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Initial Values for the Navier-Stokes Equations in Spaces with Weights in Time

By

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Abstract. We consider the nonstationary Navier-Stokes system in a smooth bounded domain $\Omega \subset \boldsymbol{R}^3$ with initial value $u_0 \in L^2_\sigma(\Omega)$. It is an important question to determine the optimal initial value condition in order to prove the existence of a unique local strong solution satisfying Serrin's condition. In this paper, we introduce a weighted Serrin condition that yields a necessary and sufficient initial value condition to guarantee the existence of local strong solutions $u(\cdot)$ contained in the weighted Serrin class $\int_0^T (\tau^\alpha \|u(\tau)\|_q)^s \mathrm{d}\tau < \infty$ with $2/s + 3/q = 1 - 2\alpha$, $0 < \alpha < 1/2$. Moreover, we prove a restricted weak-strong uniqueness theorem in this Serrin class.

Key Words and Phrases. Instationary Navier-Stokes system, Initial values, Local strong solutions, Weighted Serrin condition, Well-chosen weak solutions, Restricted Serrin's uniqueness theorem.

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1. Introduction

We consider the initial value problem

$$\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f, \qquad \operatorname{div} u = 0 \text{ in } (0,T) \times \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \qquad u(0) = u_0$$

in a bounded domain $\Omega \subset \boldsymbol{R}^3$ with boundary $\partial\Omega$ of class $C^{2,1}$ and a time interval $[0,T)$, $0 < T \le \infty$.

First we recall the definitions of weak and strong solutions to (1.1) and we define a new type of a strong solution, the "$L^s_\alpha(L^q)$-strong solution".

Definition 1.1. Let $u_0 \in L^2_\sigma(\Omega)$ be an initial value and let $f = \operatorname{div} F$ with $F = (F_{ij})_{i,j=1}^3 \in L^2(0,T;L^2(\Omega))$ be an external force. A vector field

$$u \in L^\infty(0,T;L^2_\sigma(\Omega)) \cap L^2(0,T;W^{1,2}_0(\Omega)) \tag{1.2}$$

is called a weak solution (in the sense of Leray-Hopf) of the Navier-Stokes system (1.1) with data u_0, f, if the relation

$$-\langle u, w_t\rangle_{\Omega,T} + \langle \nabla u, \nabla w\rangle_{\Omega,T} - \langle uu, \nabla w\rangle_{\Omega,T} = \langle u_0, w(0)\rangle_\Omega - \langle F, \nabla w\rangle_{\Omega,T} \tag{1.3}$$

holds for each test function $w \in C_0^\infty([0,T); C_{0,\sigma}^\infty(\Omega))$, and if the energy inequality

$$\frac{1}{2}\|u(t)\|_2^2 + \int_0^t \|\nabla u\|_2^2 \mathrm{d}\tau \le \frac{1}{2}\|u_0\|_2^2 - \int_0^t (F, \nabla u)\mathrm{d}\tau \tag{1.4}$$

is satisfied for $0 \le t < T$.

A weak solution u of (1.1) is called an $L_\alpha^s(L^q)$-strong solution with exponents $2 < s < \infty$, $3 < q < \infty$ and weight τ^α in time, $0 < \alpha < 1/2$, where $2/s + 3/q = 1 - 2\alpha$ such that additionally the weighted Serrin condition

$$u \in L_\alpha^s(0,T; L^q(\Omega)), \qquad \text{i.e.,} \qquad \int_0^T (\tau^\alpha \|u(\tau)\|_q)^s \mathrm{d}\tau < \infty \tag{1.5}$$

is satisfied. If in (1.5) $\alpha = 0$ and $2/s + 3/q = 1$, then u is called a strong solution ($L^s(L^q)$-strong solution).

In this definition we use the usual Lebesgue and Sobolev spaces, $L^q(\Omega)$ with norm $\|\cdot\|_{L^q(\Omega)} = \|\cdot\|_q$ and $W^{k,q}(\Omega)$ with norm $\|\cdot\|_{W^{k,q}(\Omega)} = \|\cdot\|_{k;q}$, respectively for $1 < q < \infty$ and $k \in \boldsymbol{N}$. Let $L^s(0,T; L^q(\Omega)) = L^s(L^q)$, $1 < q, s < \infty$, with norm $\|\cdot\|_{L^s(0,T;L^q(\Omega))} = \|\cdot\|_{q,s;T} = (\int_0^T \|\cdot\|_q^s \mathrm{d}t)^{1/s}$ denote the classical Bochner spaces. Similarly, for $1 < q, s < \infty$ and $\alpha \ge 0$ we define the weighted (in time) Bochner spaces $L_\alpha^s(0,T; L^q(\Omega)) = L_\alpha^s(L^q)$ with norm

$$\|\cdot\|_{L_\alpha^s(0,T;L^q(\Omega))} = \|\cdot\|_{L_\alpha^s(L^q)} = \left(\int_0^T (t^\alpha \|\cdot\|_q)^s \mathrm{d}t\right)^{1/s}.$$

