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On Oscillation of a Second Order Nonlinear Delay Differential Equation

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Introduction

Recently the oscillatory and nonoscillatory behavior of solutions of functional differential equations with delay has drawn increasing attention. The main reason is that delay differential equations provide a natural description of a number of real world problems arising in astrophysics, atomic physics, gas and fluid mechanics, etc. (see, for example, [14, 15]). Among numerous papers dealing with the subject we refer in particular to [2, 3, 7, 9, 11, 12, 17, 19, 23, 27, 28], to the monographs [4, 5, 13, 20], and to the references cited there.

In this paper, we study oscillatory behavior of solutions of the nonlinear delay differential equation

$$(r(t)\psi(x(t))x'(t))' + q(t)f(x(\tau(t))) = 0, \tag{1}$$

and the ordinary differential equation

$$(r(t)\psi(x(t))x'(t))' + q(t)f(x(t)) = 0, \tag{2}$$

where $t \in I = [t_0, \infty)$, $t_0 \in \boldsymbol{R} = (-\infty, +\infty)$.

In what follows, we always assume without mentioning that

(A1) $r : I \to \boldsymbol{R}_+ = (0, \infty)$ is continuously differentiable;

(A2) $q : I \to \boldsymbol{R}$ is continuous and $q(t)$ does not eventually vanish; that means that there exists a sequence $\{t_k\}$ of real numbers, $t_k \to \infty$ as $t \to \infty$ such that $q(t_k) \neq 0$;

(A3) $\psi : \boldsymbol{R} \to \boldsymbol{R}$ is continuously differentiable and $\psi(x) > 0$ for $x \neq 0$;

(A4) $f : \boldsymbol{R} \to \boldsymbol{R}$ is continuous and $xf(x) > 0$ for $x \neq 0$;

(A5) $\tau : I \to \boldsymbol{R}$ is continuously differentiable with $\tau'(t) > 0$ for all $t \in I$, $\tau(t) \leq t$ for $t \geq t_0$, and $\lim_{t\to\infty} \tau(t) = \infty$.

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Let $\chi : [\tau(t_0), t_0] \to \boldsymbol{R}$ be a continuous function. By a solution of Eq. (1), we mean a twice continuously differentiable function $x(t) : [\tau(t_0), \infty) \to \boldsymbol{R}$ such that $x(t) = \chi(t)$ for $\tau(t_0) \le t \le t_0$, and $x(t)$ satisfies Eq. (1) for all $t \ge t_0$. By a solution of Eq. (2), we mean a twice continuously differentiable function $x(t) : [t_0, \infty) \to \boldsymbol{R}$ that satisfies Eq. (2) for all $t \ge t_0$.

In what follows, we restrict our attention to proper solutions of Eqs. (1), (2), i.e., to those nonconstant solutions which exist on some ray $[T, \infty)$, where $T \ge t_0$, and satisfy condition $\sup_{t \ge T}\{|x(t)|\} > 0$. A proper solution $x(t)$ of Eq. (1) (resp., Eq. (2)) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. A nonoscillatory solution $x(t)$ of Eq. (1) (Eq. (2)) is said to be weakly oscillatory if $x'(t)$ changes sign for arbitrarily large values of t. Finally, Eq. (1) (resp., Eq. (2)) is called oscillatory if all its proper solutions are oscillatory.

Very recently, oscillatory behavior of solutions for Eqs. (1) and (2) has been studied by Cecchi and Marini [2]. They have proved, among other results on asymptotic behavior of solutions of Eqs. (1) and (2), the following oscillation criteria.

Theorem A [2, Theorem 3]. *Assume that*

(i) $q : I \to \boldsymbol{R}$ *is continuous and* $q(t)$ *does not eventually vanish;*

(ii) $r : I \to \boldsymbol{R}_+$ *is continuously differentiable;*

(iii) $\psi : \boldsymbol{R} \to \boldsymbol{R}$ *is continuously differentiable and* $\psi(x) > 0$ *for* $x \ne 0$*;*

(iv) $f : \boldsymbol{R} \to \boldsymbol{R}$ *is continuous,* $xf(x) > 0$ *and* $f'(x) \ge 0$ *for* $x \ne 0$*;*

(v) $\tau : I \to \boldsymbol{R}_+$ *is continuously differentiable such that* $\tau'(t) \ge 0$ *for* $t \ge t_0$, *and* $\lim_{t\to\infty} \tau(t) = \infty$*;*

(vi) $\lim_{t\to+\infty} \int_{t_0}^t q(s)ds = +\infty$*;*

(vii) $\lim_{t\to+\infty} \int_{t_0}^t \dfrac{1}{r(s)} ds = +\infty$.

Then every solution of Eq. (2) *is oscillatory and every solution of Eq.* (1) *is either oscillatory or weakly oscillatory.*

We point out that Theorem A can be applied not only to functional differential equations with retarded argument, but to advanced and mixed type equations as well. The following result is concerned only with delay differential equations.

Theorem B [2, Corollary 1]. *Assume that conditions* (i)–(v) *of Theorem A hold, and suppose also that*

(viii) $\tau(t) \le t$*;*

(ix) $\limsup_{t\to\infty} \int_{t_0}^t q(s)ds = +\infty$*;*

(x) *the function* $\psi(u)/f(u)$ *is locally integrable on* $(0, c)$ *and* $(-c, 0)$ *for some* $c > 0$*; that is,*

$$\int_0^c \frac{\psi(u)}{f(u)} du < +\infty, \qquad \int_{-c}^0 \frac{\psi(u)}{f(u)} du > -\infty;$$

(xi) $\limsup_{t\to\infty} \int_T^t \frac{1}{r(s)} \int_T^s q(r)drds = +\infty$, *for any* $T \geq t_0$.

Then every solution of Eq. (2) *is oscillatory and every solution of Eq.* (1) *is either oscillatory or weakly oscillatory.*

The aforementioned results have motivated the present research and the principal reasons are the following:

a) Theorems A and B depend heavily on the assumption that the function f is nondecreasing, and it may be somewhat restrictive for applications. So, as Wong [30, p. 675] has pointed out, "it will be useful to prove results which do not require $f(x)$ to be monotone." We mention here that the monotonicity condition on f has been also required, for example, by Das [3], Grace, Lalli and Yeh [12], and Kusano and Onose [19].

b) As it has been stressed by Wong [29, p. 228], condition (vii) in Theorem A "is not usually satisfied in practical problems." For example, the Emden-Fowler equation encountered in astrophysics and the Fermi-Thomas equation in atomic physics have the following form:

$$\frac{d}{dt}\left(t^p \frac{dx}{dt}\right) + t^\lambda x^\gamma = 0, \qquad t \geq 0, \tag{3}$$

where p, λ, γ are positive constants, and in some cases $p > 1$. Therefore, it is important for applications to get rid of the aforementioned assumption. We note that condition (vii) has been also assumed, for example, by Bradley [1], Grace [7, 9], Grace, Lalli and Yeh [12], and Philos and Sficas [23].

The purpose of this paper is to establish new oscillation criteria for Eqs. (1) and (2) which complement and extend results in [1, 2, 3, 6, 9, 10, 12, 18, 19, 23, 25, 27, 31]. To that end, we employ the integral averaging technique similar to that exploited by Grace [6, 7, 8], Grace and Lalli [10, 11], Kirane and Rogovchenko [18], Li [21], Philos [22], and Rogovchenko [25, 26, 27].

The paper is organized as follows. In the next section, we present three new sets of sufficient conditions which guarantee oscillation of all proper solutions of the nonlinear delay differential equation (1). Corresponding theorems for the nonlinear ordinary differential equation (2) are presented in Section 2. Though the form of the theorems as well as the proofs are very similar for each set of results, the nature of the assumptions on the functions f and ψ is different. Thus our results apply to wide classes of equations which may be overlapping but definitely distinct. We discuss a number of carefully chosen examples which clarify the relevance of our results. It is well known that theorems which guarantee the existence of oscillatory solutions are not available for most nonlinear differential equations (see, for instance, [4, 5, 13, 20]). Therefore, as opposed to the existing papers on oscillation, which provide mostly illustrative examples to the theorems proved without showing the existence of proper oscillatory solutions, our examples have been intentionally selected so that each nonlinear differential equation with or without retarded argument has an exact oscillatory solution. Finally, in the last section we compare our results to those known in the literature.

