the first inequality in (5) is an straightforward consequence of the hypothesis **H1**). We have

$$\int_I a_n(u+h_n)du = \int_J \mu_n(t)a_n(u)du = \langle \mu_n, a_n \rangle,$$

where $I_n = I - h_n \subset J$, $\mu_n(u)$ is the indicator function of I_n and $\langle \cdot, \cdot \rangle$ is a usual pairing operation in $L_p(J)$. Since $\mu_n \to \mu_0$ in the norm topology of $L_p(J)$ and $a_n \to a_0$ in the weak topology, we obtain that

$$\text{(6)} \qquad |\langle \mu_n, a_n \rangle - \langle \mu_0, a_0 \rangle| \leq |\langle \mu_n - \mu_0, a_n \rangle| + |\langle a_n - a_0, \mu_0 \rangle|$$
$$\leq C\|\mu_n - \mu_0\|_p + |\langle a_n - a_0, \mu_0 \rangle| \to 0,$$

which completes the proof of Lemma 2. □

Now let us consider the direct product $\mathscr{E} = H(F) \times C$. In a standard way, we define the linear semiflow $\pi^t : H(F) \times C \to H(F) \times C, t \geq 0$ by the formula

$$\pi^t(G, \varphi) = (\theta^t G, x_t(\varphi, 0, G)),$$

where $G = (\mu(t), \nu(t)) \in H(F)$ and $x_t(\varphi, 0, G)$ is the solution of equation (4) satisfying the initial condition $x_0(\varphi, 0, G) = \varphi$.

Lemma 3. *Let $a(t), b(t)$ satisfy either assumption* **H1** *or assumption* **H2**. *Then the semiflow π^t has the following continuity properties:*

(a) *$x_t(\varphi, 0, G) \to \varphi$ as $t \to +0$ uniformly in G for every φ;*

(b) *for each $t \geq 0$ the mapping $x_t(\varphi, 0, G) : H(F) \times C \to C$ is continuous;*

(c) *$x_t(B, 0, G)$ is a compact set for every bounded set $B \subset C$, each $t \geq \tau$ and $G \in H(F)$.*

Lemma 3 implies, according to the definitions from the previous subsection, that the linear evolutionary system π^t is uniformly α-contracting.

Proof of Lemma 3. (a) Let $\sigma_\phi(\delta)$ be a continuity modulus of $\phi(u)$. By the definition of $x_t(\phi, 0, G)(u)$ and the variation of constants formula we obtain for all positive values of $t \leq \tau$

$$x_t(\phi, 0, G)(u) = \begin{cases} \phi(t+u) \text{ if } u \in [-\tau, -t]; \text{ otherwise} \\ \omega(t, 0, \mu)\phi(0) - \int_0^t \omega(t, s, \mu)\nu(s)\phi(s-\tau)ds \end{cases}.$$

Therefore

$$\begin{aligned} &|x_t(\phi, 0, G)(u) - \phi(u)| \\ &\quad \leq \begin{cases} \sigma_\phi(t) \text{ if } u \in [-\tau, -t]; \text{ otherwise} \\ \sigma_\phi(t) + |\omega(t, 0, \mu) - 1|\,|\phi(0)| + \int_0^t \omega(t, s, \mu)\nu(s)|\phi(s-\tau)|ds \end{cases} \\ &\quad \leq \begin{cases} \sigma_\phi(t) \text{ if } u \in [-\tau, -t]; \\ \sigma_\phi(t) + |\phi|_C k(c) t^{1/p} \text{ if } u \in [-t, 0], \end{cases} \end{aligned}$$

where $k(c)$ depends only on the positive constant c from **H1** or **H2**.

(c) Assume $t = \tau, |B|_C \leq c_2$ Then using the variation of constant formula again we prove the boundness of $x_\tau(B, 0, G)$ in C, say by some positive constant c_1. Since $|x'_\tau(\phi, 0, G)(u)| \leq c_1\mu(u) + c_2\nu(u)$ for all $\phi \in B$ we deduce the compactness of $x_\tau(B, 0, G)$. The reasoning for $t \geq \tau$ is the same with the use of step by step continuation method.

