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Oscillation of Linear Functional Equations of the Second Order

By

Wiesława GOLDA and Jarosław WERBOWSKI
(Technical University of Poznań, Poland)

§1. Introduction

Let $\boldsymbol{R}$ be the set of real numbers and let I denote an unbounded subset of $\boldsymbol{R}_+ = (0, \infty)$. By g^m we denote the m-th iterate of the function $g: I \to I$, i.e.

$$g^0(t) = t,\ g^{m+1}(t) = g(g^m(t)), \qquad t \in I,\ m = 0, 1, \ldots.$$

In the whole of this paper upper indices at the sign of a function will denote iterations. In each instance we have the relation $g^1(t) = g(t)$. Exponents of a power of a functions will be written after a bracket containing the whole expression for the function.

In this paper we are concerned with the oscillatory behavior of solutions of functional equations of the form

$$a_2(t)x(g^2(t)) + a_1(t)x(g(t)) + a_0(t)x(t) = 0, \tag{1}$$

where $a_i: I \to \boldsymbol{R}$ $(i = 0, 1, 2)$ and $g: I \to I$ are given functions and x is an unknown real valued function. In the sequel we assume that

$$g(t) \not\equiv t \quad \text{and} \quad \lim_{t \to \infty} g(t) = \infty, \qquad t \in I. \tag{2}$$

By a solution of equation (1) we mean a function $x: I \to \boldsymbol{R}$ such that $\sup\{|x(s)|: s \in I_{t_0} = [t_0, \infty) \cap I\} > 0$ for any $t_0 \in \boldsymbol{R}_+$ and x satisfies (1) on I.

A solution x of equation (1) is called oscillatory if there exists a sequence of points $\{t_n\}_{n=1}^{\infty}$, $t_n \in I$, such that $\lim_{n \to \infty} t_n = \infty$ and $x(t_n)x(t_{n+1}) \leq 0$ for $n = 1, 2, \ldots$. Otherwise it is called nonoscillatory.

In contrast with the extensive development of the oscillation theory of differential equations (for example see [4] and the references contained therein), the authors are of the opinion that at this time in the literature there are no known oscillation criteria for functional equations. The purpose of this paper is to obtain sufficient conditions under which all solutions of (1) are oscillatory.

First let us observe that existence of oscillatory solutions of equation (1) is connected with the sign of the functions a_i $(i = 0, 1, 2)$ on I. For example, it is easy to prove that in each case:

(i) $a_i(t) > 0$ [or $a_i(t) < 0$] $(i = 0, 1, 2)$, $t \in I$,

(ii) $a_i(t) > 0$, $a_j(t) > 0$ [or $a_i(t) < 0$, $a_j(t) < 0$] and $a_k(t)$ is oscillatory, where $i \neq j \neq k$, $i, j, k \in \{0, 1, 2\}$ and $t \in I$,

equation (1) possesses only oscillatory solutions. However, in the case

(iii) $a_i(t) > 0$, $a_j(t) > 0$ and $a_k(t) < 0$, $(i \neq j \neq k)$, on I,

equation (1) can possess both oscillatory and nonoscillatory solutions. For example, the functional equation

$$x(t + 2\pi) - (e^{\pi} + 1)x(t + \pi) + e^{\pi}x(t) = 0, \qquad t \in \boldsymbol{R}_+,$$

possesses an oscillatory solution cos2t and a nonoscillatory solution e^t. So, a question arises: In case (iii) holds, under what additional conditions on the coefficients a_i every solution of (1) will be oscillatory. Some answers to this question will be presented in this paper.

Further we consider equation (1) with the following assumptions

$$a_2(t) > 0, \ a_1(t) < 0 \quad \text{and} \quad a_0(t) > 0 \qquad \text{for } t \in I.$$

If we denote

$$P(t) = -\frac{a_0(t)}{a_1(t)} > 0 \quad \text{and} \quad Q(t) = -\frac{a_2(t)}{a_1(t)} > 0 \qquad \text{for } t \in I,$$

then equation (1) takes the form

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)). \tag{L}$$

Now we present a very simple condition under which every solution of (L) is oscillatory.

Lemma 1. *If*

$$\limsup_{I \ni t \to \infty} Q(t)P(g(t)) > 1, \tag{3}$$

then every solution of (L) *is oscillatory.*

Proof. Suppose that (L) has a nonoscillatory solution x. Since $-x$ is also a solution of (L), without loss of generality we may assume that $x(t) > 0$ for $t \in I_{t_1}$, $t_1 > 0$. Then, in view of (2), there exists a point $t_2 \in I_{t_1}$ such that $x(g^i(t)) > 0$ $(i = 1, 2)$ for $t \in I_{t_2}$. Therefore from (L) we have for $t \in I_{t_2}$

$$x(g(t)) \geq P(t)x(t)$$

which gives

$$x(g^2(t)) \geq P(g(t))x(g(t)). \tag{4}$$

Using now (4) in (L) we obtain for $t \in I_{t_2}$

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)) \geq Q(t)P(g(t))x(g(t)),$$

which contradicts (3). Thus the proof is complete.

