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Penalty Method for Variational Inequalities and Its Error Estimates

By

Masayoshi TSUTSUMI and Takatoshi YASUDA

(Waseda University, Japan)

1. Introduction

The study of variational inequalities goes back to the work of Fichera ([7]) and Stampacchia ([16], [17]) in the sixties. After the fundamental work of Lions and Stampacchia ([13]), the theory of variational inequalities was studied by many researchers (e.g. Brézis ([2], [3]), Browder ([4], [5]), Kinderlehrer ([10], [11]), Duvaut and Lions ([6]), Friedman ([8]), Baiocchi and Capelo ([1]), and others) and became an important subject in nonlinear analysis. Detailed presentation and surveys of the theory of variational inequalities and their applications may be found in [1], [6], [8], [11], [12] and [14].

In this paper we study the penalty method for the variational inequalities, which was proposed in the book of Lions ([12]). In order to investigate the precise behavior of solutions of variational inequalities or to obtain numerical solutions, it is very convenient to consider solutions of the approximate equations with penalty terms, since we can apply the established methods of nonlinear partial differential equations and those of numerical analysis to them. We need error estimates to justify numerical calculus. When we get error estimates, it is important to distinguish the theoretical errors by approximation of penalty term from the numerical one by descritization. The study of error estimates of penalty method is still open problem so far.

Our purpose of this paper is to obtain error estimates on the penalty parameter of the approximated problem by penalty for solving the variational inequalities described by the nonlinear p-Laplacian. First in section 2 we discuss some generalization of the results of Lions ([12]). Section 3 is devoted to discuss error estimates for the penalty.

2. Existence Theorem

Let V and W be two reflexive Banach spaces continuously embedded in a linear Hausdorff space X, and satisfying that $V \cap W \neq \phi$ is dense in both V and W. The norms of V and W are denoted by $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. Then $V \cap W$ is a reflexive Banach space with norm $\|\cdot\|_{V \cap W} = \|\cdot\|_V + \|\cdot\|_W$. We denote by $\langle \cdot, \cdot \rangle$ the pairing between V and V' and between W and W'. The following theorem is a generalization of the well-known theorem in the book [12]:

Theorem 1. *Let $K \neq \phi$ be a closed convex subset of $V \cap W$ and β be a corresponding penalty operator from $V \cap W$ into W', say, $K = \{u \in W \cap V \mid \beta(u) = 0\}$. Let A be a pseudo-monotone operator from $V \to V'$. Assume that there exists a $v_0 \in K$ such that for any fixed $\varepsilon > 0$*

$$\frac{1}{\|u\|_{V \cap W}} \langle A(u) + \frac{1}{\varepsilon}\beta(u), u - v_0 \rangle \to \infty$$

as $\|u\|_{V \cap W} \to \infty$. Let $f \in V' + W'$. Then, there exists a sequence $\{u_n\}$, where each $u_n \in V \cap W$ satisfies

$$A(u_n) + \frac{1}{\varepsilon(n)}\beta(u_n) = f$$

with $\varepsilon(n) \to 0+$ as $n \to \infty$, which converges weakly in $V \cap W$ towards a solution $u \in V \cap W$ to the problem

$$(1) \qquad \langle A(u) - f, v - u \rangle \geq 0 \qquad \forall v \in K.$$

We suppose that if $\langle A(u) - A(v), u - v \rangle \leq 0$ then $u = v$. Then u is unique. Moreover we suppose

$$\langle A(u) - A(v), u - v \rangle \geq \varphi(\|u - v\|_V)$$

where $\varphi(\cdot)$ is a nonnegative function such that if $\varphi(\|x\|_V) \to 0$ then $x \to 0$ in V. Then u_n converges to u strongly in V.

Remark. We don't have to assume any injection between V and W, which was required in Lions's results ([12]), so this result is an extension of them.

Proof of theorem 1. By a perturbation theorem of pseudo-monotone operators (see [12]) it is easily seen that for any $\varepsilon > 0$ the operator $u \mapsto A(u) + \frac{1}{\varepsilon}\beta(u)$ is a pseudo-monotone operator from $V \cap W \to V' + W'$. By assumption it is also coercive. Hence the well-known theorem of pseudo-monotone operators yields that for any $f \in V' + W'$ there exists at least one element $u_\varepsilon \in V \cap W$ satisfying

$$(2) \qquad A(u_\varepsilon) + \frac{1}{\varepsilon}\beta(u_\varepsilon) = f.$$

Moreover, the coerciveness assumption implies that $\|u_\varepsilon\|_{V \cap W} \leq C$ where and in the sequel by C we denote a various positive constant independent of