(b) Let $\phi_n \to \phi_0$ in C and $G_n \to G_0$ in $H(F)$. Then, similarly to the arguments in part (c) above, we conclude that the set $x_t(\phi_n, 0, G_n)$ is compact in C for every fixed $t \geq 0$ (if $t \leq \tau$ we use the condition $\phi_n \to \phi_0$ to prove uniform convergence on $[-\tau, t-\tau]$). Next, by using the variation of constant formula, we easily obtain, by employing the Lebesgue dominated convergence theorem and relations analogous to (6), that the sequence $x_t(\phi_n, 0, G_n)(u), t \in [-\tau, 0]$ converges pointwise to $x_t(\phi_0, 0, G_0)(u)$. Taking into account the compactness of this sequence we conclude that the convergence must be uniform on $[-\tau, 0]$. Lemma 3 is proved. □

3. Main results: Case $a(t) \equiv 0$

We shall demonstrate first how our idea works in a simpler case of $a(t) \equiv 0$. By $\tau(s) \in [0, \tau]$ we denote the smallest real number r such that $\int_{s-r}^{s} b(u)du = 1$ if such a number exists and we set $\tau(s) = \tau$ otherwise.

Theorem 1. *Let* $b(t) \geq 0$ *satisfy either condition* **H1** *or condition* **H2** *and* $0 \notin H(b)$. *If additionally*

$$(7) \qquad \varepsilon = \sup_{s \in R} \left\{ \int_{s}^{s+\tau-\tau(s)} b(u)du + \int_{s+\tau-\tau(s)}^{s+\tau} b(u)du \int_{u-\tau}^{s} b(\sigma)d\sigma \right\} < 1,$$

then equation (2) *is uniformly exponentially stable. Equation* (2) *shares this property with all equations from* $H(b)$.

Proof. Clearly the inequality (7) holds for each $c \in H(b)$ with $b(u)$ replaced by $c(u)$. We claim that the corresponding equation

$$(8) \qquad x'(t) + c(t)x(t - \tau) = 0,$$

does not have bounded nontrivial solutions for any $c \in H(b)$.

Indeed, let $z(t) : R \to R$ be a bounded nonzero solution of equation (8) and $\zeta_t(u) = z(t + u), u \in [-\tau, 0]$. Then it follows from **H1**, **H2** that

$$\mathscr{K} = \text{Cl}\{(\theta^t c, \zeta_t), t \in R\} \subset H(b) \times C$$

is a compact invariant set. Thus there exist $w \in H(b)$ and some $\varphi \in C$ such that

$$(w, \varphi) \in \mathscr{K}, \quad 0 < M = |\varphi(0)| = \|\varphi\|_C = \max_{\psi}\{\|\psi\|_C : (e, \psi) \in \mathscr{K} \text{ for some } e\}.$$

Let, for example, $w = w(t), x^*(t) = x(t, \tau, \varphi), \quad x^*(\tau) = \varphi(0) > 0$. Since $0 \notin H(b)$, we conclude that $x^*(t) \not\equiv M$. Therefore we can assume that $x^*(t)$ is not a constant on each of the intervals $[\tau - v, \tau], v > 0$.

We claim that without loss of generality we can set $x^*(0) = 0$. Indeed, integrating the equation $x'(t) + w(t)x(t - \tau) = 0$, we obtain

$$x^*(t) = M - \int_{\tau}^{t} w(s)x^*(s - \tau)ds.$$

If $x^*(0) > 0$, we have $x^*(t) \geq M$ for small $t < \tau$, a contradiction. Let us assume now that $x^*(0) < 0$ and $\tau_1 \in [0, \tau]$ is such that $x^*(t) < 0$ for all $t \in [0, \tau_1)$ and $x^*(\tau_1) = 0$. Then necessarily $w(s) = 0$ almost everywhere on $[\tau, \tau + \tau_1]$ and, consequently, $x^*(t) \equiv M$ on $[\tau, \tau + \tau_1]$. Finally, taking the pair $(w_1, \varphi_1) = (\theta^{\tau_1} d, x_{\tau_1}(\varphi))$ instead of (w, φ) we obtain $x_{\tau_1}(\varphi)(0) = x^*(\tau_1) = 0$.

