Funkcialaj Ekvacioj, **41** (1998) 67–78

On a Quasilinear Wave Equation with Strong Damping

By

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1. Introduction

We consider the initial boundary value problem for the following nonlinear hyperbolic equation with strong damping

(1.1) $$u'' - f(|\nabla u|^2)\Delta u - \Delta u' = 0 \quad \text{in } \Omega \times \boldsymbol{R}_+,$$

(1.2) $$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma \times \boldsymbol{R}_+,$$

(1.3) $$u(0) = u_0, \quad u'(0) = u_1 \quad \text{in } \Omega.$$

Here Ω is a bounded domain in $\boldsymbol{R}^N$ with sufficiently smooth boundary Γ, ν is the unit outward normal to Γ and $f(\cdot)$ a real valued $C^1([0,\infty))$-function.

The equation (1.1) has its motivation in the mathematical description of small amplitude vibrations of an elastic stretched string with strong damping.

The equation (1.1) with Dirichlet condition was studied by several authors [3], [5], [6] and [7] (to name but a few). In the previous papers, the authors have shown the existence and uniqueness of global solutions by using the Galerkin method, the operator $-\Delta$ with Dirichlet boundary condition has an infinite sequence of eigenvalues (λ_j^2) with

$$0 < \lambda_1^2 \leq \lambda_2^2 \leq \dots, \quad \lambda_j^2 \to \infty \quad \text{as } j \to \infty$$

and there exists a complete orthonormal system (w_j) in $L^2(\Omega)$, each w_j being an eigenvector to λ_j^2.

To the best of my knowledge, there is no result concerning the global existence and decay properties in the case of Neumann boundary condition. The purpose of the present paper is to examine whether there exists a global solution u to the Neumann problem (1.1)–(1.3) and to study its asymptotic behavior.

The proof of the global solvability is carried out by a Galerkin method. A key point of the proof is to obtain a complete orthonormal system (w_j) in a closed subspace of L^2, then the equation (1.1) can be solved in the closed subspace.

The contents of this paper are as follows, in section 2 we prove the global in time solvability of (1.1)–(1.3) (theorem 2.2). In the section 3, we study the asymptotic behavior in the two cases $f(x) \geq m_0, f'(x) \geq 0\ \forall x \in \boldsymbol{R}$ by using an integral inequality due to Komornik [2] (theorem 3.1), and when $f(x) = |x|^\gamma, \gamma \geq 0$, that is the equation (1.1) is degenerate because $f(0) = 0$, we use a device due to Nakao [4] (theorem 3.2). Finally in theorem 3.3 we prove that the polynomial decay rate obtained in theorem 3.2 is optimal by deriving the decay estimate from below of the solution.

2. Global existence

We define the linear operator A in $L^2(\Omega)$ as follows

$$D(A) = \left\{u \in H^2(\Omega), \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \Gamma\right\}$$

$$Au = -\Delta u \quad \forall u \in D(A).$$

It is well known that A is nonnegative selfadjoint operator with compact resolvent $(I + \lambda A)^{-1}$ for all $\lambda > 0$.

We have the following lemma

Lemma 2.1. *Let H be a real space with inner product $(,)$ and norm $|\cdot|$. Let A be a nonnegative selfadjoint operator with domain $D(A)$ and range $R(A)$ in H. Suppose that $(I + A)^{-1}$ is a compact operator. Then*

(a) *$R(A)$ is closed and $H = N(A) \oplus R(A)$,*

(b) *$(A|_{R(A)})^{-1} : R(A) \to R(A)$ is compact, where $A|_{R(A)}$ is the restriction of A to $R(A)$.*

Proof. The outline of a proof of (a) will be found in [1]. So we will show only (b). Assume that $y_n \in R(A)$ satisfies $|y_n| \leq 1$. Let $x_n \in D(A) \cap R(A)$ such that $y_n = Ax_n$. Then we have $|Ax_n| \leq 1$. We want to show that x_n is bounded. To see this, suppose $|x_n| \to \infty$ as $n \to \infty$ and set $u_n = x_n/|x_n|$. Then $|u_n| = 1$ and $Au_n \to 0$ $(n \to \infty)$. Since we can write $u_n = (I + A)^{-1}(u_n + Au_n)$, it follows from the compactness of $(I + A)^{-1}$ that there is a subsequence of u_n (we denote it by the same symbol) such that $u_n \to u \in R(A)$ and $|u| = 1$. Since A is closed, it follows that $u \in D(A)$ and $Au = 0$. This shows that $u \in N(A) \cap R(A)$. So we get $u = 0$. This contradicts to $|u| = 1$. Therefore x_n is bounded. Since x_n and Ax_n are bounded, it follows from the compactness of $(I + A)^{-1}$ that there is a subsequence of x_n (we denote it by the same symbol) such that $(A|_{R(A)})^{-1}y_n = x_n \to x$ in $R(A)$. $(A|_{R(A)})^{-1}$ is therefore compact.