1. Oscillation of delay differential equation

We begin with the following proposition, providing also for the convenience of the reader the elegant original proof.

Lemma 1 ([2, Theorem 2 (b)]). *If* $q(t) \geq 0$ *for all large* t*, then Eq.* (1) *has no weakly oscillatory solutions.*

Proof. Let $x(t)$ be a weakly oscillatory solution of Eq. (1). Without loss of generality, we assume that there exists a $t_1 \geq t_0$ such that, for all $t \geq t_1$, we have $x(t) > 0$ and $x(\tau(t)) > 0$. Define the function F as follows:

$$F(t) = r(t)\psi(x(t))x'(t).$$

Then, for $t \geq t_1$, we have that

$$F'(t) = -q(t)f(x(\tau(t))) \leq 0,$$

and hence F is nonincreasing, which contradicts to the fact that F is an oscillatory function. □

Since throughout this section we always suppose that assumption (6) given below holds, our results are concerned only with oscillatory proper solutions of Eq. (1).

Following Philos [22], we introduce a class of functions $\mathscr{P}$. Let $D_0 = \{(t,s) : t > s \geq t_0\}$ and $D = \{(t,s) : t \geq s \geq t_0\}$.

The function $H \in C(D; \boldsymbol{R})$ is said to belong to the class $\mathscr{P}$ if

(H_1) $H(t,t) = 0$ for $t \geq t_0$, $H(t,s) > 0$ on D_0;

(H_2) H has a continuous and nonpositive partial derivative on D_0 with respect to the second variable.

Theorem 1. *Assume that for* $x \neq 0$

$$f'(x) \geq K, \tag{4}$$

$$\psi(x) \leq L^{-1}, \tag{5}$$

where K and L are positive constants, and suppose that

$$q(t) \geq 0 \tag{6}$$

for all $t \geq T_*$, *where* $T_* \geq t_0$ *is a real number. Let* $h, H : D \to \mathbf{R}$ *be continuous functions such that H belongs to the class* $\mathscr{P}$ *and*

$$-\frac{\partial H}{\partial s}(t,s) = h(t,s)\sqrt{H(t,s)} \qquad \text{for all } (t,s) \in D_0. \tag{7}$$

Assume also that there exists a continuously differentiable function $\rho : I \to \mathbf{R}_+$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)} Q^2(t,s) \right] ds = \infty, \tag{8}$$

where

$$Q(t,s) = h(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}.$$

Then Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1) and let $T_0 \geq t_0$ be such that $x(t) \neq 0$, for all $t \geq T_0$. Without loss of generality, we may assume that, for all $t \geq T_0$, one has $x(t) > 0$ and $x(\tau(t)) > 0$ since the similar argument holds also for the case when $x(t)$ is eventually negative. Then, by (A4), (6), and (1), we conclude that

$$(r(t)\psi(x(t))x'(t))' \leq 0, \qquad \text{for } t \geq T_1 = \max\{T_0, T_*\}. \tag{9}$$

Let us define the function $w(t)$ as follows

$$w(t) = \rho(t)r(t)\psi(x(t))\frac{x'(t)}{f(x(\tau(t)))}. \tag{10}$$

Differentiating (10) and making use of Eq. (1), we obtain

$$w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)q(t) - \frac{f'(x(\tau(t)))x'(\tau(t))\tau'(t)}{f(x(\tau(t)))}w(t). \tag{11}$$

By (9) and (A5), we have

$$r(\tau(t))\psi(x(\tau(t)))x'(\tau(t)) \geq r(t)\psi(x(t))x'(t),$$

and, consequently, by (4), (5), and (11), for $t \geq T_1$, we obtain that

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)q(t) - \frac{KL\tau'(t)}{r(\tau(t))\rho(t)}w^2(t). \tag{12}$$

Hence, by (1) and (12), for all $t \geq T \geq T_1$, we have

$$\begin{aligned}
&\int_T^t H(t,s)\rho(s)q(s)ds \\
&\quad \leq \int_T^t H(t,s)\frac{\rho'(s)}{\rho(s)}w(s)ds - \int_T^t H(t,s)w'(s)ds \\
&\quad - \int_T^t H(t,s)\frac{KL\tau'(s)}{r(\tau(s))\rho(s)}w^2(s)ds = -H(t,s)w(s)|_T^t \\
&\quad - \int_T^t \left[-\frac{\partial H}{\partial s}(t,s)w(s) - H(t,s)\frac{\rho'(s)}{\rho(s)}w(s) + H(t,s)\frac{KL\tau'(s)}{r(\tau(s))\rho(s)}w^2(s)\right]ds \\
&\quad = H(t,T)w(T) - \int_T^t \left[\sqrt{\frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)}}w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}}Q(t,s)\right]^2 ds \\
&\quad + \int_T^t \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)}Q^2(t,s)ds.
\end{aligned}$$

Thereby, for all $t \geq T \geq T_1$, we conclude that

$$\begin{aligned}
\int_T^t \left[H(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)}Q^2(t,s)\right]ds &\leq H(t,T)w(T) \\
- \int_T^t \left[\sqrt{\frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)}}w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}}Q(t,s)\right]^2 ds. \qquad &(13)
\end{aligned}$$

By virtue of (13) and (H_2), for every $t \geq T_1$, we obtain

$$\begin{aligned}
&\int_{T_1}^t \left[H(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)}Q^2(t,s)\right]ds \\
&\qquad \leq H(t,T_1)|w(T_1)| \leq H(t,t_0)|w(T_1)|. \qquad (14)
\end{aligned}$$

Thus, by (14) and (H_2), we have

$$\begin{aligned}
&\int_{t_0}^t \left[H(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)}Q^2(t,s)\right]ds \\
&\qquad \leq H(t,t_0)\left[\int_{t_0}^{T_1}\rho(s)q(s)ds + |w(T_1)|\right]. \qquad (15)
\end{aligned}$$

Inequality (15) yields

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)} Q^2(t,s) \right] ds$$
$$\leq \int_{t_0}^{T_1} \rho(s)q(s)ds + |w(T_1)| < +\infty,$$

and the latter inequality contradicts assumption (8) of the theorem. Hence, Eq. (1) is oscillatory. □

Corollary 1. *Assume that the assumptions of Theorem* 1 *hold with* (8) *replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds = \infty,$$
$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \frac{r(\tau(s))\rho(s)}{\tau'(s)} Q^2(t,s)ds < \infty.$$

Then Eq. (1) *is oscillatory.*

Remark 1. We note that corollaries similar to Corollary 1 can be easily deduced also from Theorems 2 and 3 stated below.

With the appropriate choice of functions H and h, it is possible to derive from Theorem 1 a number of oscillation criteria for Eq. (1). Defining, for example, for some integer $n > 2$, the function $H(t,s)$ by

(16) $$H(t,s) = (t-s)^{n-1}, \qquad (t,s) \in D,$$

we can easily check that $H \in \mathscr{P}$. Furthermore, the function

(17) $$h(t,s) = (n-1)(t-s)^{(n-3)/2}, \qquad (t,s) \in D$$

is continuous and satisfies condition (7). Therefore, as a consequence of Theorem 1, we obtain the following oscillation criterion.