Thus we end up with

$$(9) \qquad M = \int_{\tau}^{0} w(s)x^*(s-\tau)ds \qquad \text{and} \qquad x^*(s-\tau) = \int_{s-\tau}^{0} w(u)x^*(u-\tau)du.$$

Next we are going to estimate $x^*(s-\tau)$ for $s \in [0,\tau]$ in terms of M and obtain a contradiction with the assumption $M > 0$. The simplest way is to use the inequality $|x^*(s-\tau)| \le M$ in (9), which immediately leads to a contradiction

$$(10) \qquad \int_{\tau}^{0} w(s)ds < 1 \qquad \text{implies } M = 0.$$

However, when s is close to τ and $|x^*(s-\tau)|$ is close to 0, we can obtain a more precise upper bound. Indeed, by the definition of $\tau(0)$, we deduce from (9) that

$$|x^*(s-\tau)| \le \int_{s-\tau}^{0} w(u)duM \le M \qquad \text{for all } s \in [\tau-\tau(0),\tau].$$

Therefore, combining the last two inequalities with (9) we obtain

$$(11) \qquad \begin{aligned} M &\le \int_0^{\tau} w(s)|x^*(s-\tau)|ds \le \int_0^{\tau-\tau(0)} w(s)|x^*(s-\tau)|ds \\ &+ \int_{\tau-\tau(0)}^{\tau} w(s)|x^*(s-\tau)|ds \\ &\le \left\{\int_0^{\tau-\tau(0)} w(u)du + \int_{\tau-\tau(0)}^{\tau} w(u)du \int_{u-\tau}^{0} w(\sigma)d\sigma\right\} M < M, \end{aligned}$$

a necessary contradiction.

We see that the condition $\varepsilon < 1$ is sufficient for the nonexistence of bounded solutions $z(t) : \boldsymbol{R} \to \boldsymbol{R}$ to any system from the hull of equation (2). In view of Lemma 3 this signifies the weak hyperbolicity of π, and in view of the first part of the compatibility theorem—the dichotomy of π over all minimal subsets in $H(F)$. Since our proof is valid for any $b_\varepsilon(t) = \delta\tau^{-1} + \varepsilon b(t), \varepsilon \in [0,1]$, where $0 < \delta < (1-\varepsilon)3/2$, by applying the roughness theorem we obtain the same type of the dichotomy over every minimal set M_γ in $H(F)$ for $\varepsilon = 1$ and $\varepsilon = 0$, that is the exponential stability with dim $\mathscr{U}(G) = 0$ for all $G \in M_\gamma$. By the second part of the compatibility theorem this implies the exponential stability of the linear evolutionary system π associated with (2). This completes the proof of Theorem 1. □

It should be noted that Theorem 1 can be stated in the following form which do not make use of function $\tau(s)$:

Theorem 1′. *Let* $b(t) \geq 0$ *satisfy either condition* **H1** *or condition* **H2** *and* $0 \notin H(b)$. *If additionally*

$$\sup_{s \in R} \min_{y \in [0,\tau]} \left\{ \int_s^{s+y} b(u)du + \int_{s+y}^{s+\tau} b(u)du \int_{u-\tau}^{s} b(\sigma)d\sigma \right\} < 1, \tag{12}$$

then equation (2) *is uniformly exponentially stable together with any other equation from* $H(b)$.

Theorem 1′ shows that in (7) it is possible to replace the function $\tau(s)$ with any (not necessary measurable) function $y(s) : \boldsymbol{R} \to [0, \tau]$. However, the sharpest estimate is reached when $y(s) = \tau(s)$.

Application of our results in the form of Theorem 1′ gives the following corollary.

Corollary 1. *Let the nonzero T-periodic function* $b(t) \geq 0$ *satisfy either condition* **H1** *or condition* **H2**. *Then the inequality*

$$\max_{s \in [0,T]} \min_{y \in [0,\tau]} \left\{ \int_s^{s+y} b(u)du + \int_{s+y}^{s+\tau} b(u)du \int_{u-\tau}^{s} b(\sigma)d\sigma \right\} < 1, \tag{13}$$

implies the uniform exponential stability of equation (2).

The following result shows that, in general, condition (7) is sharper than the 3/2-condition.

Corollary 2. *Let* $b(t) \geq 0$ *satisfy either condition* **H1** *or condition* **H2**. *Then the zero solution of equation* (2) *is uniformly exponentially stable whenever*

$$\sup_{t \in R} \int_{t-\tau}^{t} b(s)ds < \frac{3}{2} \qquad \text{and} \qquad \inf_{t \in R} \int_{t-L}^{t} b(s)ds > 0 \qquad \text{for some } L > 0. \tag{14}$$

Proof. Obviously the second inequality in (14), which is also valid for all $c \in H(b)$, is equivalent to the condition $0 \notin H(b)$.

