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Euler-Cauchy Polygons and the Local Existence of Solutions to Abstract Ordinary Differential Equations

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Abstract: A local existence theorem, of Peano type, is proved for the initial-value problem for the ordinary differential equation, $y'=F(x,y)$, where F is continuous as a function of (x,y), and has values in a compact subset of a real Banach space B. The proof employs Euler-Cauchy polygons in B. The proof does not require any previous knowledge of any theory of definite integration of Banach-valued functions of a real variable.

I. Introduction

The local existence theory for solutions to the initial-value problem $y'=F(x,y)$, $y(x_0)=y_0$, where F has values in an arbitrary real Banach space B, and is defined in a neighborhood of the initial point (x_0, y_0), where x and x_0 are real, differs fundamentally, according to whether the space B in question is finite dimensional or not. In the classical case, that is, when B is a finite dimensional Euclidean space, the theorem of Peano [1] assures the existence of at least one solution, when F is assumed to be continuous in a neighborhood of (x_0, y_0). However, when B is infinite dimensional, the (strong) continuity of F does not guarantee the existence of a solution, as has been shown by Dieudonné [2, p. 287, ex. 5]; see also Yorke [3] for an example when B is a Hilbert space. Therefore, in order to obtain a valid local existence theorem for an arbitrary Banach space, something beyond continuity must be assumed for F.

It is known that if strict enough conditions are imposed upon F, then it is possible to prove the existence of a **unique** solution. For results along these lines, see Browder [4], Kato [5], Hille and Phillips [6, p. 67], and Lusternik and Sobolev [7, p. 322]. However, the main concern of the present paper is with the determination of conditions on F which guarantee the existence of a not necessarily unique solution. It is of interest that the method of proof here employs Euler-Cauchy polygons in the given Banach space, which is analogous to the procedure originally followed by Peano in his classical paper. In the

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present paper, when B is an arbitrary real Banach space, the proof of the convergence of the Euler-Cauchy polygons presents the difficulty that the unit sphere in B may contain a sequence of vectors without a converegent subsequence. This particular difficulty will be circumvented here by an application of a theorem of Mazur [8].

The basic additional assumption made here on F is that its range be compact; see the precise statement of the theorem below. Krasnoselskii and Krein [9] state, but do not prove, a more general existence theorem; they only mention, in passing, that "the proof follows by usual methods" (presumably, the authors have in mind fixed-point methods). Corduneanu [10] also proved an existence theorem, using fixed point methods on an equivalent integral operator. However, in so doing, the additional assumption that F be uniformly continuous was required in the proof. This particular additional assumption also occurs in Dieudonné [2, p. 287]. It appears that the assumption of uniform continuity is unavoidable when one employs the Schauder-Tychonoff fixed point theorem for the corresponding integral operator T, defined by

$$Tf(x)=y_0+\int_{x_0}^{x} F(s, f(s))ds,$$

in the case when B is infinite dimensional. In passing, it is also noted here that the existence of a solution to the initial-value problem may be proved by assuming that F is merely continuous and bounded, provided there exists a certain positive-definite, continuously (Fréchet) differentiable, auxiliary function, which, in conjunction with F, satisfies rather restrictive properties; see Murakami [11] or Lakshmikantham and Leela [12, p. 237]. Also, Chow and Schuur [13] have proved the existence of a weakly differentiable solution when the space B is reflexive.

The proof in the present paper requires neither the uniform continuity of F, nor the existence of any auxiliary function. Also, this proof does not require previous knowledge of any theory of definite integration of Banach-valued functions of a real variable, whereas some such theory is indispensable for any proof based on an integral operator.

(Added May 29, 1972: After this paper was completed, Professor R. S. Phillips kindly drew our attention to his paper [14], where, as an application of his general theory of integration, he obtained a "Carathéodory, almost everywhere, type" theorem [15, p. 672], for the initial value problem $y'=F(x, y)$, $y(x_0)=y_0$, in a sequentially complete, linear, convex topological space satisfying the first countability axiom. The **existence** of a solution of the initial value problem, considered in the present paper, could be deduced from the result of

Phillips, but **not** (without an additional argument) the uniform convergence of the Euler-Cauchy polygons, which is the main purpose in the present paper, simply bacause neither Carathéodory (see [15, p. 667]) nor Phillips (see [14, p. 139]) makes use of "strict" Euler-Cauchy polygons as their approximating functions in the case of a general "non-autonomous" F. [However, it must be admitted that the case of a non-autonomous $F(x, y)$, which "actually depends on both x and y", can be "reduced" to that of an autonomous $F(x, y)$, that is to say to that of an F which does not depend at all upon x, but only on y, merely by increasing the dimension of the Banach space by one, namely, by considering the "new" Banach space of all pairs (x, y), where x is a real number and y is an element of the original, or "old", Banach space B.] The direct proof in the present paper, which does not require any previous theory of integration, is still of independent interest. It is remarkable that the authors, mentioned earlier in this introduction, seem to have been entirely unaware of the general "Carathéodory type" existence theorem of Phillips, of 1940, while these authosr, never theless, were directly concerned with the abstract initial value problem $y'=F(x, y)$, $y(x_0)=y_0$.)