Corollary 2. *Let assumptions* (4), (5), *and* (6) *hold. Assume also that there exists a continuously differentiable function* $\rho : I \to \boldsymbol{R}_+$ *such that, for some integer* $n > 2$,

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \left[(t-s)^{n-1}\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)} (t-s)^{n-3} \left(n-1-\frac{\rho'(s)}{\rho(s)}(t-s) \right)^2 \right] ds = \infty.$$

Then Eq. (1) *is oscillatory.*

Remark 2. We point out that we can deduce corollaries similar to Corollary 2 from Theorems 2 and 3 as well. Of course, we are not limited only to the choice of functions H and h defined, respectively, by (16) and (17), which has become standard and goes back to the well-known paper by Kamenev [16]. With a different choice of these functions it is possible to derive from any theorem presented in this paper other sets of corollaries. In fact, another possibility is to choose the functions H and h as follows:

$$H(t,s) = \left(\ln \frac{t}{s}\right)^n, \qquad t \geq s \geq t_0, \tag{18}$$

$$h(t,s) = \frac{n}{s}\left(\ln \frac{t}{s}\right)^{n/2-1}, \qquad t \geq s \geq t_0. \tag{19}$$

One may also choose the more general forms for the functions H and h:

$$H(t,s) = \left(\int_s^t \frac{du}{\theta(u)}\right)^n, \qquad t \geq s \geq t_0, \tag{20}$$

$$h(t,s) = \frac{n}{\theta(s)}\left(\int_s^t \frac{du}{\theta(u)}\right)^{n/2-1}, \qquad t \geq s \geq t_0, \tag{21}$$

where $n > 1$ is an integer, and $\theta : [t_0, \infty) \to \boldsymbol{R}_+$ is a continuous function satisfying condition

$$\lim_{t\to\infty} \int_{t_0}^t \frac{du}{\theta(u)} = \infty. \tag{22}$$

It is a simple matter to check that in both cases assumptions (H_1) and (H_2) are verified, as well as condition (7), which has been evidently used to determine the function $h(t,s)$.

Remark 3. For the sake of simplicity, in all the examples we tacitly define the functions $H(t,s)$ and $h(t,s)$ by (16) and (17) letting $n = 3$:

$$H(t,s) = (t-s)^2, \qquad h(t,s) = 2, \qquad t \geq s \geq 1. \tag{23}$$

Example 1. Consider the nonlinear delay differential equation

$$\left((1+\sin^2 2t)\frac{1}{1+x^2(t)}x'(t)\right)' + \frac{4}{1+\sin^2 2t}x(t-\pi)\left(1+x^2(t-\pi)\right) = 0, \tag{24}$$

where $t \geq 1$. Assumptions (A1)–(A5), (4), and (5) are easily verified, and we can apply Corollary 2 letting, for example, $\rho(t) = t^2$, to show that Eq. (24) is oscillatory. In fact, $x(t) = \sin 2t$ is an oscillatory solution of Eq. (24).

It may happen that either assumption (4) or assumption (5) in Theorem 1 and Corollaries 1 and 2 fails to hold. Consequently, the aforementioned results do not apply, for example, to equations with $f(x)$ nonmonotonous or $\psi(x)$ unbounded. Furthermore, for some equations both (4) and (5) may fail to hold simultaneously. The following result gives the possibility to consider new classes of equations under the unique assumption on f and ψ.

Theorem 2. *Let h, H be as in Theorem* 1, *the function q satisfy* (6), *and suppose that for $x \neq 0$ and for some positive constant γ*

$$\frac{f'(x)}{\psi(x)} \geq \gamma. \tag{25}$$

Furthermore, assume that there exists a continuously differentiable function $\rho : I \to \boldsymbol{R}_+$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4\gamma\tau'(s)} Q^2(t,s) \right] ds = \infty. \tag{26}$$

Then Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). As in Theorem 1, without loss of generality, we may assume that, for all $t \geq T_0$, one has $x(t) > 0$ and $x(\tau(t)) > 0$, so that (9) holds. Defining again the function $w(t)$ by (10), after differentiation we obtain (11). By (9) and (25), for $t \geq T_1$, we conclude from (11) that

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)q(t) - \frac{\gamma\tau'(t)}{r(\tau(t))\rho(t)} w^2(t), \tag{27}$$

Hence, by (1) and (27), for all $t \geq T \geq T_1$, we have

$$\begin{aligned} \int_{T}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4\gamma\tau'(s)} Q^2(t,s) \right] ds \leq H(t,T)w(T) \\ - \int_{T}^{t} \left[\sqrt{\frac{\gamma H(t,s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{\gamma\tau'(s)}} Q(t,s) \right]^2 ds. \end{aligned} \tag{28}$$

The rest of the proof runs as in Theorem 1. □

Example 2. Consider the nonlinear delay differential equation

$$\left(\frac{1}{1 + 5\cos^4 t} (1 + 5x^4(t))x'(t) \right)' + \frac{1}{1 + \cos^4 t} x(t - 2\pi)\big(1 + x^4(t - 2\pi)\big) = 0, \tag{29}$$

where $t \geq 1$. Clearly, assumptions (A1)–(A5), (6), and (25) hold, so we can apply Theorem 2 with $\rho(t) = t^2$ to check that Eq. (29) is oscillatory. Observe that $x(t) = \cos t$ is an oscillatory solution of this equation.

In the following theorem, we do not require the function f to satisfy the assumption (4), so it is not necessarily monotonous on $\boldsymbol{R}$. This enables us to apply the result to new classes of equations which are not covered by the known theorems.

Theorem 3. *Let assumptions* (5) *and* (6) *hold, h, H be as in Theorem* 1, *and suppose that for $x \neq 0$ and for some positive constant K*

$$\frac{f(x)}{x} \geq K. \tag{30}$$

Assume also that there exists a continuously differentiable function $\rho : I \to \boldsymbol{R}_+$ such that

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[KH(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4L\tau'(s)} Q^2(t,s) \right] ds = \infty. \tag{31}$$

Then Eq. (1) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (1). As before, without loss of generality, we may assume that, for all $t \geq T_0$, we have $x(t) > 0$ and $x(\tau(t)) > 0$. Therefore, (9) holds. We define the function $w(t)$ letting

$$w(t) = \rho(t)r(t)\psi(x(t)) \frac{x'(t)}{x(\tau(t))}. \tag{32}$$

Differentiating (32) and making use of Eq. (1), we obtain

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)q(t) \frac{f(x(\tau(t)))}{x(\tau(t))} - \frac{x'(\tau(t))\tau'(t)}{x(\tau(t))} w(t). \tag{33}$$

By (9), (25), and (5), for $t \geq T_1$, we conclude from (33) that

$$w'(t) \leq \frac{\rho'(t)}{\rho(t)} w(t) - K\rho(t)q(t) - \frac{L\tau'(t)}{r(\tau(t))\rho(t)} w^2(t). \tag{34}$$

Hence, by (1) and (34), for all $t \geq T \geq T_1$, we obtain

$$\begin{aligned} \int_T^t \left[KH(t,s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4L\tau'(s)} Q^2(t,s) \right] ds &\leq H(t,T)w(T) \\ - \int_T^t \left[\sqrt{\frac{LH(t,s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s) + \frac{1}{2} \sqrt{\frac{r(\tau(s))\rho(s)}{L\tau'(s)}} Q(t,s) \right]^2 ds. \end{aligned} \tag{35}$$

The rest of the proof follows the same lines as that of Theorem 1. □

Example 3. Consider the nonlinear delay differential equation

$$\left((1+\sin^2 t)\frac{2+x^2(t)}{1+x^2(t)}x'(t)\right)' + \frac{27\sin^2 t(1+\sin^2 t)}{10+\sin^2 t}x(t-2\pi)\left(\frac{1}{9}+\frac{1}{1+x^2(t-2\pi)}\right)=0, \tag{36}$$

where $t \geq 1$. Since conditions (A1)–(A5), (5), (6), and (30) are verified easily, we can apply Theorem 3 with $\rho(t)=1$ to conclude that Eq. (36) is oscillatory. In fact, $x(t)=\sin t$ is an oscillatory solution of this equation.