§ 2. Main results

In this section we will assume that condition (3) is not satisfied, i.e. $Q(t)P(g(t)) \leq 1$ for all sufficiently large $t \in I$. Moreover, for convenience, we will assume that inequalities about values of functions are satisfied eventually for all large $t \in I$.

Theorem 1. *Assume that*

$$\liminf_{I \ni t \to \infty} Q(t)P(g(t)) > \frac{1}{4}. \tag{5}$$

Then every solution of (L) *oscillates.*

Proof. Assume, for the sake a contradiction, that x is an eventually positive solution of (L). Then, as in proof of Lemma 1, x satisfies (4). Using now (4) in (L) we have

$$x(g(t)) \geq P(t)x(t) + Q(t)P(g(t))x(g(t))$$

and

$$Q(t)P(g(t)) \leq 1 - P(t)\frac{x(t)}{x(g(t))}. \tag{6}$$

From (5) it follows that there exists $\varepsilon > 0$ such that

$$Q(t)P(g(t)) \geq \frac{1+\varepsilon}{4} > \frac{1}{4}. \tag{7}$$

Therefore from (6) and (7) we obtain

$$\frac{1+\varepsilon}{4} \leq 1 - P(t)\frac{x(t)}{x(g(t))}$$

which gives

$$P(t)\frac{x(t)}{x(g(t))} \leq 1 - \frac{1+\varepsilon}{4} \leq \frac{1}{1+\varepsilon}\max_{\varepsilon>0}(1+\varepsilon)\left[1 - \frac{1+\varepsilon}{4}\right] = \frac{1}{1+\varepsilon}.$$

Thus

$$(1+\varepsilon)P(t)x(t) \leq x(g(t))$$

and by iteration

$$(1+\varepsilon)P(g(t))x(g(t)) \leq x(g^2(t)). \tag{8}$$

By using (8) in (L) and then by repeating the above arguments we find that

$$[1+\varepsilon]^2 P(t)x(t) \leq x(g(t))$$

and

$$[1+\varepsilon]^2 P(g(t))x(g(t)) \leq x(g^2(t)).$$

Thus, by induction, we have for every $k = 1, 2, \ldots$

$$[1+\varepsilon]^k P(g(t))x(g(t)) \leq x(g^2(t)). \tag{9}$$

Choose k such that

$$[1+\varepsilon]^{k+1} > 4. \tag{10}$$

Using now (9) in (L) we obtain

$$x(g(t)) = P(t)x(t) + Q(t)x(g^2(t)) \geq [1+\varepsilon]^k Q(t)P(g(t))x(g(t)).$$

Thus

$$1 \geq [1+\varepsilon]^k Q(t)P(g(t))$$

which, by (7), gives

$$4 \geq [1+\varepsilon]^{k+1}.$$

The last inequality contradicts (10) and so the proof is complete.

Remark 1. It is worth noticing that the constant on the right-hand side of (5) cannot be improved, i.e. condition (5) connot be replaced by the weaker condition

$$\liminf_{I \ni t \to \infty} Q(t)P(g(t)) > \frac{1-\varepsilon}{4} \tag{11}$$

for some $\varepsilon \in (0, 1]$. For example, we consider the functional equation

$$2x(g(t)) = (1 + A)[t]^m x(t) + (1 - A)[t]^{-2m} x(g^2(t)),$$

where $m \in \boldsymbol{R}$, $g(t) = [t]^2$, $t \in \boldsymbol{R}_+$ and $A \in (0, 1)$. Let $\varepsilon \in (0, 1]$. Then condition (11) is fulfilled for all $A \in (0, \sqrt{\varepsilon})$ since

$$\liminf_{t\to\infty} Q(t)P(g(t)) =$$

$$= \liminf_{t\to\infty} \frac{1-A}{2}[t]^{-2m}\frac{1+A}{2}[t]^{2m} = \frac{1-A^2}{4} > \frac{1-\varepsilon}{4}.$$

However, the equation has a nonoscillatory solution $[t]^m$.

Theorem 2. *Suppose that for some integer $m \geq 0$ the following condition is satisfied*

$$\limsup_{I\ni t\to\infty} \{Q(t)P(g(t)) + \sum_{i=0}^{m}\prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t))\} > 1. \tag{12}$$

Then every solution of (L) *is oscillatory.*

Proof. Assume that x is an eventually positive solution of (L). Then, as in the proof of Lemma 1, we have

$$x(g^i(t)) > 0 \qquad \text{for } i \geq 1.$$

Then from (L) we obtain the following inequalities

$$x(g(t)) \geq Q(t)x(g^2(t)) \tag{13}$$

and

$$x(g(t)) \geq P(t)x(t).$$

Using the last inequality one can prove by induction the formula

$$x(g^i(t)) \geq x(t)\prod_{j=0}^{i-1} P(g^j(t)), \qquad (i = 1, 2, \ldots). \tag{14}$$