Let $\lambda > 0$ be such that

$$\sup_{t \in R} \int_{t-\tau}^{t} b(s)ds = \lambda.$$

Firstly, we consider the case when, for some fixed $s \in \boldsymbol{R}$, $\int_{s-\tau}^{s} b(u)du = \mu \leq 1$, and therefore $\tau(s) = \tau$. In this case, integrating by parts and taking into account the relation

$$\int_s^u b(v)dv + \int_{u-\tau}^{s} b(v)dv \leq \lambda$$

which is valid for all $u - \tau \leq s \leq u$, we obtain that

$$\begin{aligned}\int_s^{s+\tau} b(u)du \int_{u-\tau}^{s} b(\sigma)d\sigma &= \int_s^{s+\tau} b(\sigma-\tau)\int_s^{\sigma} b(v)dvd\sigma \\ &\le \int_s^{s+\tau} b(\sigma-\tau)[\lambda - \int_{\sigma-\tau}^{s} b(v)dv]d\sigma \\ &= \lambda\mu - \int_s^{s+\tau} b(\sigma-\tau)\int_{\sigma-\tau}^{s} b(v)dvd\sigma \\ &= \lambda\mu + \int_s^{s+\tau}\int_{\sigma-\tau}^{s} b(v)dvd\int_{\sigma-\tau}^{s} b(v)dv = \lambda\mu - \frac{\mu^2}{2} < 1\end{aligned}$$

is true for every $\mu \in [0,1]$ if and only if $\lambda < 3/2$.

Next we assume that $\int_{s-r}^{s} b(u)du = 1$ for some nonnegative $r = \tau(s) \le \tau$. We have

$$\begin{aligned} i_1 &= \int_s^{s+\tau-r} b(u)du + \int_{s+\tau-r}^{s+\tau} b(u)du \int_{u-\tau}^{s} b(\sigma)d\sigma \\ &= \int_s^{s+\tau-r} b(u)du + \int_{s+\tau-r}^{s+\tau}\int_{u-\tau}^{s} b(\sigma)d\sigma d\int_{s+\tau-r}^{u} b(\sigma)d\sigma \\ &= \int_s^{s+\tau-r} b(u)du + \int_{s+\tau-r}^{s+\tau} b(u-\tau)\int_{s+\tau-r}^{u} b(\sigma)d\sigma du.\end{aligned}$$

Since

$$\int_{s+\tau-r}^{u} b(\sigma)d\sigma + \int_{u-\tau}^{s+\tau-r} b(\sigma)d\sigma \le \lambda$$

for all $u \in [s+\tau-r, s+\tau]$, we find that

$$\begin{aligned} i_1 &\le \int_s^{s+\tau-r} b(u)du + \int_{s+\tau-r}^{s+\tau}\left\{\lambda - \int_{u-\tau}^{s+\tau-r} b(\sigma)d\sigma\right\} b(u-\tau)du \\ &= \int_s^{s+\tau-r} b(u)du + \lambda + \int_{s+\tau-r}^{s+\tau}\int_{u-\tau}^{s+\tau-r} b(\sigma)d\sigma d\int_{u-\tau}^{s+\tau-r} b(\sigma)d\sigma \\ &= \int_s^{s+\tau-r} b(u)du + \lambda + \frac{1}{2}\left\{\left(\int_s^{s+\tau-r} b(u)du\right)^2 - \left(\int_{s-r}^{s+\tau-r} b(u)du\right)^2\right\} \\ &= \int_s^{s+\tau-r} b(u)du + \lambda + \frac{1}{2}\int_s^{s-r} b(u)du\left\{\int_s^{s+\tau-r} b(u)du + \int_{s-r}^{s+\tau-r} b(u)du\right\} \\ &= -\frac{1}{2} + \lambda < 1\end{aligned}$$

whenever $\lambda < 1.5$. Summing up, we obtain that (7) holds if

$$\lambda = \sup_{t \in R} \int_{t-\tau}^{t} b(s)ds < \frac{3}{2}$$

which proves Corollary 2. □

In some cases the following version of Corollary 2 can be useful:

Corollary 2′. *Let* $b(t) \geq 0$ *satisfy either condition* **H1** *or condition* **H2** *on* R_+. *The zero solution of equation* (2) *is asymptotically stable provided*

$$\limsup_{t \to +\infty} \int_{t-\tau}^{t} b(s)ds < \frac{3}{2} \quad \text{and} \tag{15}$$

$$\liminf_{t \to +\infty} \int_{t-L}^{t} b(s)ds > 0 \quad \text{for some } L > 0.$$

Theorem 2. *Let* $b(t) \geq 0$ *satisfy either condition* **H1** *or condition* **H2** *and* $0 \notin H(b)$. *If additionally*

$$\sup_{s \in R} \left\{ \int_{s}^{2\tau+s} b(u)du \right\} < 2, \tag{16}$$

then equation (2) *is uniformly exponentially stable. It shares this property with all equations from* $H(b)$.