ε. Therefore, $\|A(u_\varepsilon)\|_{V'} \le C$. From (2) we have

$$|\langle \beta(u_\varepsilon), u_\varepsilon \rangle| \le \varepsilon(\|f\|_{V'+W'}\|u_\varepsilon\|_{V\cap W} + \|A(u_\varepsilon)\|_{V'}\|u_\varepsilon\|_V)$$
$$\le C\varepsilon.$$

Hence, as $\varepsilon \to 0$,

$$\langle \beta(u_\varepsilon), u_\varepsilon \rangle \to 0.$$

Analogously, for any $v \in V \cap W$, as $\varepsilon \to 0$,

$$\langle \beta(u_\varepsilon), v \rangle \to 0.$$

Since any bounded set in a reflexive Banach space is weakly sequentialy compact, we can find a sequence $\{\varepsilon(n)\}$ such that $\varepsilon(n) \to 0$ and $u_n = u_{\varepsilon(n)} \rightharpoonup u$ in $V \cap W$ as $n \to \infty$.

Let $v \in V \cap W$ be arbitrary fixed. Then

$$\langle \beta(u_n) - \beta(v), u_n - v \rangle \ge 0$$

yields

$$\langle \beta(v), u - v \rangle \le 0 \qquad \forall v \in V \cap W.$$

Taking $v = u - \lambda w, \lambda > 0$ and $w \in V \cap W$, and letting $\lambda \to 0+$, we have $\langle \beta(u), w \rangle \le 0$. Hence $\beta(u) = 0$, that is, $u \in K$.

We fix $v \in K$ so that $\beta(v) = 0$. From the equation we deduce

$$\langle A(u_n) - f, v - u_n \rangle = \frac{1}{\varepsilon(n)} \langle \beta(v) - \beta(u_n), v - u_n \rangle \ge 0$$

from which it follows that

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \le \limsup_{n\to\infty} \langle f, u_n - u \rangle = 0.$$

Finally by pseudo-monotonicity we have

$$\liminf_{n\to\infty} \langle A(u_n), u_n - v \rangle \ge \langle A(u), u - v \rangle.$$

Hence

$$\langle A(u), v - u \rangle \ge \langle f, v - u \rangle \qquad \forall v \in K.$$

The strong convergence follows from the fact that

$$\limsup_{n\to\infty} \langle A(u_n) - A(u), u_n - u \rangle$$
$$\le \limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle - \lim_{n\to\infty} \langle A(u), u_n - u \rangle \le 0.$$

The uniqueness is as follows. If u_1 and u_2 are two solutions of (1), then we have

$$-\langle A(u_1) - A(u_2), u_1 - u_2 \rangle \geq 0,$$

and it implies the uniqueness.

3. Error estimates

Our aim of this paper is to obtain the error estimates of solutions of the penalized problem. Since construction of the general theory of error estimations in an abstract form is rather difficult, we here treat some concrete examples.

Example 1. Let Ω be a bounded domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega$ and let $V = W_0^{1,p}(\Omega)$ $(1 < p < \infty)$ and $W = L^q(\Omega)$ $(1 < q < \infty)$. Suppose that A is a monotone hemicontinuous operator from $V \to V'$, given by

$$A(\varphi) = -\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\left|\frac{\partial\varphi}{\partial x_j}\right|^{p-2} \frac{\partial\varphi}{\partial x_j}\right).$$

For $\varphi \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$, let K be a closed convex subset of $L^q(\Omega)$ given by

$$K = \{v | v \in W_0^{1,p}(\Omega) \cap L^q(\Omega), \quad v \geq \varphi \quad \text{a.e. in } \Omega\}.$$

Then, choosing a duality map from $W \to W'$ by $J(v) = |v|^{q-2}v$, we have

$$\beta(v) = J(v - P_K v) = -|(v-\varphi)^-|^{q-2}(v-\varphi)^-$$

where P_K is the projection on K in W, say, $P_K v = (v-\varphi)^+ + \varphi$. Here $(v-\varphi)^-$ is defined by

$$(v-\varphi)^+(x) = \begin{cases} v(x) - \varphi(x) & \text{if } v(x) \geq \varphi(x), \\ 0 & \text{else,} \end{cases}$$

that is, $v = (v-\varphi)^+ - (v-\varphi)^- + \varphi$.