By lemma 2.1 we get:

$$R(A) = \left\{ u \in L^2(\Omega), \quad \int_\Omega u(x)\,dx = 0 \right\} \quad \text{is closed in } L^2(\Omega),$$

$$L^2(\Omega) = N(A) \oplus R(A) \quad \text{where } N(A) = \{u(x) = \text{constant } a.e. \text{ in } \Omega\},$$

$$(A|_{R(A)})^{-1} : \quad R(A) \to R(A) \quad \text{is compact.}$$

Let $P : L^2(\Omega) \to R(A)$ be an orthogonal projection of $L^2(\Omega)$ onto $R(A)$, then we get the explicit representation as follows

$$Pu(x) = u(x) - \frac{1}{|\Omega|} \int_\Omega u(x)\,dx = u(x) - \bar{u} \quad \text{for all } u \in L^2(\Omega),$$

where $|\Omega|$ is the volume of Ω and $\bar{u}$ the mean value of u.

Moreover we know that $D(A^{1/2}) = H^1(\Omega)$ and $|A^{1/2}|^2_{L^2(\Omega)} = |\nabla u|^2_{L^2(\Omega)}$ for all $u \in H^1(\Omega)$.

Note that $V = H^1(\Omega) \cap R(A)$ and $H = L^2(\Omega) \cap R(A)$ are the Hilbert spaces with norms $\|w\|_V = |\nabla w|_{L^2}$ and $|v|_H = |v|_{L^2}$ respectively.

In L^2 we consider the following problem

$$\text{(2.1)} \qquad \begin{cases} u''(t) + f(|A^{1/2}u(t)|^2)Au(t) + Au'(t) = 0, \\ u(0) = u_0 \quad u'(0) = u_1, \end{cases}$$

where

$$\text{(2.2)} \qquad f \in C^1([0,\infty)) \quad \text{with} \quad f(x) \geq 0, \quad \forall x \geq 0.$$

The equation (2.1) is an abstract model of (1.1)–(1.3).

If $u(t)$ is a solution to (2.1), then by lemma 2.1 we get $u(t) = u_1(t) + u_2(t)$, where $u_1(t) \in N(A)$ and $u_2(t) \in D(A) \cap R(A)$ furthermore we have

$$\begin{cases} u_1''(t) + u_2''(t) + f(|A^{1/2}u_2(t)|^2)Au_2(t) + Au_2'(t) = 0, \\ u(0) = u_1(0) + u_2(0) = u_{01} + u_{02}, \\ u'(0) = u_1'(0) + u_2'(0) = u_{11} + u_{12}, \end{cases}$$

where we have used the fact that

$$Au(t) = Au_2(t), \quad Au'(t) = Au_2'(t) \quad \text{and} |A^{1/2}u(t)|^2 = |A^{1/2}u_2(t)|^2,$$

then we get the equation in each of $N(A)$ and $R(A)$:

$$\text{(2.3)} \qquad \begin{cases} u_1''(t) = 0 \quad \text{in } N(A), \\ u_1(0) = u_{01}, \\ u_1'(0) = u_{11}. \end{cases}$$

$$
(2.4) \qquad \begin{cases} u_2''(t) + f(|A^{1/2}u_2(t)|^2)Au_2(t) + Au_2'(t) = 0 & \text{in } R(A), \\ u_2(0) = u_{02}, \\ u_2'(0) = u_{12}. \end{cases}
$$

If we can solve (2.3) and (2.4), we will get the solution $u(t) = u_1(t) + u_2(t)$ to (1.1)–(1.3), so we have to solve (2.3) and (2.4).

For (2.3), we get the following explicit solution:

$$
u_1(t) = u_{01} + u_{11}t.
$$

Now we will consider the solvability of (2.4) which is considered as an equation in the closed subspace $R(A)$ as follows:

$$
(2.5) \qquad \begin{cases} u''(t) + f(|A_0^{1/2}u(t)|^2)A_0u(t) + A_0u'(t) = 0 & \text{in } R(A), \\ u(0) = u_0 \quad \in V, \\ u'(0) = u_1 \quad \in H, \end{cases}
$$

where $A_0 = A|_{R(A)}$.