Theorem 4. *Let H and h be as in Theorem* 1, *f, ψ and q satisfy* (4), (5), *and* (6), *respectively, and assume that*

$$0 < \inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] \leq \infty. \tag{37}$$

Suppose also that there exist two functions $\rho \in C^1([t_0,\infty);(0,\infty))$ and $\phi \in C([t_0,\infty);(-\infty,\infty))$ such that, for all $t > t_0$ and for any $T \geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\frac{\rho(s)r(\tau(s))}{\tau'(s)}Q^2(t,s)ds < \infty, \tag{38}$$

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\rho(s)q(s)-\frac{\rho(s)r(\tau(s))}{4KL\tau'(s)}Q^2(t,s)\right]ds \geq \phi(T), \tag{39}$$

$$\limsup_{t\to\infty}\int_{t_0}^{t}\frac{\phi_+^2(s)\tau'(s)}{\rho(s)r(\tau(s))}ds = \infty, \tag{40}$$

where $\phi_+(t)=\max(\phi(t),0)$. Then Eq. (1) *is oscillatory.*

Proof. As above, we assume that there exists a solution $x(t)$ of Eq. (1) such that, for some $T_0 \geq t_0$, we have $x(t) > 0$ and $x(\tau(t)) > 0$ on $[T_0,\infty)$. Defining $w(t)$ by (10), we obtain (13), which, for any $T \geq T_1$, yields

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(\tau(s))\rho(s)}{4KL\tau'(s)}Q^2(t,s)\right]ds \leq w(T)$$
$$-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}\left[\sqrt{\frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)}}w(s)+\frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}}Q(t,s)\right]^2 ds.$$

By (39), for any $T \geq T_1$, we conclude from the latter inequality that

$$w(T) \geq \phi(T) + \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}} Q(t,s) \right]^2 ds.$$

Consequently, we have

$$w(T) \geq \phi(T), \tag{41}$$

and

$$\liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t \left[\sqrt{\frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}} Q(t,s) \right]^2 ds \leq M < \infty,$$

where

$$M \stackrel{\text{def}}{=} w(T_1) - \phi(T_1). \tag{42}$$

The latter inequality yields

$$\begin{aligned} \infty &> \liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t \left[\sqrt{\frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}} Q(t,s) \right]^2 ds \\ &\geq \liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^t \left[\frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)} w^2(s) + \sqrt{H(t,s)}Q(t,s)w(s) \right] ds \\ &= \liminf_{t\to\infty} [\alpha(t) + \beta(t)], \end{aligned} \tag{43}$$

where $\alpha(t)$ and $\beta(t)$ are defined by

$$\alpha(t) = \frac{1}{H(t,T_1)} \int_{T_1}^t \frac{KLH(t,s)\tau'(s)}{r(\tau(s))\rho(s)} w^2(s)ds,$$

$$\beta(t) = \frac{1}{H(t,T_1)} \int_{T_1}^t \sqrt{H(t,s)}Q(t,s)w(s)ds.$$

Suppose now that

$$\int_{T_1}^{\infty} \frac{w^2(s)\tau'(s)}{\rho(s)r(\tau(s))} ds = \infty. \tag{44}$$

Assumption (37) implies that there exists a positive constant η such that

(45) $$\inf_{s\geq t_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right] > \eta > 0.$$

On the other hand, by (44), for any constant $\mu > 0$, there is a $T_2 > T_1$ such that

$$\int_{T_1}^{t}\frac{w^2(s)\tau'(s)}{\rho(s)r(\tau(s))}ds \geq \frac{\mu}{\eta}, \qquad \text{for all } t \geq T_2.$$

Thus, after integration by parts, for all $t \geq T_2$, we obtain that

(46) $$\begin{aligned}\alpha(t) &= \frac{1}{H(t,T_1)}\int_{T_1}^{t} KLH(t,s)d\left[\int_{T_1}^{s}\frac{w^2(u)\tau'(u)}{r(\tau(u))\rho(u)}du\right]\\ &= \frac{1}{H(t,T_1)}\int_{T_1}^{t} KL\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_{T_1}^{s}\frac{w^2(u)\tau'(u)}{r(\tau(u))\rho(u)}du\right]ds\\ &\geq \frac{1}{H(t,T_1)}\int_{T_2}^{t} KL\left[-\frac{\partial H(t,s)}{\partial s}\right]\left[\int_{T_1}^{s}\frac{w^2(u)\tau'(u)}{r(\tau(u))\rho(u)}du\right]ds\\ &\geq \frac{\mu}{\eta}\frac{1}{H(t,T_1)}\int_{T_2}^{t} KL\left[-\frac{\partial H(t,s)}{\partial s}\right]ds = \frac{KL\mu}{\eta}\frac{H(t,T_2)}{H(t,T_1)}.\end{aligned}$$

It follows from (45) that

$$\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)} > \eta > 0.$$

Hence, there exists a $T_3 \geq T_2$ such that

$$\frac{H(t,T_2)}{H(t,t_0)} \geq \eta, \qquad \text{for all } t \geq T_3.$$

Thus, by (46) and (H_2), we have that

$$\alpha(t) \geq KL\mu, \qquad \text{for all } t \geq T_3.$$

Since μ is an arbitrary positive constant, we conclude that

(47) $$\lim_{t\to\infty}\alpha(t) = \infty.$$

Let us consider a sequence of real numbers $\{t_n\}_{n=1}^{\infty} \in (T_1,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$\lim_{n\to\infty}[\alpha(t_n)+\beta(t_n)] = \liminf_{t\to\infty}[\alpha(t)+\beta(t)].$$

By (43), there exists a natural number N such that

(48) $$\alpha(t_n)+\beta(t_n) \leq M, \qquad \text{for all } n > N,$$

where the constant M is defined by (42). It follows from (47) and (48) that

$$\lim_{n\to\infty} \beta(t_n) = -\infty. \tag{49}$$

Thus, by (48) and (49), for n large enough, we conclude that

$$\frac{\beta(t_n)}{\alpha(t_n)} + 1 < \varepsilon,$$

where $\varepsilon \in (0,1)$ is a constant. By he latter inequality an (49),

$$\lim_{n\to\infty} \frac{\beta(t_n)}{\alpha(t_n)} \cdot \beta(t_n) = \infty. \tag{50}$$

On the other hand, application of the Schwarz inequality implies that, for any natural number n,

$$\begin{aligned}\beta^2(t_n) &= \frac{1}{H^2(t_n, T_1)}\left[\int_{T_1}^{t_n} \sqrt{H(t_n,s)}Q(t_n,s)w(s)ds\right]^2 \\ &\le \left[\frac{1}{H(t_n,T_1)}\int_{T_1}^{t_n} \frac{KLH(t_n,s)\tau'(s)w^2(s)}{\rho(s)r(\tau(s))}\right] \\ &\quad\times \left[\frac{1}{H(t_n,T_1)}\int_{T_1}^{t_n} \frac{\rho(s)r(\tau(s))}{KL\tau'(s)}Q^2(t_n,s)ds\right] \\ &\le \alpha(t_n)\left[\frac{1}{H(t_n,T_1)}\int_{T_1}^{t_n} \frac{\rho(s)r(\tau(s))}{KL\tau'(s)}Q^2(t_n,s)ds\right].\end{aligned}$$

It follows from (H_2) and from the latter inequality that, for sufficiently large values of n,

$$\frac{\beta^2(t_n)}{\alpha(t_n)} \le \frac{1}{\eta}\frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n} \frac{\rho(s)r(\tau(s))}{KL\tau'(s)}Q^2(t_n,s)ds.$$

By (50), we obtain

$$\lim_{n\to\infty} \frac{1}{H(t_n,t_0)}\int_{t_0}^{t_n} \frac{\rho(s)r(\tau(s))}{\tau'(s)}Q^2(t_n,s)ds = \infty. \tag{51}$$

Consequently,

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)}\int_{t_0}^{t} \frac{\rho(s)r(\tau(s))}{\tau'(s)}Q^2(t,s)ds = \infty,$$

but the latter equality contradicts assumption (38) of the theorem. Therefore, (44) fails to hold, and we conclude that

$$\int_{T_1}^{\infty} \frac{\tau'(s)w^2(s)}{\rho(s)r(\tau(s))} ds < \infty. \tag{52}$$

It follows from (41) and (52) that

$$\int_{T_1}^{\infty} \frac{\tau'(s)\phi_+^2(s)}{\rho(s)r(\tau(s))} ds < \int_{T_1}^{\infty} \frac{\tau'(s)w^2(s)}{\rho(s)r(\tau(s))} ds < \infty, \tag{53}$$

but this contradicts assumption (40) of the theorem. Hence, we conclude that Eq. (1) is oscillatory. □

Corollary 3. *Assume that* (4)–(6) *hold. Suppose also that there exist functions* $\rho \in C^1([t_0, \infty); (0, \infty))$ *and* $\phi \in C([t_0, \infty); (-\infty, \infty))$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \frac{\rho(s)r(\tau(s))}{\tau'(s)} (t-s)^{n-3} \left(n - 1 - \frac{\rho'(s)}{\rho(s)}(t-s) \right)^2 ds < \infty,$$

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[(t-s)^{n-1} \rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)} (t-s)^{n-3} \right.$$
$$\left. \times \left(n - 1 - \frac{\rho'(s)}{\rho(s)}(t-s) \right)^2 \right] ds \geq \phi(T),$$

and (40) *holds. Then Eq.* (1) *is oscillatory.*

Proof. The proof is evident since

$$\liminf_{t\to\infty} \frac{H(t,s)}{H(t,t_0)} = \liminf_{t\to\infty} \frac{(t-s)^{n-1}}{(t-t_0)^{n-1}} = 1,$$

for any $s \geq t_0$. □

Remark 4. We note that it is straightforward to deduce from Theorems 5–9 corollaries similar to Corollary 3.