The corresponding penalized problem is as follows:

$$u_\varepsilon \in W_0^{1,p}(\Omega) \cap L^q(\Omega),$$

(3) $$\sum_{j=1}^{N} \frac{\partial}{\partial x_j}\left(\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2} \frac{\partial u_\varepsilon}{\partial x_j}\right) + \frac{1}{\varepsilon}|(u_\varepsilon - \varphi)^-|^{q-2}(u_\varepsilon - \varphi)^- = f.$$

By Theorem 1 we see that there exists a unique solution $u \in K$ such that

(4) $$\langle A(u), v - u \rangle \geq \langle f, v - u \rangle \qquad \forall v \in K.$$

Theorem 2. *Suppose that* $f, A(\varphi) \in L^{q'}(\Omega)$. *Then if* $p \geq 2$,

$$\|u_\varepsilon - u\|_{W_0^{1,p}} \leq C\varepsilon^{p'/p^2(q-1)}.$$

Theorem 3. *Suppose that* $f \in L^{q'}(\Omega)$ *and* $\varphi = 0$. *Then if* $p \geq 2$,

$$\|u_\varepsilon - u\|_{W_0^{1,p}} \leq C\varepsilon^{1/p(q-1)}.$$

Remark. Even if $\varphi \neq 0$, we can reduce the problem to the case $\varphi = 0$ when A is linear.

Proof of theorem 2. Note that $(u_\varepsilon - \varphi)^- \in W_0^{1,p}(\Omega) \cap L^q(\Omega)$ (see [9]). Multiplying (3) by $(u_\varepsilon - \varphi)^-$ and integrating by parts over Ω, we have

$$\sum_{j=1}^{N} \int_{u_\varepsilon \leq \varphi} \left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2} \frac{\partial u_\varepsilon}{\partial x_j}\left(\frac{\partial u_\varepsilon}{\partial x_j} - \frac{\partial \varphi}{\partial x_j}\right) dx + \frac{1}{\varepsilon}\int_\Omega |(u_\varepsilon - \varphi)^-|^q dx$$
$$= \int_\Omega -f(u_\varepsilon - \varphi)^- dx.$$

Then we have

$$\sum_{j=1}^{N} \int_{u_\varepsilon \leq \varphi} \left(\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2} \frac{\partial u_\varepsilon}{\partial x_j} - \left|\frac{\partial \varphi}{\partial x_j}\right|^{p-2} \frac{\partial \varphi}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_j} - \frac{\partial \varphi}{\partial x_j}\right) dx + \frac{1}{\varepsilon}\int_\Omega |(u_\varepsilon - \varphi)^-|^q dx$$
$$= \int_\Omega (-f - A\varphi)(u_\varepsilon - \varphi)^- dx \leq \|f + A\varphi\|_{L^{q'}} \|(u_\varepsilon - \varphi)^-\|_{L^q}$$
$$\leq C\varepsilon^{q'-1} + \frac{1}{2\varepsilon}\|(u_\varepsilon - \varphi)^-\|_{L^q}^q.$$

Hence for $p \geq 2$,

$$\sum_{j=1}^{N} \int_{u_\varepsilon \leq \varphi} \left|\frac{\partial u_\varepsilon}{\partial x_j} - \frac{\partial \varphi}{\partial x_j}\right|^p dx \leq C\varepsilon^{1/(q-1)}, \tag{5}$$

$$\int_\Omega |(u_\varepsilon - \varphi)^-|^q dx \leq C\varepsilon^{q'}. \tag{6}$$

Next, multiplying (3) by $(u_\varepsilon - \varphi)^+ - (u - \varphi)$ and integrating by parts over Ω, we have

$$\sum_{j=1}^{N}\int_{\Omega}\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2}\frac{\partial u_\varepsilon}{\partial x_j}\frac{\partial}{\partial x_j}((u_\varepsilon-\varphi)^+-(u-\varphi))dx \tag{7}$$
$$+\frac{1}{\varepsilon}\int_{\Omega}|(u_\varepsilon-\varphi)^-|^{q-2}(u_\varepsilon-\varphi)^-(u-\varphi)dx$$
$$=\int_{\Omega}f((u_\varepsilon-\varphi)^+-(u-\varphi))dx.$$

Here, since $u\in K$, we remark

$$\frac{1}{\varepsilon}\int_{\Omega}|(u_\varepsilon-\varphi)^-|^{q-2}(u_\varepsilon-\varphi)^-(u-\varphi)dx\geq 0. \tag{8}$$

Applying $(u_\varepsilon-\varphi)^+ +\varphi\in K$ to (4), we have

$$\sum_{j=1}^{N}\int_{\Omega}\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_j}((u_\varepsilon-\varphi)^+-(u-\varphi))dx \tag{9}$$
$$\geq\int_{\Omega}f((u_\varepsilon-\varphi)^+-(u-\varphi))dx.$$