The main result of this section is the following:

Theorem 2.2. *Under the hypothesis* (2.2), *if* $u_0 \in H^1(\Omega)$ *and* $u_1 \in L^2(\Omega)$ *then there exists a unique function* $u : [0, T] \to L^2$ *in the class*

$$
\begin{cases} u \in L^\infty(0, T; H^1) \\ u' \in L^\infty(0, T; L^2) \cap L^2(0, T; H^1) \end{cases}
$$

and satisfies

$$
\begin{cases} u'' + f(|A^{1/2}u|^2)Au + Au' = 0 & \text{in } L^2(0, T; (H^1)'), \\ u(0) = u_0, \quad u'(0) = u_1. \end{cases}
$$

Proof. As it sufficies to solve (2.5), we shall denote in the sequel A_0 by A for simplicity.

Let $(u_{0k})_{k \in N}$ and $(u_{1k})_{k \in N}$ be sequences in $D(A) \cap R(A)$ and $D(A^{1/2}) \cap R(A)$ respectively, such that

$$
(2.6) \qquad u_{0k} \to u_0 \quad \text{strongly in } V,
$$

and

$$
(2.7) \qquad u_{1k} \to u_1 \quad \text{strongly in } H.
$$

For each $k \in N$, let u_k be the solution of the problem (2.5) with initial data u_{0k}

and u_{1k}. Then u_k satisfies (see [5])

$$
\begin{gathered}
u_k \in L^\infty(0,T;D(A)\cap R(A)),\\
u_k' \in L^\infty(0,T;V)\cap L^2(0,T;D(A)\cap R(A)),\\
u_k'' \in L^2(0,T;R(A)),\\
u_k'' + f(|A^{1/2}u_k|^2)Au_k + Au_k' = 0 \quad \text{in } L^2(0,T;R(A)), \qquad (2.8)\\
u_k(0) = u_{0k}, \quad u_k'(0) = u_{1k}.
\end{gathered}
$$

Taking the inner product of the equation (2.8) with $2u_k'(t)$ we obtain

$$\frac{d}{dt}\left(|u_k'(t)|^2 + \int_0^{|A^{1/2}u_k|^2} f(s)\,ds\right) + 2|A^{1/2}u_k'(t)|^2 = 0.$$

Integrating from 0 to $t \le T$ we have

$$|u_k'(t)|^2 + \int_0^{|A^{1/2}u_k(t)|^2} f(s)\,ds + 2\int_0^t |A^{1/2}u_k'(s)|^2\,ds = |u_{1k}|^2 + \int_0^{|A^{1/2}u_{0k}|^2} f(s)\,ds.$$

By hypothesis (2.2) and the convergences (2.6)–(2.7) it follows that:

$$(u_k') \text{ is bounded in } L^\infty(0,T;R(A))\cap L^2(0,T;V), \qquad (2.9)$$

and then

$$(u_k) \text{ is bounded in } L^\infty(0,T;V). \qquad (2.10)$$

Now we are interested in the convergence of the nonlinear term. Let

$$M_k(t) = |A^{1/2}u_k(t)|^2, \quad t\in[0,T].$$

Multiplying the equation (2.8) by $Au_k(t)$ we have

$$\frac{d}{dt}\left\{\frac{1}{2}|Au_k(t)|^2 + (A^{1/2}u_k'(t), A^{1/2}u_k(t))\right\} \le |A^{1/2}u_k'(t)|^2$$

(note that $f \ge 0$) and hence we have

$$
\begin{aligned}
&\frac{1}{2}|Au_k(t)|^2 + (u_k'(t), Au_k(t))\\
&\qquad \le \int_0^\infty |A^{1/2}u_k'(s)|^2\,ds + \frac{1}{2}|Au_k(0)|^2 + (u_k'(0), Au_k(0)) \le C < \infty,
\end{aligned}
$$

which together with the uniform boundedness of $|u_k'(t)|$ implies

$$|Au_k(t)|^2 \le C < \infty.$$

Since $\int_0^\infty |A^{1/2}u_k'(t)|^2\,dt \le C < \infty$ (or $|u_k'(t)| \le C < \infty$) and A^{-1} is compact, by Ascoli-Arzela's lemma (or Aubin's lemma) we have

$$A^{1/2}u_k(t) \to A^{1/2}u(t) \text{ in } L^2 \text{ uniformly on } [0,T] \tag{2.11}$$