Proof. Our arguments repeat those in the proof of Theorem 1 except the part of deriving a contradiction between formulas (9) and (11). Indeed, let us use

$$M - N = \int_0^{\tau} b(s)(-x^*(s-\tau))ds + \int_{-\zeta}^{0} b(u)(-x^*(u-\tau))du, \tag{17}$$

where $-\zeta \in [-\tau, 0]$ is a point where the solution $x^*(t)$ attains its minimal value on $[-\tau, 0]$. Obviously $x^*(-\zeta) = N \geq -M$. Therefore, by (17)

$$M \leq -(-N) + \int_0^{\tau} b(s)(-N)ds + \int_{-\zeta}^{0} b(u)Mdu,$$

and

$$M\left(1 - \int_{-\zeta}^{0} b(u)du\right) \leq \left(\int_0^{\tau} b(s)ds - 1\right)(-N).$$

Now, since $M > 0$ we conclude that $\int_0^{\tau} b(s)ds > 1$ due to (10). Therefore,

$$M\left(1-\int_{-\zeta}^{0} b(u)du\right) \le \left(\int_0^\tau b(s)ds-1\right)(-N) \le \left(\int_0^\tau b(s)ds-1\right)M,$$

a contradiction to (16). □

Remark 1. Theorem 2 improves Proposition 1.2.6 from [3] by relaxing the condition of the continuity of $b(t)$ together with the requirement that $\inf_{t\ge 0} b(t) > 0$.

The following example shows the exact nature of the sufficient condition of Theorem 1. The aim of this example is to demonstrate that, in general, the inequality (12) is sharper than the 3/2-condition and, moreover, the bounds in Theorems 1 and 2 are the best possible. The latter means that violations of the strict inequalities in (7) or (16) can result in existence of solutions not converging to 0.

Example 1. Consider equation (2) where $\tau = 1$ and $b(t)$ is a 2-periodic piecewise constant function defined by

$$b(t) = 0 \quad \text{for } t \in [0,1) \quad \text{and}$$

$$b(t) = \xi \quad \text{for } t \in [1,2) \quad \text{for some } \xi \in [1.5, 2).$$

Corollary 2 fails to apply because $\sup_{t\in R} \int_{t-\tau}^{t} b(s)ds = \xi \ge 1.5$ in this case. Nevertheless, the conditions of Theorem 1 are satisfied, which shows the exponential stability of equation (2). Obviously, $0 \notin H(b) = \{b(t+h), h \in \boldsymbol{R}\}$. Let us evaluate

$$\varepsilon(a) = \int_a^{a+\tau-\tau(a)} b(u)du + \int_{a+\tau-\tau(a)}^{a+\tau} b(u)du \int_{a-\tau}^{a} b(\sigma)d\sigma.$$

By the periodicity of $b(t)$ it is sufficient to choose a from $[0,2]$. We have:

(1) If $a \in [0, 1-1/\xi]$ then $r = \tau(a) = a + 1/\xi$ and

$$\varepsilon(a) = \int_a^{1-1/\xi} b(s)ds + \int_{1-1/\xi}^{a+1} b(u)du \int_{u-1}^{a} b(\sigma)d\sigma = 0.$$

(2) If $a \in [1-1/\xi, 1]$ then $r = \tau(a) = 1$ and

$$\varepsilon(a) = \int_1^{a+1} b(u)du \int_{u-1}^{a} b(\sigma)d\sigma = 0.$$

(3) If $a \in [1, 1+1/\xi]$ then also $r = \tau(a) = 1$ and

$$\varepsilon(a) = \int_a^2 b(u)du \int_1^a b(\sigma)d\sigma = \xi^2(2-a)(a-1) \le \frac{\xi^2}{4} < 1.$$

(4) If $a \in [1 + 1/\xi, 2]$ then $r = \tau(a) = 1/\xi$ and

$$\varepsilon(a) = \int_a^{a+1-1/\xi} b(u)du = (2 - a)\xi \le (1 - 1/\xi)\xi = \xi - 1 < 1.$$

Therefore $\varepsilon = \max \varepsilon(a) < 1$.