Example 4. Consider the nonlinear delay differential equation

$$\left(\frac{1 + \exp 2t \sin^2 t}{2 + \exp 2t \sin^2 t} \frac{2 + x^2(t)}{1 + x^2(t)} x'(t) \right)' + 2\exp(\pi/2)x(t - \pi/2) = 0, \tag{54}$$

where $t \geq 1$. It is not difficult to check that conditions (A1)–(A5), (4), and (5) are satisfied. Hence, we can apply Corollary 3 choosing, for example, $\rho(t) = t^{-2}$. A straightforward calculation verifies assumptions (38)–(39), and we conclude that Eq. (54) is oscillatory. Indeed, $x(t) = \exp t \sin t$ is an oscillatory solution of Eq. (54).

Theorem 5. *Let H and h be as in Theorem* 1, f, ψ *and* q *satisfy* (25) *and* (6), *respectively, and suppose that* (37) *holds. Assume also that there exist functions* $\rho \in C^1([t_0, \infty); (0, \infty))$ *and* $\phi \in C([t_0, \infty); (-\infty, \infty))$ *such that, for all* $t > t_0$ *and for any* $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4\gamma\tau'(s)} Q^2(t,s) \right] ds \geq \phi(T), \tag{55}$$

and conditions (38) *and* (40) *are satisfied. Then Eq.* (1) *is oscillatory.*

Proof. Starting with the inequality (28), we proceed as in the proof of Theorem 4. □

Example 5. Consider the nonlinear delay differential equation

$$\left(\frac{1}{1+3\sin^2 t}(1+3x^2(t))x'(t)\right)' + \frac{1}{1+\sin^2 t} x(t-2\pi)\left(1+x^2(t-2\pi)\right) = 0, \tag{56}$$

where $t \geq 1$. Clearly, conditions (A1)–(A5), (6), and (25) are satisfied, so we can apply Theorem 5. With the same choice of $\rho(t)$ as in Example 4, after a routine computation we conclude that Eq. (56) is oscillatory. It is not difficult to check that $x(t) = \sin t$ is an oscillatory solution of this equation.

Theorem 6. *Let H and h be as in Theorem* 1, f, ψ *and* q *satisfy* (30), (5), *and* (6), *respectively, and assume that* (37) *holds. Suppose also that there exist functions* $\rho \in C^1([t_0, \infty); (0, \infty))$ *and* $\phi \in C([t_0, \infty); (-\infty, \infty))$ *such that, for all* $t > t_0$ *and for any* $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[KH(t,s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4L\tau'(s)} Q^2(t,s) \right] ds \geq \phi(T), \tag{57}$$

and conditions (38) *and* (40) *are satisfied. Then Eq.* (1) *is oscillatory.*

Proof. The proof follows the same lines as that of Theorem 4 with the only difference that we start with the inequality (35). □

Example 6. Consider the nonlinear delay differential equation

$$\left(\frac{1+\sin^2 t}{2+\sin^2 t}\frac{2+x^2(t)}{1+x^2(t)}x'(t)\right)' + \frac{1+\sin^2 t}{13+\sin^2 t}x(t-2\pi)\left(1+\frac{12}{1+x^2(t-2\pi)}\right) = 0, \tag{58}$$

where $t \geq 1$. Assumptions (A1)–(A5), (5), (6), and (30) are easily verified, and we can apply Theorem 6 with $\rho(t) = t^{-2}$ to show that Eq. (58) is oscillatory. Observe that $x(t) = \sin t$ is an oscillatory solution of this equation.

Theorem 7. *Let H and h be as in Theorem* 1, f, ψ *and* q *satisfy* (4), (5), *and* (6), *respectively, and assume that* (37) *holds. Suppose also that there exist functions* $\rho \in C^1([t_0, \infty); (0, \infty))$ *and* $\phi \in C([t_0, \infty); (-\infty, \infty))$ *such that, for all* $t > t_0$ *and for any* $T \geq t_0$,

$$\liminf_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} H(t, s)\rho(s)q(s)ds < \infty, \tag{59}$$

$$\liminf_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[H(t, s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)} Q^2(t, s)\right] ds \geq \phi(T), \tag{60}$$

and (40) *holds. Then Eq.* (1) *is oscillatory.*

Proof. As above, we assume that there exists a solution $x(t)$ of Eq. (1) such that, for some $T_0 \geq t_0$, we have $x(t) > 0$ and $x(\tau(t)) > 0$ on $[T_0, \infty)$. Defining $w(t)$ by (10), we obtain (13), which, in turn, yields

$$\liminf_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[H(t, s)\rho(s)q(s) - \frac{r(\tau(s))\rho(s)}{4KL\tau'(s)} Q^2(t, s)\right] ds \leq w(T)$$

$$- \limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[\sqrt{\frac{KLH(t, s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}} Q(t, s)\right]^2 ds,$$

for any $T \geq T_1$. It follows from (60) and from the latter inequality that

$$w(T) \geq \phi(T) + \limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[\sqrt{\frac{KLH(t, s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s)\right.$$

$$\left. + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}} Q(t, s)\right]^2 ds.$$

Consequently, for any $T \geq T_1$, the inequality (41) holds. Hence,

$$\limsup_{t\to\infty} \frac{1}{H(t, T_1)} \int_{T_1}^{t} \left[\sqrt{\frac{KLH(t, s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s) + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}} Q(t, s)\right]^2 ds$$

$$\leq M < \infty,$$

where the constant M is defined by (42). By the latter inequality,

$$\limsup_{t\to\infty} [\alpha(t) + \beta(t)] \leq \limsup_{t\to\infty} \frac{1}{H(t, T_1)} \int_{T_1}^{t} \left[\sqrt{\frac{KLH(t, s)\tau'(s)}{r(\tau(s))\rho(s)}} w(s)\right. \tag{61}$$

$$\left. + \frac{1}{2}\sqrt{\frac{r(\tau(s))\rho(s)}{KL\tau'(s)}} Q(t, s)\right]^2 ds \leq M,$$

where $\alpha(t)$ and $\beta(t)$ are defined in the proof of Theorem 4. By (60), we have

$$(62)\quad \phi(T_1) \le \liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} \left[H(t,s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)} Q^2(t,s) \right] ds$$

$$\le \liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} H(t,s)\rho(s)q(s)ds$$

$$- \liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)} Q^2(t,s)ds.$$

Inequalities (59) and (62) yield

$$(63)\qquad \liminf_{t\to\infty} \frac{1}{H(t,T_1)} \int_{T_1}^{t} \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)} Q^2(t,s)ds < \infty.$$

By (63), there exists a sequence of real numbers $\{t_n\}_{n=1}^{\infty} \in (T_0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$ and

$$(64)\qquad \lim_{n\to\infty} \frac{1}{H(t_n,T)} \int_{T}^{t_n} \frac{\rho(s)r(\tau(s))}{4KL\tau'(s)} Q^2(t_n,s)ds < \infty.$$

To complete the proof, we assume that (44) holds. Using the same argument as in the proof of Theorem 4, we can verify condition (47). It follows from (61) that there exists a natural number N such that (48) is satisfied. Proceeding as in the proof of Theorem 4, we arrive at (51) which contradicts inequality (64). Hence, (44) fails to hold. It follows from (41) and (52) that (53) holds, but this contradicts assumption (40) of the theorem. Thus, we conclude that Eq. (1) is oscillatory. □

Example 7. Consider the nonlinear delay differential equation

$$(65)\qquad \left(\frac{2+\exp 2t\sin^2 t}{3+\exp 2t\sin^2 t} \frac{3+x^2(t)}{2+x^2(t)} x'(t) \right)' + 2\exp(\pi/2)x(t-\pi/2) = 0,$$

where $t \ge 1$. It is easy to check that conditions (A1)–(A5), (4), and (5) are satisfied. Therefore, with $\rho(t) = t^{-2}$, we can check that assumptions (38)–(39) hold, and Eq. (65) is oscillatory by Theorem 7. Observe that $x(t) = \exp t \sin t$ is an oscillatory solution of Eq. (65).