II. An Existence Theorem for the Initial Value Problem

The main purpose of this section is the proof of the following theorem.

Theorem. Hypotheses:

1. x_0 is a real number and y_0 is an element of a real Banach space B; R is an open rectangle, centered at (x_0, y_0), and contained in the cross product of the real numbers with B; that is, there exist positive real numbers a, b such that

$$R=\{(x, y):\ |x-x_0|<a \quad \text{and} \quad \|y-y_0\|<b\},$$

where x is real, y is in B, and $\|\cdot\|$ denotes the norm for B;

2. F is a function defined on R with values in B, that is, F: $R\to B$. The range $F(R)$ is compact in the strong topology for B, that is to say, every sequence of vectors in $F(R)$ contains a subsequence which converges strongly to some vector in B; consequently, there is an $M>0$ such that $\|F(x, y)\|\leqq M$ for all (x, y) in R.

3. F is (strongly) continuous on R; that is, given any (x_1, y_1) in R, and any $\varepsilon>0$, there exists $\delta=\delta(\varepsilon, x_1, y_1)>0$ such that, whenever (x, y) is in R, and

$$|x-x_1|<\delta \quad \text{and} \quad \|y-y_1\|<\delta,$$

then

$$\|F(x, y)-F(x_1, y_1)\|<\varepsilon;$$

4. r satisfies the inequality

$$0<r<\min\left(a, \frac{b}{M}\right).$$

Conclusion: There exists a strongly differentiable function f: $(x_0-r, x_0+r)\to B$ such that

$$f'(x)=F(x, f(x)), \text{ for } |x-x_0|<r,$$

and

$$f(x_0)=y_0.$$

Moreover, this function f is constructed as the uniform limit of a sequence of piecewise linear, continuous functions, defined on (x_0-r, x_0+r) to B (these functions are "Euler-Cauchy polygons", which are described precisely in parts A and B of the proof below).

The proof of this theorem requires the following modification of the Ascoli lemma, see, for example, Royden [16, Theorem on p. 155]; notice that in the statement of that theorem, the word "compact" means what was previously referred to here as compact, **plus** closed. This convergence lemma will be stated precisely in the form needed here.

Lemma. Hypotheses:

1. $\{f_m\}_{m=1}^{\infty}$ is a sequence of functions, defined on the finite closed interval $[\alpha, \beta]$, with values in a real Banach space B; that is, for each positive integer m and each x satisfying $\alpha\leqq x\leqq\beta$, the vector $f_m(x)$ is an element of B;
2. $\{f_m\}_{m=1}^{\infty}$ is an equally uniformly continuous sequence of functions on $[\alpha, \beta]$; that is, given $\varepsilon>0$, there exists a $\delta=\delta(\varepsilon)>0$, independent of m, such that whenever $|x_2-x_1|<\delta$, with x_1, x_2 in $[\alpha, \beta]$, then $\|f_m(x_2)-f_m(x_1)\|<\varepsilon$ for every positive integer m;
3. the sequence of functions $\{f_m\}_{m=1}^{\infty}$ is pointwise compact on $[\alpha, \beta]$; that is, for each $x\in[\alpha, \beta]$, the sequence of vectors $\{f_m(x)\}_{m=1}^{\infty}$ contains a convergent subsequence $\{f_{k(m)}(x)\}_{m=1}^{\infty}$, where k is a strictly increasing function, from the positive integers to the positive integers.

Conclusion: The sequence of functions $\{f_m\}_{m=1}^{\infty}$ contains a subsequence $\{f_{p(m)}\}_{m=1}^{\infty}$, which converges uniformly to a continuous function f on $[\alpha, \beta]$; that is, there is a continuous function f: $[\alpha, \beta]\to B$, and a strictly increasing function p, from the positive integers to the positive integers, such that

$$\lim_{m\to\infty} \|f_{p(m)}(x)-f(x)\|=0,$$

uniformly for x in $[\alpha, \beta]$.

Proof of the theorem:

The existence proof will be given for the interval $[x_0, x_0+r]$, rather than $[x_0-r, x_0+r]$, because the extension to $[x_0-r, x_0]$ is obvious. For convenience, the proof will be divided into several steps.