The same result is straightforward from Theorem 2.

Finally, it should be noted that for $\xi = 2$ the equation has a 4-periodic piecewise linear solution $x(t)$ defined by

$$x(t) = \begin{cases} 1 & \text{if } t \in [0, 1], \\ 3 - 2t & \text{if } t \in [1, 2], \\ -1 & \text{if } t \in [2, 3], \\ 2t - 7 & \text{if } t \in [3, 4]. \end{cases}$$

Therefore the right-hand side bounds in (7) or (16) can not be decreased.

Example 2. The delay differential equation

$$x'(t) + p(2 + \cos t)x(t - 2\pi) = 0 \tag{18}$$

was studied in [8] where its asymptotic stability was proved for $p \in (0, 0.125)$. This stability interval for p is the best possible since equation (18) is reducible, via the change of variables $y(s) = x(t)$, $s = p\int_0^t (2 + \cos u)du$, to the following equation with constant coefficients

$$y'(s) + y(s - 4p\pi) = 0.$$

It is easy to check that for delay differential equations with constant coefficients the 3/2-condition and Theorem 1 give the same result. One can expect that their application to equation (18) will give the same result too. Computations confirm this giving the same stability domain $p \in (0, p^*)$, where

$$p^* = \frac{3}{8\pi} = 0.119\ldots = \min_{s \in [0, 2\pi]} \max_{x \in [0, 2\pi]} \frac{2}{f(s, x) + \sqrt{f(s, x)^2 + 4g(s, x)}},$$

$$f(s, x) = 4\pi - 2x + \sin(s - x) - \sin s,$$

$$g(s, x) = 2x \sin s - \sin s \sin(s - x) + \sin^2 s + 2x^2$$
$$- 2x \sin(s - x) + 0.25 \cos 2s - 0.25 \cos(2s - 2x).$$

Example 3. The delay differential equation

$$x'(t) + p(2 + \cos t)x(t - \pi) = 0$$

was studied in [8] where the uniform asymptotic stability was proved for

$p \in (0, 0.1207)$. The 3/2-condition gives $p \in (0, 0.1810)$ and Theorem 2 is applicable for $p \in (0, 0.1591)$. However, by applying Corollary 1 we obtain a sharper estimate here:

$$p \in (0, p^*), \quad \text{where } p^* = 0.1930.. = \min_{s \in [0,2\pi]} \max_{x \in [0,\pi]} \frac{2}{f(s,x) + \sqrt{f(s,x)^2 + 4g(s,x)}},$$

and

$$f(s,x) = 2\pi - 2x - \sin(s-x) - \sin s,$$

$$g(s,x) = 2x \sin s + \sin s \sin(s-x) - \sin^2 s + 2x^2 + 4\cos s$$
$$- 4\cos(s-x) + 2x\sin(s-x) - 0.25\cos 2s + 0.25\cos(2s - 2x).$$

4. Main results: Case of general $a(t) \geq 0$

Theorem 3. *Let* $a(t), b(t) \geq 0$ *satisfy either condition* **H1** *or condition* **H2** *and* $(0,0) \notin H((a,b))$. *If additionally*

$$\varepsilon_1 = \sup_{s \in R} \min_{v \in [0,\tau]} \left\{ \int_s^{s+v} \omega(s+\tau, u) b(u) du \right.$$
$$\left. + \int_{s+v}^{s+\tau} \omega(s+\tau, u) b(u) du \int_{u-\tau}^{s} \omega(u-\tau, \sigma) b(\sigma) d\sigma \right\} < 1, \tag{19}$$

then equation (1) *is uniformly exponentially stable. All equations from the hull* $H((a,b))$ *have the same stability property. In particular the inequality*

$$\limsup_{s \to +\infty} \int_s^{s+\tau} b(u) du < 1 \tag{20}$$

guaranties the exponential stability for every $a(t) \geq 0$ *if for some* $L > 0$

$$\liminf_{t \to +\infty} \int_{t-L}^{t} (a(s) + b(s)) ds > 0.$$

Proof. The proof of this more general case goes very closely along main lines of the proof of Theorem 1. Some modifications, due to a more complex nature of equation (1), are necessary to add. First, we note that inequality (19) holds for each $(a_1, b_1) \in H((a,b))$ with $a(u), b(u)$ replaced by $a_1(u), b_1(u)$. We claim that the corresponding equation

$$x'(t) + a_1(t)x(t) + b_1(t)x(t-\tau) = 0, \tag{21}$$

does not have bounded nontrivial solutions for every $(a_1, b_1) \in H((a,b))$.