for any $T > 0$. By (2.9)–(2.10), there exists a subsequence of u_k, still denoted u_k such that

$$u_k \to u \quad \text{weakly-star in } L^\infty(0,T;V), \tag{2.12}$$

$$u_k' \to u' \quad \text{weakly-star in } L^\infty(0,T;R(A)), \tag{2.13}$$

$$u_k' \to u' \quad \text{weakly in } L^\infty(0,T;V). \tag{2.14}$$

By (2.8), (2.11) and (2.12)–(2.14), we obtain that

$$\begin{cases} u'' + f(|A^{1/2}u(t)|^2)Au + Au' = 0 & \text{in } L^2(0,T;V'), \\ u(0) = u_0, \quad u'(0) = u_1, \end{cases}$$

and the theorem is now proved. Note that in this proof the assumption $f \in C^1$ is relaxed as $f \in C$.

The uniqueness is obtained in the standard way so we will omit the proof here. Of course, for the uniqueness, the Lipshitz continuity of f is required.

3. Asymptotic behavior

We define the energy of the solution of (1.1)–(1.3) by:

$$E(u(t)) := \frac{1}{2}\int_\Omega \left(|u'|^2 + \int_0^{|\nabla u|^2} f(s)\,ds \right) dx,$$

A simple computation gives:

$$E'(t) = -\int_\Omega |\nabla u'|^2\,dx \le 0,$$

then the energy is non-increasing, and

$$E(t) + \int_0^t |\nabla u'(s)|^2\,ds = E(0).$$

We will study the asymptotic behavior of the solution to (1.1)–(1.3) in the two cases:

case 1:

$$f(x) \ge m_0 > 0 \quad \text{and} \quad f'(x) \ge 0 \quad \forall x \in \boldsymbol{R}_+, \tag{3.1}$$

case 2:

$$f(x) = x^{\gamma} \quad \text{with } \gamma \geq 0. \tag{3.2}$$

In the case 2, the equation is degenerate because $f(0) = 0$.

From now on we denote by v the term $u - \bar{u}$, where u is the solution of (1.1)–(1.3) and $\bar{u}$ its mean value that is $\bar{u}(t) = (1/|\Omega|)\int_{\Omega} u\,dx$.

3.1. The non degenerate case

The main result of this subsection is

Theorem 3.1. *Under the hypothesis* (3.1), *we have*

$$E(v(t)) \leq E(0)e^{1-t/c(\Omega)} \quad \forall t \geq 0.$$

$c(\Omega)$ is a constant which depends only of Ω.

Proof. We multiply (1.1) with v, we have then

$$0 = \int_0^T \int_{\Omega} v(v'' - f(|\nabla v|^2)\Delta v - \Delta v')\,dx\,dt$$

After integration by parts, we obtain

$$2\int_0^T E(t)\,dt = -\left[\int_{\Omega} vv'\right]_0^T + 2\int_0^T \int_{\Omega} (v')^2\,dx\,ds - \int_0^T \int_{\Omega} \nabla v \nabla v'\,dx\,ds$$
$$+ \int_0^T \int_{\Omega} \left(\int_0^{|\nabla v|^2} f(s)\,ds\right) dx\,ds - \int_0^T \int_{\Omega} f(|\nabla v|^2)|\nabla v|^2\,dx\,ds,$$

for all $0 < T < +\infty$.

By hypothesis (3.1) we have then,

$$2\int_0^T E(t)\,dt \leq -\left[\int_{\Omega} vv'\right]_0^T + 2\int_0^T \int_{\Omega} v'^2\,dx\,ds - \int_0^T \nabla v \nabla v'\,dx\,ds.$$

Using the Sobolev imbedding $H^1(\Omega) \subset L^2(\Omega)$, the definition of E, the hypothesis (3.1) and the Cauchy-Schwarz inequality we have

$$\left|\int_{\Omega} vv'\,dx\right| \leq cE,$$

and

$$\int_0^T \int_{\Omega} v'^2\,dx\,ds \leq cE,$$

here and in the sequel we denote by c positive constants which may be different at different occurencies.