A. The Sequence of Partitions of $[x_0, x_0+r]$.

Let ϕ be a function from the positive integers to the positive integers, and, for each positive integer m, let $[x_0, x_0+r]$ be partitioned into $\phi(m)$ subintervals,

$$x_0=x(0, m)<x(1, m)<x(2, m)<\cdots<x(\phi(m)-1, m)<x(\phi(m), m)=x_0+r.$$

B. Definition of the Sequence of Euler-Cauchy Polygons $\{f_m\}_{m=1}^{\infty}$.

In terms of the given sequence of partitions of $[x_0, x_0+r]$, the sequence of "polygons $\{f_m\}_{m=1}^{\infty}$ is defined inductively, on the interval $[x_0, x_0+r]$, as follows. Let m be a positive integer, and define

$$f_m(x)=y_0+F(x_0, y_0)(x-x_0) \text{ for } x_0 \leqq x \leqq x(1, m).$$

Clearly, $(x, f_m(x))$ is in R for $x_0 \leqq x \leqq x(1, m)$, because

$$\|f_m(x)-y_0\|=\|F(x_0, y_0)(x-x_0)\| \leqq M(x-x_0)<b.$$

In particular, $(x(1, m), f_m(x(1, m)))$ is in R. If $x(1, m)<x_0+r$, one may then define $f_m(x)$, on the next subinterval $[x(1, m), x(2, m)]$, by

$$f_m(x)=y_0+F(x_0, y_0)(x(1, m)-x_0)+F(x(1, m), f_m(x(1, m)))(x-x(1, m)),$$

for $x(1, m) \leqq x \leqq x(2, m)$. Notice that

$$\|f_m(x)-y_0\| \leqq M[(x(1, m)-x_0)+(x-x(1, m))]=M[x-x_0] \leqq Mr<b.$$

Hence, for $x(1, m) \leqq x \leqq x(2, m)$, it is true that $(x, f_m(x))$ is in R.

Using mathematical induction, suppose that $f_m(x)$ has been defined, by this stepwise procedure, for $x_0 \leqq x \leqq x(j, m)$, where $1 \leqq j<\phi(m)$, and that $(x, f_m(x))$ is in R for all such x. In particular, $(x(j, m), f_m(x(j, m)))$ is in R. Therefore, one may then define $f_m(x)$, on the next subinterval $[x(j, m), x(j+1, m)]$, by

$$\text{(P)} \qquad \begin{aligned} f_m(x)=y_0+\sum_{k=1}^{j} F(x(k-1, m), f_m(x(k-1, m)))(x(k, m)-x(k-1, m)) \\ +F(x(j, m), f_m(x(j, m)))(x-x(j, m)), \end{aligned}$$

for $x(j, m) \leqq x \leqq x(j+1, m)$. As before, notice that

$$\begin{aligned} \|f_m(x)-y_0\| &\leqq M\left[\sum_{k=1}^{j}(x(k, m)-x(k-1, m))+x-x(j, m)\right] \\ &=M[x-x_0] \leqq Mr<b. \end{aligned}$$

Hence, by this last inequality, together with the induction hypothesis, it follows, for $x_0 \leqq x \leqq x(j+1, m)$, that $(x, f_m(x))$ is in R. This means that $f_m(x)$ is well defined by this stepwise procedure, and is given by (P) for $j \geqq 1$; actually, (P)

also holds for j=0, with the understanding that the sum $\sum_{k=1}^{0} \cdots$ is taken to be equal to zero.

Since,

$$\|f_m(x)-y_0\|<b$$

for every positive integer m, the sequence $\{f_m(x)\}_{m=1}^{\infty}$ is equally bounded on $[x_0, x_0+r]$.

C. Equal Uniform Continuity of the Sequence of Functions $\{f_m\}_{m=1}^{\infty}$.

Let ξ_1 and ξ_2 be numbers in $[x_0, x_0+r]$, where, without loss, it is assumed that $x_0 \leqq \xi_1 \leqq \xi_2 \leqq x_0+r$. There are two cases to consider, for each positive integer m: (i) ξ_1 and ξ_2 are in the same subinterval of the m^{th} partition; (ii) ξ_1 and ξ_2 are in different subintervals.