Indeed, let $z(t) : \boldsymbol{R} \to \boldsymbol{R}$ be a bounded nonzero solution of equation (21) and $\zeta_t(u) = z(t+u), u \in [-\tau, 0]$. Then it follows from **H1**, **H2** that

$$\mathscr{K} = \mathrm{Cl}\{(\theta^t a_1, \theta^t b_1, \zeta_t),\ \ t \in \boldsymbol{R}\} \subset H((a,b)) \times C$$

is a compact invariant set. Thus there exist $(v_1, v_2) \in H((a,b))$ and some $\varphi \in C$ such that $((v_1, v_2), \varphi) \in \mathscr{K}$,

$$0 < M = |\varphi(0)| = \|\varphi\|_C = \max_{\psi}\{\|\psi\|_C : ((e,f)\psi) \in \mathscr{K} \text{ for some } (e,f)\}.$$

Let, for example, $d = d(t), x^*(t) = x(t, \tau, \varphi)$, $x^*(\tau) = \varphi(0) > 0$. Since $(0,0) \notin H((a,b))$, we have $x^*(t) \not\equiv M$, and therefore we can assume that $x^*(t)$ is not a constant on each of the intervals $[\tau - \nu, \tau], \nu > 0$.

We claim that there exists $h \in [0, \tau]$ such that $x^*(h) = 0$. Indeed, if not then we get $x^*(t) > 0$ for all $t \in [0, \tau]$ implying $x^{*\prime}(t) \le 0$ for all t close to τ. Since $x^{*\prime}(t) \not\equiv 0$ in a left neighborhood of τ we get a contradiction to the assumption of maximality of M. Thus $x^*(h) = 0$ for some $h \in [0, \tau]$. Moreover, by shifting the pair $(v_1, v_2) \in H((a,b))$ we can set $x^*(m) = M, x^*(0) = 0$ for some $m \in [0, \tau]$.

Applying the variation of constant formula we obtain

$$M = \int_m^0 \omega(m, s, v_1) v_2(s) x^*(s - \tau) ds,$$

$$x^*(s - \tau) = \int_{s-\tau}^0 \omega(s - \tau, u, v_1) v_2(u) x^*(u - \tau) du.$$

By using the latter we derive

$$\begin{aligned} M &\le \int_0^m \omega(m, s, v_1) v_2(s) |x^*(s - \tau)| ds \le \int_0^{m-\nu} \omega(m, s, v_1) v_2(s) |x^*(s - \tau)| ds \\ &+ \int_{m-\nu}^m \omega(m, s, v_1) v_2(s) |x^*(s - \tau)| ds \le \left\{ \int_0^{m-\nu} \omega(m, s, v_1) v_2(s) ds \right. \\ &\left. + \int_{m-\nu}^m \omega(m, s, v_1) v_2(s) ds \int_{s-\tau}^0 \omega(s - \tau, \sigma, v_1) v_2(\sigma) d\sigma \right\} M < M, \end{aligned}$$

a contradiction.

We see that the condition $\varepsilon_1 < 1$ is sufficient for the nonexistence of bounded solutions $z(t) : \boldsymbol{R} \to \boldsymbol{R}$ to any equation from the hull generated by equation (1). By Lemma 3, this gives the weak hyperbolicity of π^t, and the first part of the compatibility theorem implies the existence of dichotomies for π over all minimal sets in $H(F)$. To prove that π admits a dichotomy over all

$H(F)$, we need to homotope π to an elementary system. Here, in contrast to the proof of Theorem 4, we have to apply such homotopy transformation twice.