Theorem 8. *Let H and h be as in Theorem* 1, *f, ψ and q satisfy* (25) *and* (6), *respectively, and assume that* (37) *holds. Suppose also that there exist functions $\rho \in C^1([t_0,\infty);(0,\infty))$ and $\phi \in C([t_0,\infty);(-\infty,\infty))$ such that, for all $t > t_0$ and for any $T \ge t_0$,*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4\gamma\tau'(s)} Q^2(t,s) \right] ds \geq \phi(T), \tag{66}$$

and conditions (40) *and* (59) *hold. Then Eq.* (1) *is oscillatory.*

Proof. Starting with the inequality (28), we proceed as in the proof of Theorem 7. □

Example 8. Consider the nonlinear delay differential equation

$$\left(\frac{1}{1+5\cos^4 4t}(1+5x^4(t))x'(t)\right)' + \frac{16}{1+\cos^4 4t} x\left(t-\frac{\pi}{2}\right)\left(1+x^4\left(t-\frac{\pi}{2}\right)\right) = 0, \tag{67}$$

where $t \geq 1$. Clearly, assumptions (A1)–(A5), (6), and (25) hold, so we can apply Theorem 8 choosing $\rho(t)$ as in the previous example. A straightforward verification of conditions of this criterion yields the oscillatory character of Eq. (67). Indeed, $x(t) = \sin t$ is an oscillatory solution of this equation.

Theorem 9. *Let H and h be as in Theorem* 1, *f, ψ and q satisfy* (30), (5), *and* (6), *respectively, and assume that* (37) *holds. Suppose also that there exist functions $\rho \in C^1([t_0,\infty);(0,\infty))$ and $\phi \in C([t_0,\infty);(-\infty,\infty))$ such that, for all $t > t_0$ and for any $T \geq t_0$,*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[KH(t,s)\rho(s)q(s) - \frac{\rho(s)r(\tau(s))}{4L\tau'(s)} Q^2(t,s) \right] ds \geq \phi(T), \tag{68}$$

and assumptions (40) *and* (59) *hold. Then Eq.* (1) *is oscillatory.*

Proof. The starting point for the proof is the inequality (35). The rest of the proof resembles that of Theorem 7. □

Example 9. Consider the nonlinear delay differential equation

$$\left(\frac{4+\cos^2 2t}{1+\cos^2 2t}\frac{1+x^2(t)}{4+x^2(t)}x'(t)\right)' + \frac{4(4+\cos^2 2t)}{40+\cos^2 2t} x(t-\pi)\left(1+\frac{36}{4+x^2(t-\pi)}\right) = 0, \tag{69}$$

where $t \geq 1$. It is not difficult to verify the assumptions (A1)–(A5), (5), (6), and (30). Therefore, we can apply Theorem 9 with the same choice of $\rho(t)$ as in Example 7 to demonstrate that Eq. (69) is oscillatory by Theorem 9. In fact, $x(t) = \cos 2t$ is an oscillatory solution of this equation.

2. Oscillation of ordinary differential equation

In this section, we present oscillation criteria for Eq. (2) which may be viewed as a particular case of Eq. (1). Thus some results which require assumption (6) (namely, Theorems 12, 15, 18 and eventual corollaries) for the former equation can be obtained directly from the corresponding theorems for the latter one. We stress that we do not need this assumption for other results presented in this section. Finally, we note that throughout this section we do not require condition (A5) related only to the delay differential equation (1), while assumptions (A1)–(A4) are tacitly supposed to hold.

Theorem 10. *Assume that assumptions* (4) *and* (5) *hold, and let* h, H *be as above. Assume that there exists a continuously differentiable function* $\rho : I \to \boldsymbol{R}_+$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4KL} Q^2(t,s) \right] ds = \infty. \tag{70}$$

Then Eq. (2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (2) and let $T_0 \ge t_0$ be such that $x(t) \ne 0$, for all $t \ge T_0$. Without loss of generality, we may assume that, for all $t \ge T_0$, we have $x(t) > 0$ since the similar argument holds also for the case when $x(t)$ is eventually negative. Define the function $w(t)$ by

$$w(t) = \rho(t)r(t)\psi(x(t))\frac{x'(t)}{f(x(t))}. \tag{71}$$

Differentiating (71) and making use of Eq. (2), we obtain

$$w'(t) = \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)q(t) - \frac{f'(x(t))}{\rho(t)r(t)\psi(x(t))} w^2(t). \tag{72}$$

By (4), (5), and (72), for $t \ge T_0$, we conclude that

$$w'(t) \le \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)q(t) - \frac{KL}{r(t)\rho(t)} w^2(t). \tag{73}$$

The remainder of the proof proceeds as that of Theorem 1. □

Corollary 4. *Assume that the assumptions of Theorem* 10 *hold with* (70) *replaced by*

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} H(t,s)\rho(s)q(s)ds = \infty,$$

$$\limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} r(s)\rho(s)Q^2(t,s)ds < \infty.$$

Then Eq. (2) *is oscillatory.*

Remark 5. We note that it is straightforward to deduce corollaries similar to Corollary 4 from Theorems 11 and 12 stated below.

Corollary 5 ([6, Theorem 2]. *Let assumptions* (4) *and* (5) *hold. Assume that there exists a continuously differentiable function* $\rho : I \to R_+$ *such that for some integer* $n > 2$

$$\limsup_{t\to\infty} t^{1-n} \int_{t_0}^{t} \left[(t-s)^{n-1} \rho(s) q(s) - \frac{r(s)\rho(s)}{4KL} (t-s)^{n-3} \left(n - 1 - \frac{\rho'(s)}{\rho(s)} (t-s) \right)^2 \right] ds = \infty.$$

Then Eq. (2) *is oscillatory.*

Example 10. Choosing $\rho(t) = 1$, it is not difficult to check that the nonlinear differential equation

$$\left((1 + 2\sin^2 t) \frac{1 + x^2(t)}{1 + 2x^2(t)} x'(t) \right)' + (3\sin^2 t - 1)x(t) = 0, \qquad t \ge 1. \tag{74}$$

is oscillatory by Corollary 5. In fact, $x(t) = \sin t$ is an oscillatory solution of Eq. (74).

Theorem 11 (cf. [8, Theorem 1], [12, Theorem 4]). *Let* h, H *be as in Theorem* 1, *and suppose that* (25) *holds. Assume also that there exists a continuously differentiable function* $\rho : I \to \boldsymbol{R}_+$ *such that*

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \left[H(t, s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4\gamma} Q^2(t, s) \right] ds = \infty. \tag{75}$$

Then Eq. (2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (2). As in Theorem 10, without loss of generality, we may assume that, for all $t \ge T_0$, we have $x(t) > 0$. Define again the function $w(t)$ by (71), obtaining in the same manner (72). By (25), we conclude from (72) that

$$w'(t) \le \frac{\rho'(t)}{\rho(t)} w(t) - \rho(t)q(t) - \frac{\gamma}{r(t)\rho(t)} w^2(t), \tag{76}$$

for $t \ge T_0$. The rest of the proof runs as in Theorem 10. □

Example 11. With the same choice of $\rho(t)$ as in the previous example, the nonlinear differential equation

$$(77)\qquad \left(\frac{11+7\sin^2 t}{11}(1+3x^2(t))x'(t)\right)' + \frac{3}{11}(23-35\sin^2 t)x(t)\big(1+x^2(t)\big) = 0,$$

where $t \geq 1$, is oscillatory by Theorem 11. Observe that $x(t) = \sin t$ is an oscillatory solution of this equation.