(i) Suppose $x(j,m) \leqq \xi_1 \leqq \xi_2 \leqq x(j+1,m)$, for some j with $0 \leqq j < \phi(m)$. Then, using the definition, (P), of the "polygonal" function f_m, one has

$$\|f_m(\xi_2)-f_m(\xi_1)\| = \|F(x(j,m), f_m(x(j,m)))(\xi_2-\xi_1)\| \leqq M(\xi_2-\xi_1).$$

(ii) Suppose

$$x(j,m) \leqq \xi_1 \leqq x(j+1,m) \leqq x(j+2,m) \leqq \cdots \leqq x(j+l,m) \leqq \xi_2 \leqq x(j+l+1,m),$$

for some j with $0 \leqq j < \phi(m)$, where the positive integer l satisfies $l \geqq 1$ and $0 < j+l+1 \leqq \phi(m)$. The proof, in this case (ii), consists of a repeated application of the inequality already obtained in case (i). First, from case (i), it follows that

$$\|f_m(\xi_2)-f_m(x(j+l,m))\| \leqq M(\xi_2-x(j+l,m));$$
$$\|f_m(x(k,m))-f_m(x(k-1,m))\| \leqq M(x(k,m)-x(k-1,m)),$$

for $k=j+2,\ j+3, \cdots, j+l$; and

$$\|f_m(x(j+l,m))-f_m(\xi_1)\| \leqq M(x(j+1,m)-\xi_1).$$

Next, since

$$\begin{aligned} f_m(\xi_2)-f_m(\xi_1) = {} & f_m(\xi_2)-f_m(x(j+l,m)) \\ & + \sum_{k=j+2}^{j+l} [f_m(x(k,m))-f_m(x(k-1,m))] \\ & + f_m(x(j+1,m))-f_m(\xi_1), \end{aligned}$$

it follows, from the triangle inequality, that

$$\begin{aligned} \|f_m(\xi_2)-f_m(\xi_1)\| \leqq M\Big[& (\xi_2-x(j+l,m)) + \sum_{k=j+2}^{j+l} (x(k,m)-x(k-1,m)) \\ & + (x(j+1,m)-\xi_1) \Big] = M(\xi_2-\xi_1). \end{aligned}$$

Thus, it has been shown that if ξ_1 and ξ_2 are numbers in $[x_0, x_0+r]$, then, for any positive integer m, one has

$$\|f_m(\xi_2)-f_m(\xi_1)\| \leqq M|\xi_2-\xi_1|,$$

which proves the equal uniform continuity of the sequence of functions $\{f_m\}_{m=1}^{\infty}$ on $[x_0, x_0+r]$.

D. Pointwise Compactness of the Sequence of Functions $\{f_m\}_{m=1}^{\infty}$.

Let x be in $[x_0, x_0+r]$. It will be shown that the sequence of vectors $\{f_m(x)\}_{m=1}^{\infty}$ is compact; that is, it contains a convergent subsequence. If $x=x_0$, then $f_m(x_0)=y_0$ for every m, and there is nothing to prove. Therefore, it will be supposed that $x_0<x<x_0+r$.

For each positive integer m, there is a unique, nonnegative integer $j(m)=j(m, x)$, with $0 \leqq j(m)<\phi(m)$, such that

$$x(j(m), m) \leqq x \leqq x(j(m)+1, m).$$

Consider the sequence of vectors

$$\left\{\frac{f_m(x)-y_0}{x-x_0}\right\}_{m=1}^{\infty}.$$

In view of the definition (P) of the polygonal function f_m, one has that

$$\frac{f_m(x)-y_0}{x-x_0}=\sum_{k=1}^{j(m)} F(x(k-1, m), f_m(x(k-1, m))) \cdot \frac{(x(k, m)-x(k-1, m))}{(x-x_0)}$$
$$+F(x(j(m), m), f_m(x(j(m), m))) \cdot \frac{(x-x(j(m), m))}{(x-x_0)}.$$

Notice that, letting

$$\lambda_k=\frac{x(k, m)-x(k-1, m)}{x-x_0}, \quad \text{for } k=1,2, \cdots, j(m);$$

$$\lambda_{j(m)+1}=\frac{x-x(j(m), m)}{x-x_0};$$

and

$$y_k=F(x(k-1, m), f_m(x(k-1, m))), \quad \text{for } k=1,2, \cdots, j(m)+1,$$

one has

$$\frac{f_m(x)-y_0}{x-x_0}=\sum_{k=1}^{j(m)+1} \lambda_k y_k,$$

where $0 \leqq \lambda_k \leqq 1$ and $\sum_{k=1}^{j(m)+1} \lambda_k=1$. That is, for each positive integer m, the vector,

$$\frac{f_m(x)-y_0}{x-x_0},$$

is a convex combination of vectors lying in the range of F. Thus, the sequence

$$\left\{\frac{f_m(x)-y_0}{x-x_0}\right\}_{m=1}^{\infty}$$

lies in the smallest convex set which contains $F(R)$. Since $F(R)$ is compact, it follows, by a theorem of Mazur [8], that the smallest convex set which contains $F(R)$ is also compact. Consequently, the sequence of vectors

$$\left\{\frac{f_m(x)-y_0}{x-x_0}\right\}_{m=1}^{\infty}$$

contains a convergent subsequence, and this means that the sequence of vectors $\{f_m(x)\}_{m=1}^{\infty}$ also contains a convergent subsequence. This completes the proof that the sequence of functions $\{f_m\}_{m=1}^{\infty}$ is pointwise compact on $[x_0, x_0+r]$.