Firstly, let us assume (20). Since $\varepsilon_1 \le \sup_{s\in \boldsymbol{R}} \int_s^{s+\tau} b(u)du$, for any nonnegative $a(t)$, inequality (20) implies the exponential dichotomy of π over minimal subsets in H. Moreover, in this case it is not difficult to see the precise character of the dichotomy, since (20) also holds for all $a_\varepsilon(t) = \varepsilon a(t)$; $b_\varepsilon(t) = \delta\tau^{-1} + \varepsilon b(t), \varepsilon \in [0,1]$, where $0 < \delta < 1 - \sup_{\boldsymbol{R}} \int_t^{t+\tau} b(u)du$. Indeed, by the roughness theorem, we obtain the same type of the dichotomic behavior over minimal subsets of H for $\varepsilon = 1$ and $\varepsilon = 0$, that is the exponential stability. Applying the second part of the compatibility theorem, we conclude that inequality (20) implies the exponential stability of π.

Now, we are able to prove that the inequality $\varepsilon_1 < 1$ also implies exponential stability of π even if $\varepsilon_0^{-1} = \limsup_{s\in \boldsymbol{R}} \int_s^{s+\tau} b(u)du$ is bigger than 1. Indeed, since (19) holds for all $b_\varepsilon(t) = \varepsilon b(t)$, $\varepsilon \in [\varepsilon_0, 1]$, we get the weak hyperbolicity of the corresponding linear evolutionary system π_ε for every $\varepsilon \in [\varepsilon_0, 1]$. On the other hand, for $\varepsilon = \varepsilon_0$, the condition (20) is already satisfied and therefore the system π_{ε_0} is exponentially stable. Now, by roughness theorem, we get again the exponential stability for all values of ε. This completes the proof of Theorem 3. □

Corollary 3. *Let conditions of Theorem* 3 *be satisfied and the coefficient* $a(t) \not\equiv 0$ *be τ-periodic. Then the inequality*

$$\limsup_{t\to+\infty} \int_{t-\tau}^{t} b(s)ds < \frac{3}{2}$$

is sufficient for the exponential stability of equation (1).

Proof. Note that in formula (19) we have $\omega(s+\tau, u) \le 1$ and $\omega(s+\tau, u) \cdot \omega(u-\tau, \sigma) = \omega(s+\tau, \sigma+\tau) \le 1$ with the latter holds due to the τ-periodicity of $a(t)$. The remaining part of the proof repeats that of Corollary 2. □

Remarks 2.

2.1. As it can easily be seen our main results, Therems 1, 1′, 2, and 3, as well as the corollaries, remain valid if the equations' coefficients are defined on the simiaxis R $:= \{t : t \ge t_0\}$ only and the respective conditions (7), (12), (16), and (19) are satisfied asymptotically as $t \to +\infty$, i.e. the "$\sup_{t\in \boldsymbol{R}}$" in the conditions is replaced by the "$\limsup_{t\to+\infty}$". To carry over the same proof to this case one has to chose a sufficiently large $T \ge t_0$ and to appropriately redefine $a(t)$ and $b(t)$ for $t \le T$ as constants.

2.2. As we can see Corollary 3 supplements condition (20). It would be interesting to find out whether it is necessary to assume the τ-periodicity of $a(t)$ there. It is also of interest to compare Corollary 3 with the following stability condition of A. Myshkis (see [6], p. 104, Example 1.4):

$$a(t) \geq 0, \quad b(t) \geq 0, \qquad \tau \sup_{t \geq 0}(a(t) + b(t)) < \frac{3}{2}.$$

2.3. If we set $v = 0$ in the statement of Theorem 3 we obtain a generalization of Lemma 3.1 from [4] for the Carathéodory equations (compare also with [7]), with the following stability condition

$$\sup_{s \in R} \left\{ \int_s^{s+\tau} \omega(s+\tau, u) b(u) du \right\} < 1.$$

2.4. One possible way to deal with equation (1) is to use the change of variable $x = \omega(t, 0, a)y$. This transforms equation (1) into equation (2). Since $\omega(t, 0, a) \leq 1$ for $t \geq 0$ the exponential stability of the transformed equation will imply the exponential stability of the original one. Unfortunately this approach (see [10]) gives less sharp estimates since it does not take into account the stabilizing effect of the coefficient $a(t)$ in equation (1). For example, for $a(t) \equiv a$ it gives ([10], Theorem 6)

$$\limsup_{t \to +\infty} \int_{t-\tau}^t b(s) ds < \frac{3}{2} \exp(-\tau a)$$

while our Corollary 3 results in

$$\limsup_{t \to +\infty} \int_{t-\tau}^t b(s) ds < \frac{3}{2}.$$

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