Theorem 12 (cf. [18, Theorem 2.2]). *Let assumptions* (5), (6), *and* (30) *hold, and let h, H be as above. Suppose also that there exists a continuously differentiable function $\rho : I \to \boldsymbol{R}_+$ such that*

$$(78)\qquad \limsup_{t\to\infty} \frac{1}{H(t,t_0)} \int_{t_0}^{t} \left[H(t,s)K\rho(s)q(s) - \frac{r(s)\rho(s)}{4L} Q^2(t,s) \right] ds = \infty.$$

Then Eq. (2) *is oscillatory.*

Proof. Let $x(t)$ be a nonoscillatory solution of Eq. (2). Without loss of generality, we may assume that, for all $t \geq T_0$, we have $x(t) > 0$. Define the function $w(t)$ letting

$$(79)\qquad w(t) = \rho(t)r(t)\psi(x(t))\frac{x'(t)}{x(t)}.$$

Differentiating (79) and making use of Eq. (2), we obtain

$$(80)\qquad w'(t) = \frac{\rho'(t)}{\rho(t)}w(t) - \rho(t)q(t) - \frac{1}{\rho(t)r(t)\psi(x(t))}w^2(t).$$

By (5), (6), and (30), for $t \geq T_1 = \max\{T_0, T_*\}$, we conclude from (80) that

$$(81)\qquad w'(t) \leq \frac{\rho'(t)}{\rho(t)}w(t) - K\rho(t)q(t) - \frac{L}{r(t)\rho(t)}w^2(t).$$

The rest of the proof follows the same lines as that of Theorem 1. □

Example 12. Consider the nonlinear ordinary differential equation

$$(82)\qquad \left((2+\cos^2 t)\left(1+\frac{59}{1894}\cos^2 t\right)\frac{3+x^2(t)}{2+x^2(t)}x'(t)\right)'$$
$$+\frac{1}{1894}(2+\cos^2 t)(77+295\cos^2 t)x(t)\left(1+\frac{18}{2+x^2(t)}\right) = 0,$$

where $t \geq 1$. Assumptions (A1)–(A4), (5), and (30) are easily verified and Eq. (82) is oscillatory by Theorem 12 with $\rho(t) = 1$. Observe that $x(t) = \cos t$ is an oscillatory solution of this equation.

Theorem 13. *Let H and h be as in Theorem* 1, *f and ψ satisfy* (4) *and* (5), *respectively, and assume that* (37) *holds. Suppose also that there exist functions*

$\rho \in C^1([t_0, \infty); (0, \infty))$ and $\phi \in C([t_0, \infty); (-\infty, \infty))$ such that, for all $t > t_0$ and for any $T \geq t_0$,

$$\limsup_{t\to\infty} \frac{1}{H(t, t_0)} \int_{t_0}^{t} \rho(s) r(s) Q^2(t, s) ds < \infty, \tag{83}$$

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[H(t, s)\rho(s)q(s) - \frac{\rho(s)r(s)}{4KL} Q^2(t, s) \right] ds \geq \phi(T), \tag{84}$$

$$\limsup_{t\to\infty} \int_{t_0}^{t} \frac{\phi_+^2(s)}{\rho(s)r(s)} ds = \infty, \tag{85}$$

where $\phi_+(t) = \max(\phi(t), 0)$. Then Eq. (2) is oscillatory.

Proof. As above, we assume that there exists a solution $x(t)$ of Eq. (2) such that, for some $T_0 \geq t_0$, one has $x(t) > 0$ on $[T_0, \infty)$. Defining $w(t)$ by (71), we obtain inequality (73). Then, for $t > T \geq T_1$, (73) yields

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[H(t, s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4KL} Q^2(t, s) \right] ds \leq w(T)$$
$$- \liminf_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[\sqrt{\frac{KLH(t, s)}{r(s)\rho(s)}} w(s) + \frac{1}{2} \sqrt{\frac{r(s)\rho(s)}{KL}} Q(t, s) \right]^2 ds.$$

The rest of the proof follows the same lines as that of Theorem 4. □

Example 13. Consider the nonlinear differential equation

$$\left((1 + \sin^2 t)^2 (1 + \cos^2 t) \frac{1}{1 + x^2(t)} x'(t) \right)' + \sin t (7 - 5\sin^2 t) x(t) = 0, \tag{86}$$

where $t \geq 1$. Conditions (A1)–(A5), (4), and (5) are satisfied, and we can apply Theorem 13 with $\rho(t) = 1$. A direct computation verifies assumptions (38)–(39), and Eq. (86) is oscillatory by Theorem 13. We note that $x(t) = \sin t$ is an oscillatory solution of Eq. (86).

Theorem 14 ([8, Theorem 3]). *Let H and h be as in Theorem 1, f and ψ satisfy (25), and suppose that (37) holds. Assume also that there exist functions $\rho \in C^1([t_0, \infty); (0, \infty))$ and $\phi \in C([t_0, \infty); (-\infty, \infty))$ such that, for all $t > t_0$ and for any $T \geq t_0$,*

$$\limsup_{t\to\infty} \frac{1}{H(t, T)} \int_{T}^{t} \left[H(t, s)\rho(s)q(s) - \frac{\rho(s)r(s)}{4\gamma} Q^2(t, s) \right] ds \geq \phi(T), \tag{87}$$

and assumptions (83) and (85) are satisfied. Then Eq. (2) is oscillatory.

Proof. We start with the following inequality

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4\gamma}Q^2(t,s)\right]ds\le w(T)$$
$$-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{\gamma H(t,s)}{r(s)\rho(s)}}w(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{\gamma}}Q(t,s)\right]^2 ds,$$

and the proof proceeds as that of Theorem 4. □

Example 14. Consider the nonlinear differential equation

$$\left(\left(1+\frac{1}{5}x^2(t)\right)x'(t)\right)'+\frac{3}{5}x(t)(1+x^2(t))=0, \tag{88}$$

where $t\ge 1$. Conditions (A1)–(A4), and (25) are satisfied and we can apply Theorem 14 letting $\rho(t)=t^{-2}$ to conclude that Eq. (88) is oscillatory. Indeed, $x(t)=\sin t$ is an oscillatory solution of this equation.

Theorem 15. *Let H and h be as in Theorem* 1, *f, ψ and q satisfy* (30), (5), *and* (6) *respectively, and suppose that* (37) *holds. Assume also that there exist functions $\rho\in C^1([t_0,\infty);(0,\infty))$ and $\phi\in C([t_0,\infty);(-\infty,\infty))$ such that, for all $t>t_0$ and for any $T\ge t_0$,*

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[KH(t,s)\rho(s)q(s)-\frac{\rho(s)r(s)}{4L}Q^2(t,s)\right]ds\ge\phi(T), \tag{89}$$

and (83) *and* (85) *are satisfied. Then Eq.* (2) *is oscillatory.*

Proof. Defining $w(t)$ by (79), we obtain

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[KH(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4L}Q^2(t,s)\right]ds\le w(T)$$
$$-\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{LH(t,s)}{r(s)\rho(s)}}w(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{L}}Q(t,s)\right]^2 ds.$$

The rest of the proof follows the same lines as that of Theorem 4. □

Example 15. Consider the nonlinear differential equation

$$\left(\frac{1+\cos^2 t}{2+\cos^2 t}\frac{2+x^2(t)}{1+x^2(t)}x'(t)\right)'+\frac{9(1+\cos^2 t)}{10+\cos^2 t}x(t)\left(\frac{1}{9}+\frac{1}{1+x^2(t)}\right)=0, \tag{90}$$

where $t\ge 1$. Assumptions (A1)–(A4), (5), (6), and (30) are easily verified, and we can apply Theorem 15 taking $\rho(t)$ as in the previous example to prove that

Eq. (90) is oscillatory. Observe that $x(t) = \cos t$ is an oscillatory solution of this equation.