E. Application of the Convergence Lemma.

In view of what has already been shown, the convergence lemma may now be applied to the sequence of functions $\{f_m\}_{m=1}^{\infty}$ on $[x_0, x_0+r]$. Therefore, there is a subsequence of functions $\{f_{p(m)}\}_{m=1}^{\infty}$ which converges uniformly to a continuous function f on $[x_0, x_0+r]$; that is,

$$f(x)=\lim_{m\to\infty} f_{p(m)}(x),$$

the convergence being uniform on $[x_0, x_0+r]$.

F. Existence of a Solution to the Initial Value Problem.

F.0. In this section, it will be shown that if the sequence of partitions of $[x_0, x_0+r]$ is sufficiently "fine" (in the precise sense specified in F.4 below), then the function f, which was obtained as the limit of the subsequence $\{f_{p(m)}\}_{m=1}^{\infty}$ of polygonal functions, is indeed a solution to the initial value problem.

To avoid complicating the notation further, the subsequence $\{f_{p(m)}\}_{m=1}^{\infty}$ will be written simply $\{f_m\}_{m=1}^{\infty}$, as if it were the original sequence. Thus, one has

$$f(x)=\lim_{m\to\infty} f_m(x),$$

the convergence being uniform on $[x_0, x_0+r]$.

The function f, being the uniform limit of a sequence of continuous functions, is also continuous. It is now to be shown that the (strong) derivative $f'(x)$ exists for each x in $[x_0, x_0+r)$, and, in fact, equals $F(x, f(x))$. (Clearly, the initial condition $f(x_0)=y_0$ is satisfied.) This will follow from the estimate for

$$\left\|\frac{f_m(\xi)-f_m(x)}{\xi-x}-F(x,f(x))\right\|,$$

with $\xi\neq x$, which is obtained in F.4 belows. In sections F.1, F.2, and F.3, which are preliminary to F.4, given $\varepsilon>0$, three positive numbers $(\delta, \mathrm{h}, \mathrm{N})$ are chosen. The argument is divided into two cases, first $x_0<x<x_0+r$, and then $x=x_0$.

F.1. Consider x such that $x_0<x<x_0+r$ (the case $x=x_0$ will be considered in F.7). Let $\varepsilon>0$. The function F is continuous at $(x, f(x))$. Therefore, there exists $\delta=\delta(\varepsilon, x)>0$ so small that both

$$x_0<x-\delta<x<x+\delta<x_0+r$$

and

$$\delta<b-\|f(x)-y_0\|$$

hold. Further, δ is such that, whenever

$$x-\delta<\xi<x+\delta,$$

and

$$\|y-f(x)\|<\delta,$$

hold, then

$$\|F(\xi, y)-F(x, f(x))\|<\varepsilon.$$

Notice that, if (ξ, y) satisfies

$$|\xi-x|<\delta \text{ and } \|y-f(x)\|<\delta,$$

then (ξ, y) is in R, the domain of definition of the function F, because

$$x_0<\xi<x_0+r,$$

and, by the triangle inequality,

$$\|y-y_0\|\leqq\|y-f(x)\|+\|f(x)-y_0\|<\delta+\|f(x)-y_0\|<b.$$

F.2. The function f is continuous at x. Therefore, given $\delta/2>0$ (this positive number δ was determined in F.1), there exists $h=h(\varepsilon, x)>0$ such that $h<\delta$, and, whenever

$$|\xi-x|<h<\delta,$$

then

$$\|f(\xi)-f(x)\|<\frac{\delta}{2}.$$

F.3. The sequence of functions $\{f_m\}_{m=1}^{\infty}$ converges uniformly to f on the interval $[x_0, x_0+r]$. Therefore, by the uniform convergence, given $\delta/2>0$, there exists a positive integer $N=N(\varepsilon)$ such that

$$\|f_m(\xi)-f(\xi)\|<\frac{\delta}{2}$$

for **every** ξ satisfying $x_0\leqq\xi\leqq x_0+r$, and **every** integer $m>N$.