Combining (7) and (9) under (8), we have

$$\sum_{j=1}^{N}\int_{u_\varepsilon\geq\varphi}\left(\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2}\frac{\partial u_\varepsilon}{\partial x_j}-\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial u}{\partial x_j}\right)dx \tag{10}$$
$$\leq\sum_{j=1}^{N}\int_{u_\varepsilon\leq\varphi}\left(\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2}\frac{\partial u_\varepsilon}{\partial x_j}-\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}\right)\left(\frac{\partial u}{\partial x_j}-\frac{\partial \varphi}{\partial x_j}\right)dx$$
$$=\sum_{j=1}^{N}\int_{u_\varepsilon\leq\varphi}\left(\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2}\frac{\partial u_\varepsilon}{\partial x_j}-\left|\frac{\partial \varphi}{\partial x_j}\right|^{p-2}\frac{\partial \varphi}{\partial x_j}\right)\left(\frac{\partial u}{\partial x_j}-\frac{\partial \varphi}{\partial x_j}\right)dx$$
$$-\sum_{j=1}^{N}\int_{u_\varepsilon\leq\varphi}\left(\left|\frac{\partial u}{\partial x_j}\right|^{p-2}\frac{\partial u}{\partial x_j}-\left|\frac{\partial \varphi}{\partial x_j}\right|^{p-2}\frac{\partial \varphi}{\partial x_j}\right)\left(\frac{\partial u}{\partial x_j}-\frac{\partial \varphi}{\partial x_j}\right)dx.$$

Hence we have

$$\sum_{j=1}^{N}\int_{u_\varepsilon\geq\varphi}\left|\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial u}{\partial x_j}\right|^{p}dx+\sum_{j=1}^{N}\int_{u_\varepsilon\leq\varphi}\left|\frac{\partial u}{\partial x_j}-\frac{\partial \varphi}{\partial x_j}\right|^{p}dx$$
$$\leq C\left(\sum_{j=1}^{N}\int_{u_\varepsilon\leq\varphi}\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p}dx+\sum_{j=1}^{N}\int_{u_\varepsilon\leq\varphi}\left|\frac{\partial \varphi}{\partial x_j}\right|^{p}dx\right)^{(p-2)/p}$$

$$\cdot\left(\sum_{j=1}^{N}\int_{u_\varepsilon\le\varphi}\left|\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial\varphi}{\partial x_j}\right|^p dx\right)^{1/p}\left(\sum_{j=1}^{N}\int_{u_\varepsilon\le\varphi}\left|\frac{\partial u}{\partial x_j}-\frac{\partial\varphi}{\partial x_j}\right|^p dx\right)^{1/p}$$

$$\le C\left(\sum_{j=1}^{N}\int_{u_\varepsilon\le\varphi}\left|\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial\varphi}{\partial x_j}\right|^p dx\right)^{p'/p}+\frac{1}{2}\sum_{j=1}^{N}\int_{u_\varepsilon\le\varphi}\left|\frac{\partial u}{\partial x_j}-\frac{\partial\varphi}{\partial x_j}\right|^p dx.$$

Then we have

$$\sum_{j=1}^{N}\int_{u_\varepsilon\ge\varphi}\left|\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial u}{\partial x_j}\right|^p dx+\sum_{j=1}^{N}\int_{u_\varepsilon\le\varphi}\left|\frac{\partial u}{\partial x_j}-\frac{\partial\varphi}{\partial x_j}\right|^p dx\le C\varepsilon^{p'/p(q-1)}. \tag{11}$$

Combining (5) and (11), it follows that

$$\sum_{j=1}^{N}\int_{\Omega}\left|\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial u}{\partial x_j}\right|^p dx$$

$$\le\sum_{j=1}^{N}\int_{u_\varepsilon\ge\varphi}\left|\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial u}{\partial x_j}\right|^p dx+\sum_{j=1}^{N}\int_{u_\varepsilon\le\varphi}\left|\frac{\partial u_\varepsilon}{\partial x_j}-\frac{\partial\varphi}{\partial x_j}\right|^p dx$$

$$+\sum_{j=1}^{N}\int_{u_\varepsilon\le\varphi}\left|\frac{\partial u}{\partial x_j}-\frac{\partial\varphi}{\partial x_j}\right|^p dx$$

$$\le C\varepsilon^{p'/p(q-1)}.$$

Thus we have the theorem.

Proof of theorem 3. We get the following lemma.