Using these estimates, we conclude that

$$2\int_0^T E(t)\,dt \leq cE - \int_0^T \int_\Omega \nabla v \nabla v'\,dx\,ds.$$

As

$$\left|\int_0^T \int_\Omega \nabla v \nabla v'\,dx\,ds\right| \leq \frac{m_0}{2}\int_0^T \int_\Omega |\nabla v|^2\,dx\,ds + \frac{2}{m_0}\int_0^T \int_\Omega |\nabla v'|^2\,dx\,ds$$
$$\leq \int_0^T E(t)\,dt + cE,$$

then

$$2\int_0^T E(t)\,dt \leq cE + \int_0^T E(t)\,dt.$$

That is

$$\int_0^T E(t)\,dt \leq cE.$$

By a general result of Komornik [2] (th 8.1), we conclude that

$$E(v(t)) \leq E(0)e^{1-t/c(\Omega)}.$$

3.2. The degenerate case

The main results of this subsection are

Theorem 3.2. *Under the hypothesis* (3.2), *we have*

$$E(v(t)) \leq \frac{c}{(1+t)^{1+1/\gamma}} \quad \forall t \geq 0, \quad \text{if } \gamma \neq 0$$

and

$$E(v(t)) \leq ce^{-\omega t} \quad \forall t \geq 0 \quad \text{if } \gamma = 0,$$

c and ω are positive constants.

Theorem 3.3. *In addition to the assumption* (3.2), *suppose that the initial energy*

$$E(0) = \frac{1}{2}|v_1|_{L^2}^2 + \frac{1}{2(\gamma+1)}|\nabla v_0|_{L^2}^{2(\gamma+1)} \ll 1$$

and

$$A(0) := |\nabla v_0|_{L_2}^2 + 2(v_1, v_0) > 0,$$

then

$$E(v(t)) \geq \frac{c}{(1+t)^{1+1/\gamma}} \quad for\ t \geq T_0,$$

c is a positive constant.

Proof of Theorem 3.2. We have

$$v'' - |\nabla v|^{2\gamma} \Delta v - \Delta v' = 0, \tag{3.3}$$

we shall derive the decay estimate of the energy $E(v(t))$, where we set

$$E(v(t)) := |v'(t)|_{L^2}^2 + \frac{1}{\gamma+1} |\Delta v(t)|_{L^2}^{2(\gamma+1)}.$$

The proof is essentially included in Nakao [4]. For convenience of readers we reproduce it in our context.

Taking the scalar product of (3.3) with $2v'(t)$, we have

$$E'(v(t)) + 2|\Delta v'(t)|_{L^2}^2 = 0.$$

Integrating it over $[0, T]$, we obtain

$$E(T) + 2\int_0^T |\nabla v'(t)|_{L^2}^2\, dt = E(0).$$

Integrating from t to $t+1$, we obtain

$$E(t) - E(t+1) + 2\int_t^{t+1} |\nabla v'(s)|_{L^2}^2\, ds = 0.$$

That is

$$2\int_t^{t+1} |\nabla v'(s)|_{L^2}^2\, ds = E(t) - E(t+1) := F(t)^2,$$

then there exist $t_1 \in [t, t+1/4]$, $t_2 \in [t+3/4, t+1]$ such that

$$|\nabla v'(t_i)|_{L^2}^2 \leq 4F(t)^2 \quad \text{for } i = 1, 2.$$

Taking the scalar product of (3.3) with $2v$ and integrating it over $[t_1, t_2]$ we have

$$0 = \int_{t_1}^{t_2} \int_\Omega vv'' - |\nabla v|^{2\gamma} v\Delta v - v\Delta v'\, dx\, ds,$$

then

$$\int_{t_1}^{t_2} |\nabla v|_{L^2}^{2(\gamma+1)} \leq \left[\int_\Omega |v'(t)|\,|v(t)|\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} |v'(s)|_{L^2}^2\,ds$$

$$+ \int_{t_1}^{t_2}\int_\Omega |\nabla v(s)|_{L^2}\,|\nabla v'(s)|_{L^2}\,dx\,ds.$$

$$\leq c\left[F(t)^2 + F(t)\sup_{t\leq s\leq t+1}|\nabla v(s)|_{L^2}^2\right] := G(t)^2,$$

thus we obtain

$$E(t_2) \leq \frac{1}{t_2 - t_1}\int_{t_1}^{t_2} E(s)\,ds \leq cG(t)^2,$$

and hence

$$\sup_{t\leq s\leq t+1} E(s)^{1+\gamma/(\gamma+1)} \leq E(t_2) + 2\int_{t_1}^{t_2} |\nabla v'(s)|_{L^2}^2\,ds$$

$$\leq cG(t)^2 \leq c\left[F(t)^2 + F(t)\sup_{t\leq s\leq t+1} E(s)^{1/(2(\gamma+1))}\right].$$