Hence, in particular, for $|\xi-x|<h$, where h is the positive number determined in F.2, it will be true, using the triangle inequality, that

$$\|f_m(\xi)-f(x)\|\leqq\|f_m(\xi)-f(\xi)\|+\|f(\xi)-f(x)\|<\frac{\delta}{2}+\frac{\delta}{2}=\delta,$$

for $m>N$. Therefore, from Section F.1, it follows that, whenever

$$|\xi-x|<h<\delta \quad \text{and} \quad m>N,$$

one has that

$$\|F(\xi, f_m(\xi))-F(x, f(x))\|<\varepsilon.$$

F.4. So far in the argument, no particular restriction has been placed on the sequence of partitions of $[x_0, x_0+r]$. However, it will now be further assumed that as $m\to\infty$, the length of the largest subinterval of the m^{th} partition tends to zero; that is,

$$\lim_{m\to\infty}\{\max_{1\leqq k\leqq\phi(m)}[x(k,m)-x(k-1,m)]\}=0.$$

F.5. The purpose of the present section is to show that when $0<|x-\xi|$ is sufficiently small, then

$$\left\|\frac{f_m(\xi)-f_m(x)}{\xi-x}-F(x, f(x))\right\|$$

will be small for all sufficiently large m.

Suppose, for definiteness, that $x<\xi$; the argument is similar if $\xi<x$. For each positive integer m, there are nonnegative integers $j(m)$ and $J(m)$ such that

$$x(J(m), m)\leqq x<x(J(m)+1, m),$$

and

$$x(j(m), m)\leqq\xi<x(j(m)+1, m).$$

Further, by the additional assumption just made concerning the sequence of partitions, it follows that

$$\lim_{m\to\infty} x(J(m), m)=x,$$

and

$$\lim_{m\to\infty} x(j(m), m)=\xi.$$

It is, therefore, possible to choose a positive integer $\bar{N}$, greater than the N in F. 3, such that

$$J(m)<j(m)$$

for all $m>\bar{N}$. Since this last inequality implies that

$$J(m)+1\leqq j(m),$$

it follows that

$$x<x(J(m)+1, m)\leqq x(j(m), m)\leqq \xi.$$

It is important, in the following argument, that there should be at least one partition number between x and ξ. As a matter of fact, it is precisely this circumstance which motivated the additional restriction just made on the sequence of partitions.

Since $x<\xi$, it follows that, for all sufficiently large positive integers m,

$$J(m)<j(m),$$

so that

$$J(m)+1\leqq j(m);$$

consequently, the following inequality will hold

$$x<x(J(m)+1, m)\leqq x(j(m), m)\leqq \xi.$$

Notice that, for **all** m,

$$x(j(m), m)\leqq \xi.$$

It will first be assumed, in the argument, that the strict inequality $x(j(m), m)<\xi$ holds; the special case when $x(j(m), m)=\xi$ will be considered in passing.

Consider the interval $[x, \xi]$. The starting point of the argument is the following identity, which expresses a difference quotient over $[x, \xi]$ as a convex combination of difference quotients over smaller subintervals. This identity is

$$\begin{aligned}\frac{f_m(\xi)-f_m(x)}{\xi-x}=&\left(\frac{f_m(x(J(m)+1, m))-f_m(x)}{x(J(m)+1, m)-x}\right)\left(\frac{x(J(m)+1, m)-x}{\xi-x}\right)\\&+\sum_{i=J(m)+2}^{j(m)}\left(\frac{f_m(x(i, m))-f_m(x(i-1, m))}{x(i, m)-x(i-1, m)}\right)\\&\times\left(\frac{x(i, m)-x(i-1, m)}{\xi-x}\right)\end{aligned}$$

$$+\frac{f_m(\xi)-f_m(x(j(m),m))}{\xi-x(j(m),m)}\Big)\cdot\Big(\frac{\xi-x(j(m),m)}{\xi-x}\Big).$$

That this identity, indeed, expresses the "large" difference quotient as a convex combination of "smaller" difference quotients, is clear from the fact that

$$1=\Big(\frac{x(J(m)+1,m)-x}{\xi-x}\Big)+\sum_{i=J(m)+2}^{j(m)}\Big(\frac{x(i,m)-x(i-1,m)}{\xi-x}\Big)$$
$$+\Big(\frac{\xi-x(j(m),m)}{\xi-x}\Big).$$

The integer $j(m)\geqq J(m)+1$; it is understood that, if $j(m)=J(m)+1$, then $\Sigma\cdots$, in the last two equations, is taken to be zero. It is to be noticed that, if $x(j(m),m)=\xi$, then the above expression for the difference quotient remains valid, provided that the last term on the right hand side, following the summation, is replaced by zero.