Lemma 1. *Suppose that* $f\in L^q(\Omega)$. *We have*

$$\|\nabla u_\varepsilon^-\|_{L^p}\le C\varepsilon^{1/p(q-1)}. \tag{12}$$

Proof. Note that $u_\varepsilon^-\in W_0^{1,p}(\Omega)\cap L^q(\Omega)$ (see [9]). Multiplying (3) by u_ε^- and integrating by parts over Ω, we have

$$\sum_{j=1}^{N}\int_{\Omega}\left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^p dx+\frac{1}{\varepsilon}\int_{\Omega}|u_\varepsilon^-|^q dx=\int_{\Omega}fu_\varepsilon^- dx$$

$$\le\|f\|_{L^{q'}}\|u_\varepsilon^-\|_{L^q}\le C\varepsilon^{q'-1}+\frac{1}{2\varepsilon}\|u_\varepsilon^-\|_{L^q}^q$$

from which we obtain (12).

To prove the theorem 3, it suffices to show that

$$\int_\Omega |\nabla u_\varepsilon^+ - \nabla u|^p dx \le C\varepsilon^{1/(q-1)}.$$

Multiplying (3) by $u_\varepsilon^+ - u$ and integrating by parts over Ω, we have

$$\sum_{j=1}^N \int_\Omega \left|\frac{\partial u_\varepsilon}{\partial x_j}\right|^{p-2} \frac{\partial u_\varepsilon}{\partial x_j}\left(\frac{\partial u_\varepsilon^+}{\partial x_j} - \frac{\partial u}{\partial x_j}\right)dx + \frac{1}{\varepsilon}\int_\Omega |u_\varepsilon^-|^{q-2}u_\varepsilon^-(u_\varepsilon^+ - u)dx$$
$$= \int_\Omega f(u_\varepsilon^+ - u)dx.$$

Since $u_\varepsilon^+ \in K$, we have

$$\sum_{j=1}^N \int_\Omega \left|\frac{\partial u}{\partial x_j}\right|^{p-2} \frac{\partial u}{\partial x_j}\left(\frac{\partial u_\varepsilon^+}{\partial x_j} - \frac{\partial u}{\partial x_j}\right)dx \ge \sum_{j=1}^N \int_\Omega f(u_\varepsilon^+ - u)dx,$$

and it follows

$$\sum_{j=1}^N \int_\Omega \left(\left|\frac{\partial u_\varepsilon^+}{\partial x_j}\right|^{p-2} \frac{\partial u_\varepsilon^+}{\partial x_j} - \left|\frac{\partial u}{\partial x_j}\right|^{p-2} \frac{\partial u}{\partial x_j}\right)\left(\frac{\partial u_\varepsilon^+}{\partial x_j} - \frac{\partial u}{\partial x_j}\right)dx$$
$$\le \sum_{j=1}^N \int_\Omega \left|\frac{\partial u_\varepsilon^-}{\partial x_j}\right|^{p-2} \frac{\partial u_\varepsilon^-}{\partial x_j}\left(\frac{\partial u_\varepsilon^+}{\partial x_j} - \frac{\partial u}{\partial x_j}\right)dx,$$

for,

$$\int_\Omega |u_\varepsilon^-|^{q-2}u_\varepsilon^-(u_\varepsilon^+ - u)dx = -\int_\Omega |u_\varepsilon^-|^{q-2}u_\varepsilon^- u dx \le 0.$$

Hence, for $p \ge 2$ we have

$$\sum_{j=1}^N \int_\Omega \left|\frac{\partial u_\varepsilon^+}{\partial x_j} - \frac{\partial u}{\partial x_j}\right|^p dx \le C\left(\sum_{j=1}^N \int_\Omega \left|\frac{\partial u_\varepsilon^-}{\partial x_j}\right|^p dx\right)^{(p-1)/p}\left(\sum_{j=1}^N \int_\Omega \left|\frac{\partial u_\varepsilon^+}{\partial x_j} - \frac{\partial u}{\partial x_j}\right|^p dx\right)^{1/p}$$

from which it follows that

$$\sum_{j=1}^N \int_\Omega \left|\frac{\partial u_\varepsilon^+}{\partial x_j} - \frac{\partial u}{\partial x_j}\right|^p dx \le C\sum_{j=1}^N \int_\Omega \left|\frac{\partial u_\varepsilon^-}{\partial x_j}\right|^p dx \le C\varepsilon^{1/q-1}.$$

Thus we have the theorem.

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nuna adreso:
Department of Applied Physics
Waseda University
Tokyo 169-8555
Japan
E-mail: tutumi@mn.waseda.ac.jp

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