Replacing in (L) t by $g(t)$ we obtain

$$x(g^2(t)) = P(g(t))x(g(t)) + Q(g(t))x(g^3(t)). \tag{15}$$

Induction yields

$$x(g^{i+1}(t)) = P(g^i(t))x(g^i(t)) + Q(g^i(t))x(g^{i+2}(t)) \qquad \text{for } i \in \{2, 3, \ldots\}. \tag{16}$$

Using now (16) with $i = 2$ and next $i = 3$ in (15) we have

$$\begin{aligned} x(g^2(t)) &= P(g(t))x(g(t)) + Q(g(t))P(g^2(t))x(g^2(t)) \\ &\quad + Q(g(t))Q(g^2(t))x(g^4(t)) \\ &= P(g(t))x(g(t)) + Q(g(t))P(g^2(t))x(g^2(t)) \\ &\quad + Q(g(t))Q(g^2(t))P(g^3(t))x(g^3(t)) \end{aligned}$$

$$+ Q(g(t))Q(g^2(t))Q(g^3(t))x(g^5(t))$$
$$= P(g(t))x(g(t)) + \sum_{i=0}^{1} P(g^{i+2}(t))x(g^{i+2}(t)) \prod_{j=0}^{i} Q(g^{j+1}(t))$$
$$+ x(g^5(t)) \prod_{j=0}^{i} Q(g^{j+1}(t)).$$

Then induction gives for $m > 1$

$$x(g^2(t)) = P(g(t))x(g(t)) + \sum_{i=0}^{m} P(g^{i+2}(t))x(g^{i+2}(t)) \prod_{j=0}^{i} Q(g^{j+1}(t))$$
$$+ x(g^{m+4}(t)) \prod_{j=0}^{m+1} Q(g^{j+1}(t)).$$

From the above equality, in view of (13), (14) and positivity of $x(g^i(t))$, we derive

$$x(g^2(t)) \geq Q(t)P(g(t))x(g^2(t)) + \sum_{i=0}^{m} x(g^2(t)) \prod_{j=0}^{i} Q(g^{j+1}(t))P(g^{j+2}(t)).$$

Dividing now both sides of the above inequality by $x(g^2(t))$ we obtain a contradiction with (12). Thus the proof is complete.

§ 3. Applications

In this section we show an application of our results to difference and recurrence equations. First, let us consider a difference equation of the form

$$\text{(DE)} \qquad \Delta_h x(t) = Q(t)x(t + 2h),$$

where $\Delta_h x(t)$ denotes the difference of the function x with the span $h > 0$, i.e. $\Delta_h x(t) = x(t + h) - x(t)$ and $Q: \boldsymbol{R}_+ \to \boldsymbol{R}_+$ is continuous function. From results of Section 2 follows a criterion for oscillation of solutions of difference equation (DE)

Theorem 3. *All solutions of* (DE) *are oscillatory if one of the following conditions is fulfilled*

$$\liminf_{t\to\infty} Q(t) > \frac{1}{4}$$

or for some $m \geq 0$

$$\limsup_{t\to\infty} \{Q(t) + \sum_{i=0}^{m} \prod_{j=0}^{i} Q(t + (j + 1)h)\} > 1.$$

Consider now a recurrence equation of the form

$$\text{(RE)} \qquad b(n)x(n+1) = a(n)x(n) + c(n)x(n+2), \qquad n \in \boldsymbol{N} = \{1, 2, \dots\},$$

where $a, b, c: \boldsymbol{N} \to \boldsymbol{R}_+$. A nontrivial solution of (RE) is called oscillatory if for every $n_1 \in \boldsymbol{N}$ there exists $n \geq n_1$ such that $x(n)x(n+1) \leq 0$. If one nontrivial solution of (RE) is oscillatory then all nontrivial solutions are oscillatory (see [1]). So (RE) may be classified as oscillatory or nonoscillatory. Now we apply Theorems 1 and 2 to equation (RE) to obtain the following result.

Theorem 4. *Each one of the following conditions*

$$\liminf_{n \to \infty} \frac{a(n+1)c(n)}{b(n)b(n+1)} > \frac{1}{4} \tag{17}$$

or for some $m \geq 0$

$$\limsup_{n \to \infty} \left\{ \frac{a(n+1)c(n)}{b(n)b(n+1)} + \sum_{i=0}^{m} \prod_{j=0}^{i} \frac{a(n+2+j)c(n+1+j)}{b(n+1+j)b(n+2+j)} \right\} > 1 \tag{18}$$

implies that equation (RE) *is oscillatory.*

Remark 2. If in equation (RE) we take $a(n) = c(n-1)$, then, from condition (17) we get Theorem 5 of [3]. Moreover, condition (18) with $m = 0$ gives Theorem 2.3 of [2].

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nuna adreso:
Institute of Mathematics
Technical University of Poznań
ul, Piotrowo 3A
60–965 Poznań
Poland

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