In view of the recursive definition (P) for the polygonal functions, the difference quotients for the smaller subintervals may be replaced by suitable values of the function F. Thus,

$$\frac{f_m(\xi)-f_m(x)}{\xi-x}=F(x(J(m),m),f_m(x(J(m)+1,m)))\Big(\frac{x(J(m)+1,m)-x}{\xi-x}\Big)$$
$$+\sum_{i=J(m)+2}^{j(m)}F(x(i-1,m),f_m(x(i,m)))\cdot\Big(\frac{x(i,m)-x(i-1,m)}{\xi-x}\Big)$$
$$+F(x(j(m),m),f_m(x(j(m),m)))\Big(\frac{\xi-x(j(m),m)}{\xi-x}\Big).$$

This equation holds for $x(j(m),m)\leqq\xi$. Also, by multiplying the next to the last equation above by $F(x,f(x))$, it follows that

$$F(x,f(x))=F(x,f(x))\Big(\frac{x(J(m)+1,m)-x}{\xi-x}\Big)$$
$$+\sum_{i=J(m)+2}^{j(m)}F(x(x,f(x))\Big(\frac{x(i,m)-x(i-1,m)}{\xi-x}\Big)$$
$$+F(x,f(x))\Big(\frac{\xi-x(j(m),m)}{\xi-x}\Big).$$

Again, this equation holds for $x(j(m),m)\leqq\xi$.

Subtracting the last two equations, taking the norm of the left hand side, and using the triangle inequality, gives the key inequality:

$$\Big\|\frac{f_m(\xi)-f_m(x)}{\xi-x}-F(x,f(x))\Big\|$$
$$\leqq\|F(x(J(m),m),\ f_m(x(J(m)+1,m)))-F(x,f(x))\|\cdot\Big(\frac{x(J(m)+1,m)-x}{\xi-x}\Big)$$

$$+\sum_{i=J(m)+2}^{j(m)} \|F(x(i-1,m), f_m(x(i,m)))-F(x,f(x))\|\left(\frac{x(i,m)-x(i-1,m)}{\xi-x}\right)$$
$$+\|F(x(j(m),m))-F(x,f(x))\|\left(\frac{\xi-x(j(m),m)}{\xi-x}\right).$$

This inequality, which was derived under the assumption that $x<\xi$, is readily seen to hold also when $\xi<x$, essentially by interchanging the roles of ξ and x.

Now, let $\varepsilon>0$, and choose δ, h, and N as in F. 1, F. 2, and F. 3 respectively. Suppose, again for definiteness, that $x<\xi$, the argument being similar for $\xi<x$. Then, for sufficiently large m, it is true that both $m\geqq N$ and $J(m)<j(m)$. For any such m, it will be true, by F. 3, that, whenever $0<\xi-x<h$, each of the norms on the right hand side of the key inequality may be replaced by ε, and still have, a fortiori, a valid inequality. But, if this replacement is made, then the coefficient of ε, on the right hand side of the new inequality, is simply one. Therefore, for any such sufficiently large m,

$$\left\|\frac{f_m(\xi)-f_m(x)}{\xi-x}-F(x,f(x))\right\|<\varepsilon,$$

provided $0<\xi-x<h$. However, the same argument shows that the same inequality continues to hold for $0<x-\xi<h$. In summary, this inequality holds, for sufficiently large m, whenever $0<|\xi-x|<h$.

F. 6. Now, it only remains to let $m\to\infty$ in the last inequality, for fixed ξ and x. This means that, given x satisfying $x_0<x<x_0+r$, and $\varepsilon>0$, there is an $h>0$ such that, whenever $x_0<\xi<x_0+r$ and $0<|\xi-x|<h<r$, then

$$\left\|\frac{f(\xi)-f(x)}{\xi-x}-F(x,f(x))\right\|\leqq\varepsilon;$$

that is,

$$f'(x)=\lim_{\xi\to x}\frac{f(\xi)-f(x)}{\xi-x}=F(x,f(x)).$$

F. 7. In the previous argument it was assumed that $x_0<x<x_0+r$, and it was proved that, for such x,

$$f'(x)=F(x,f(x)).$$

Only slight changes are necessary, in the argument from F. 1 to F. 6, when $x=x_0$. In particular, as in F. 1, using the continuity of the function F at $(x_0,f(x_0))=(x_0,y_0)$, given $\varepsilon>0$, the number $\delta=\delta(\varepsilon,x_0)>0$ has to be chosen such that both

$$x_0<x_0+\delta<x_0+r$$

and

$$\delta < b$$

hold. Further, δ is such that, whenever

$$x_0 \leqq \xi < x_0 + \delta,$$

and

$$\|y - y_0\| < \delta,$$

hold, then

$$\|F(\xi, y) - F(x_0, y_0)\| < \varepsilon.$$

As in F. 1, the above inequalities imply that (ξ, y) is in R, the domain of F.