Theorem 16. *Let H and h be as in Theorem* 1, *f and ψ satisfy* (4) *and* (5), *respectively, and assume that* (37) *holds. Suppose also that there exist functions $\rho \in C^1([t_0, \infty); (0, \infty))$ and $\phi \in C([t_0, \infty); (-\infty, \infty))$ such that, for all $t > t_0$ and for any $T \geq t$,*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\rho(s)r(s)}{4KL} Q^2(t,s) \right] ds \geq \phi(T), \tag{91}$$

and assumptions (59) *and* (85) *hold. Then Eq.* (2) *is oscillatory.*

Proof. Again, we assume that there exists a solution $x(t)$ of Eq. (2) such that, for some $T_0 \geq t_0$, one has $x(t) > 0$ on $[T_0, \infty)$. Defining $w(t)$ by (71), we obtain (73), which, in turn, yields

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4KL} Q^2(t,s) \right] ds \leq w(T)$$
$$- \limsup_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[\sqrt{\frac{KLH(t,s)}{r(s)\rho(s)}} w(s) + \frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{KL}} Q(t,s) \right]^2 ds.$$

The rest of the proof follows the same lines as that of Theorem 7. □

Example 16. Consider the nonlinear differential equation

$$\left((1 + \sin^2 2t)\left(1 + \frac{1}{5}\sin^2 2t\right) \frac{2}{1 + x^2(t)} x'(t) \right)' + \frac{24}{5} x(t)(1 + x^2(t)) = 0, \tag{92}$$

where $t \geq 1$. Assumptions (A1)–(A4), (4), and (5) are easily verified. Hence, we can apply Theorem 16 with $\rho(t) = t^{-2}$ to conclude that Eq. (92) is oscillatory. In fact, $x(t) = \sin 2t$ is an oscillatory solution of Eq. (92).

Theorem 17 ([8, Theorem 4]). *Let H and h be as in Theorem* 1, *f and ψ satisfy* (25), *and assume that* (37) *holds. Suppose also that there exist functions $\rho \in C^1([t_0, \infty); (0, \infty))$ and $\phi \in C([t_0, \infty); (-\infty, \infty))$ such that, for all $t > t_0$ and for any $T \geq t_0$,*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{\rho(s)r(s)}{4\gamma} Q^2(t,s) \right] ds \geq \phi(T), \tag{93}$$

and (59) *and* (85) *hold. Then Eq.* (2) *is oscillatory.*

Proof. Assume that there exists a solution $x(t)$ of Eq. (2) such that, for some $T_0 \geq t_0$, one has $x(t) > 0$ on $[T_0, \infty)$. Defining $w(t)$ by (71), we obtain

(73), which yields

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4\gamma}Q^2(t,s)\right]ds\le w(T)$$

$$-\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{KLH(t,s)}{r(s)\rho(s)}}w(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{\gamma}}Q(t,s)\right]^2 ds.$$

To complete the proof, we proceed as in that of Theorem 7. □

Example 17. Consider the nonlinear differential equation

$$\left(\left(1+\frac{1}{5}x^2(t)\right)x'(t)\right)'+\frac{12}{5}x(t)(1+x^2(t))=0, \tag{94}$$

where $t\ge 1$. Clearly, assumptions (A1)–(A4), and (25) hold, so we can apply Theorem 17 with $\rho(t)=t^{-2}$. A straightforward calculation yields that Eq. (94) is oscillatory by Theorem 17. Indeed, $x(t)=\cos 2t$ is an oscillatory solution of this equation.

Theorem 18. *Let H and h be as in Theorem* 1, *f, ψ and q satisfy* (30), (5), *and* (6), *respectively, and assume that* (37) *holds. Suppose further that there exist functions $\rho\in C^1([t_0,\infty);(0,\infty))$ and $\phi\in C([t_0,\infty);(-\infty,\infty))$ such that, for all $t>t_0$ and for any $T\ge t_0$,*

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[KH(t,s)\rho(s)q(s)-\frac{\rho(s)r(\tau(s))}{4L}Q^2(t,s)\right]ds\ge\phi(T), \tag{95}$$

and assumptions (59) *and* (85) *hold. Then Eq.* (2) *is oscillatory.*

Proof. Defining $w(t)$ by (79), we obtain

$$\liminf_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[KH(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4L}Q^2(t,s)\right]ds\le w(T)$$

$$-\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t\left[\sqrt{\frac{LH(t,s)}{r(s)\rho(s)}}w(s)+\frac{1}{2}\sqrt{\frac{r(s)\rho(s)}{L}}Q(t,s)\right]^2 ds.$$

The remainder of the proof follows the lines of the proof of Theorem 7. □

Example 18. Consider the nonlinear differential equation

$$\left(\frac{1+\sin^2 t}{2+\cos^2 t}\,\frac{2+x^2(t)}{1+x^2(t)}x'(t)\right)'+\frac{9(1+\sin^2 t)}{10+\sin^2 t}x(t)\left(\frac{1}{9}+\frac{1}{1+x^2(t)}\right)=0, \tag{96}$$

where $t\ge 1$. Assumptions (A1)–(A4), (5), (6), and (30) are verified easily.

Therefore, we can apply Theorem 18 taking, for example, $\rho(t) = t^{-2}$ to conclude that Eq. (96) is oscillatory. Observe that $x(t) = \sin t$ is an oscillatory solution of this equation.

3. Discussion

In this paper, we have proposed for Eqs. (1) and (2) three sets of criteria, referred to in the sequel as (A)–(C), which guarantee oscillation of all proper solutions. For each set, we required one of the following group of assumptions

(A) (4) and (5);
(B) (25);
(C) (30) and (5).

In addition, we required (6) for all criteria concerning Eq. (1) and only for the set (C) for Eq. (2).

Some our results for the first two sets of assumptions (namely, Theorems 11, 14 and 17 and Corollary 5) for Eq. (2) are closely related to those derived by Grace [8, 9], Grace, Lalli and Yeh [12], Kirane and Rogovchenko [18], and Rogovchenko [25] and may be viewed as their natural extension and refinement. For instance, Theorem 11 generalizes the result by Grace, Lalli and Yeh [12, Theorem 4]. Furthermore, Theorem 11 improves another theorem due to Grace [8, Theorem 4] because we do not require f to satisfy

$$\int^{\infty} \frac{du}{f(u)} = \infty.$$

On the other hand, we do not require either the reverse condition

$$\int^{\infty} \frac{du}{f(u)} < +\infty, \qquad \int^{-\infty} \frac{du}{f(u)} < +\infty, \tag{97}$$

which has been mentioned by Cecchi and Marini [2, p. 1260] as the one imposed by the majority of authors. We note that condition (x) in Theorem B, which has been proposed by Cecchi and Marini as an alternative to (97), fails to hold for Eqs. (24), (29), (74), and (77).

The third set of sufficient conditions (C) is essentially new. We note that none of the theorems in [2, 3, 12, 19, 31] applies to Eqs. (36), (58), (69), (82), (90), and (96).

We do not require in this paper assumption (vii) of Theorem A. Therefore, our theorems can be applied to certain classes of equations like, for instance, Emden-Fowler type equations (3), to which the results due to Bradley [1], Cecchi and Marini [2], Grace [7, 9], Grace, Lalli and Yeh [12], and Philos and Sficas [23] fail to apply.

The results presented in this paper are of high degree of generality, which yields more complicated in comparison to [2] conditions to be verified.

Nevertheless, application of our results to specific equations requires mainly a routine computation which can be assisted by symbolic computer languages like *Maple*® or *Mathematica*®. We note that, though being very simple, assumptions (vi) in Theorem A and (ix) in Theorem **B** both fail to hold, for instance, for the differential equation

$$\left(\frac{1}{t}x'(t)\right)' + \frac{2}{t^3}x(t) = 0, \qquad t > 0, \tag{98}$$

while the oscillatory character of this equation can be easily established using our Theorem 1 with H and h defined by (23) and with $\rho(t) = t^2$. One can check that $x(t) = t\sin(\ln t)$ is an oscillatory solution of Eq. (98).

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