Using the Young inequality, we arrive at

$$\sup_{t\leq s\leq t+1} E(s)^{1+\gamma/(\gamma+1)} \leq cF(t)^2 = c(E(t) - E(t+1)).$$

By a general result of Nakao [4], we get the decay estimate of the energy $E(v(t))$ such that

$$E(v(t)) \leq \frac{c}{(1+t)^{1+1/\gamma}} \quad \forall t \geq 0 \quad \text{if } \gamma \neq 0,$$

and

$$E(v(t)) \leq ce^{-\omega t} \quad \forall t \geq 0 \quad \text{if } \gamma = 0.$$

Proof of Theorem 3.3. Multiplying (3.3) by $2v(t)$ and integrating over Ω, we have

$$A'(t) + 2B(t) = 0,$$

where we set

$$A(t) = |\nabla v(t)|_{L^2}^2 + 2(v'(t), v(t)),$$

$$B(t) = |\nabla v(t)|_{L^2}^{2(\gamma+1)} - |v'(t)|_{L^2}^2.$$

We have $B(t) \leq |\nabla v(t)|_{L^2}^{2(\gamma+1)} - (1/2)|v'(t)|_{L^2}^2$. Since $A(0) > 0$, it follows that $A(t) > 0$ for some $t > 0$. We put

$$T_1 := \sup\{t \in [0, \infty), A(s) > 0 \quad \text{for } 0 < s < t\}$$

then it holds that $T_1 > 0$ and $A(t) > 0$ for $t < T_1$.

Since for any $0 \leq k \leq \gamma$ and $0 < \varepsilon \ll 1$

$$|\nabla v|_{L^2}^{2k} |(v', v)|^{\gamma+1-k} \leq \varepsilon |\nabla v|_{L^2}^{\gamma+1} + c(\varepsilon) |\nabla v|_{L^2}^{2k} |v'|_{L^2}^{2(\gamma-k)} |v'|_{L^2}^2,$$

we get

$$\begin{aligned} A(t)^{\gamma+1} &\geq |\nabla v|_{L^2}^{2(\gamma+1)} - \left\{ \frac{1}{2} |\nabla v|_{L^2}^{2(\gamma+1)} + c \max_{0 \leq k \leq \gamma} E(0)^{\gamma/(\gamma+1)+(\gamma-k)} |v'|_{L^2}^2 \right\} \\ &\geq \frac{1}{2} |\nabla v|_{L^2}^{2(\gamma+1)} - cE(0)^\gamma |v'|_{L^2}^2, \end{aligned}$$

then

$$\begin{aligned} 2A(t)^{\gamma+1} - B(t) &\geq |\nabla v|_{L^2}^{2(\gamma+1)} - 2cE(0)^\gamma |v'|_{L^2}^2 - |\nabla v|_{L^2}^{2(\gamma+1)} + \frac{1}{2} |v'|_{L^2}^2, \\ &\geq \left(\frac{1}{2} - 2cE(0)^\gamma \right) |v'(t)|_{L^2}^2 \geq 0, \end{aligned}$$

when we have used the assumption $E(0) \ll 1$, then we obtain

$$A'(t) + 4A(t)^{\gamma+1} \geq 0,$$

and hence, we see from $A(0) > 0$ that

$$A(t) \geq (4\gamma)^{-1/\gamma} \left(t + \frac{1}{4} (\gamma)^{-1} A(0)^{-\gamma} \right)^{-1/\gamma}$$

which conclude $T_1 = \infty$. Then we have

$$\begin{aligned} |\nabla v(t)|_{L^2}^2 &= A(t) - 2(v'(t), v(t)) \\ &\geq \frac{c}{(1+t)^{1/\gamma}} - c|v'(t)|_{L^2} |\nabla v(t)|_{L^2} \\ &\geq \frac{c}{(1+t)^{1/\gamma}} - \frac{c}{(1+t)^{(\gamma+2)/(2\gamma)}} \\ &\geq \frac{c}{(1+t)^{1/\gamma}} \qquad \text{for } t \geq T_0, \end{aligned}$$

and then

$$E(v(t)) \geq \frac{c}{(1+t)^{1+1/\gamma}} \quad \text{for } t \geq T_0,$$

thus we have proved Theorem 3.2.

Acknowledgment. The author thanks very much the referee for carefully reading the previous version of the manuscript and for his comments which resulted in this revised version of the paper.

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(Ricevita la 25-an de novembro, 1996)
(Reviziita la 30-an de aprilo, 1997)