Next, as in F. 2, since f is continuous at x_0, given $\delta/2 > 0$ (the same δ as determined above), there exists $h = h(\varepsilon, x_0) > 0$ such that $h < \delta$, and, whenever

$$x_0 \leqq \xi < x_0 + h < x_0 + \delta,$$

then

$$\|f(\xi) - f(x_0)\| = \|f(\xi) - y_0\| < \frac{\delta}{2}.$$

Also, by the uniform convergence of $\{f_m\}_{m=1}^{\infty}$ on $[x_0, x_0 + r]$, the choice of N, as in F. 3, implies that, for $m > N$, it is true that

$$\|f_m(\xi) - f(\xi)\| < \frac{\delta}{2}.$$

Thus, in particular, for $x_0 \leqq \xi < x_0 + h$ (where h was chosen immediately above), and $m > N$, it follows that

$$\begin{aligned}\|f_m(\xi) - y_0\| &= \|f_m(\xi) - f(x_0)\| \leqq \|f_m(\xi) - f(\xi)\| + \|f(\xi) - f(x_0)\| \\ &< \frac{1}{2}\delta + \frac{1}{2}\delta = \delta.\end{aligned}$$

Therefore, as in F. 3, whenever

$$x_0 \leqq \xi < x_0 + h < x_0 + \delta \quad \text{and} \quad m > N,$$

one has that

$$\|F(\xi, f_m(\xi)) - F(x_0, f(x_0))\| = \|F(\xi, f_m(\xi)) - F(x_0, y_0)\| < \varepsilon.$$

The argument proceeds, now, as in F. 5. It only has to be noticed that $J(m)$ is now zero for every m. Therefore, it follows that, for any sufficiently large m,

$$\left\| \frac{f_m(\xi) - f_m(x_0)}{\xi - x_0} - F(x_0, f(x_0)) \right\| = \left\| \frac{f_m(\xi) - y_0}{\xi - x_0} - F(x_0, y_0) \right\| < \varepsilon,$$

provided $x_0<\xi<x+h$. Letting $m\to\infty$, as in F. 6, it finally follows that

$$f'(x_0)=F(x_0, f(x_0)),$$

where $f'(x_0)$ is understood to be the right hand derivative at x_0.

References

[1] G. Peano, Démonstration de l'intégrabilité des équations différentielles ordinaires, Math. Ann., **37** (1890), 182-228. (Opere Scelte, Vol. 1, 119-170, Edizioni Cremonese, Roma, 1957.)

[2] J. Dieudonné, "Foundations of Modern Analysis", Academic Press, New York, 1960.

[3] J. Yorke, A continuous differential equation in Hilbert space without existence, Funkcialaj Ekvacioj, **13**, (1970), 19-21.

[4] F. Browder, Non-linear equations of evolution, Ann. Math., **80** (1964), 485-523.

[5] T. Kato, Integration of the equation of evolution in a Banach space, J. Math. Soc. Japan, **5**, No. 2 (1953), 208-234.

[6] E. Hille and R. S. Phillips, "Functional Analysis and Semi-Groups", American Math. Soc., Providence, R. I., 1957.

[7] L. A. Lusternik and V. J. Sobolev, "Elements of Functional Analysis", Hindustan Publishing Co., Delhi, 1961.

[8] S. Mazur, Über die kleinste konvexe Menge, die eine gegebene kompakte Menge enthält, Studia Math., **2** (1930), 7-9.

[9] M. Krasnoselskii and S. Krein, Nonlocal existence and uniqueness theorems for systems of ordinary differential equations, Dokl. Acad. Nauk. SSSR, **102** (1955), 13-16.

[10] C. Corduneanu, Equazioni differenziali negli spazi di Banach, teoremi di esistenza e di prolungabilità, Lincei-Rend. Sc. fis. mat. e nat., **23** (1957), 226-230.

[11] H. Murakami, On non-linear ordinary and evolution equations, Funkcialaj Ekvacioj, **9** (1966), 151-162.

[12] V. Lakshmikantham and S. Leela, "Differential and Integral Inequalities", Vol. II, Academic Press, New York, 1969.

[13] Shui-Nee Chow and J. D. Schuur, An existence theorem for ordinary differential equations in Banach spaces, Bull. Amer. Math. Soc., **77**, No. 6 (1971), 1018-1020.

[14] R. S. Phillips, Integration in a convex linear topological space, Trans. Amer. Math. Soc., **47**, No. 1 (1940), 114-145.

[15] C. Carathéodory, Vorlesungen über reelle Funktionen, Chelsea Publishing Co., 1968.

[16] H. Royden, "Real Analysis", Macmillan and Co., New York, 1965.

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