The expression $\langle\cdot,\cdot\rangle_\Omega = \langle\cdot,\cdot\rangle$ denotes the pairing of functions on Ω, and $\langle\cdot,\cdot\rangle_{\Omega,T}$ means the corresponding pairing on $[0,T)\times\Omega$. Furthermore, to deal with solenoidal vector fields we use the smooth function spaces $C_0^\infty(\Omega)$ and $C_{0,\sigma}^\infty(\Omega) = \{v \in C_0^\infty(\Omega) : \operatorname{div} v = 0\}$, and the spaces

$$L_\sigma^q(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_q}, \qquad W_0^{1,q}(\Omega) = \overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,q}}, \qquad W_{0,\sigma}^{1,q}(\Omega) = \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{1,q}}.$$

Throughout this paper, $A = A_2$ denotes the Stokes operator in $L_\sigma^2(\Omega)$. More general, A_q, $1 < q < \infty$, means the Stokes operator in $L_\sigma^q(\Omega)$, and e^{-tA_q}, $t \ge 0$, is the semigroup generated by A_q in $L_\sigma^q(\Omega)$. Note that, with $x = (x_1, x_2, x_3) \in \Omega \subset \boldsymbol{R}^3$, for $F = (F_{ij})_{i,j=1}^3$, $u = (u_1, u_2, u_3)$ we let $\operatorname{div} F = (\sum_{i=1}^3 \partial_i F_{ij})_{j=1}^3$, $u \cdot \nabla u = (u \cdot \nabla)u = (u_1\partial_1 + u_2\partial_2 + u_3\partial_3)u$, so that $u \cdot \nabla u = \operatorname{div}(uu)$, $uu = (u_i u_j)_{i,j=1}^3$ if u is solenoidal.

For properties of weak and strong solutions to (1.1) we refer to [2, 3, 18, 19, 21, 24, 27]. We may assume in the following, without loss of generality, that each weak solution of (1.1)

$$u : [0,T) \to L_\sigma^2(\Omega) \text{ is weakly continuous} \tag{1.6}$$

(see [26, V. Theorem 1.3.1]). Therefore $u(0) = u_0$ is well-defined. Moreover, for a weak solution u, there exists a distribution p in $(0, T) \times \Omega$, the associated pressure, such that $\partial_t u - \Delta u + u \cdot \nabla u + \nabla p = f$ holds in the sense of distributions [26, V. 1.7]. Assume that u is a strong solution of (1.1), that $\partial\Omega$ is of class C^∞ and $F \in C^\infty((0, T) \times \overline{\Omega})$. Then Serrin's condition (1.5) with $\alpha = 0$ yields the regularity property

$$u \in C^\infty((0, T) \times \overline{\Omega}), \qquad p \in C^\infty((0, T) \times \overline{\Omega}), \tag{1.7}$$

and uniqueness within the class of weak solutions satisfying the energy inequality, see [26, V. Theorem 1.8.2, Theorem 1.5.1].

The existence of at least one weak solution u of (1.1) is well-known since the pioneering work of [19, 24]. The existence of a strong solution u of (1.1) could be shown up to now at least in a sufficiently small interval $[0, T)$, $0 < T \leq \infty$, and under additional smoothness conditions on the initial data u_0 and the external force f. The first sufficient condition on the initial data for a bounded domain seems to be due to [21], yielding a solution class of so-called local strong solutions. Since then many results on sufficient initial value conditions for the existence of local strong solutions have been developed, see [2, 10, 13, 14, 18, 20, 22, 25, 26, 27]. Recent results in [8, 9] yield sufficient and necessary conditions, also written in terms of (solenoidal) Besov spaces $\boldsymbol{B}^{-2/s}_{q,s}(\Omega) = \boldsymbol{B}^{-1+3/q}_{q,s}(\Omega)$ where $2/s + 3/q = 1$. See Section 4 for a definition of solenoidal Besov spaces; for a review of these results we refer to [5].

In this paper, we are interested in a weighted Serrin condition with respect to time and $L^s_\alpha(L^q)$-strong solutions. Our result yields a sufficient condition on initial data and external force to guarantee the existence of local $L^s_\alpha(L^q)$-strong solutions. The motivation for this approach is an extension of the results in [8, 9] where $2/s + 3/q = 1$ to the case $u_0 \notin \boldsymbol{B}^{-1+3/q}_{q,s}(\Omega)$, i.e.,

$$e^{-\tau A}u_0 \notin L^s(0, T; L^q(\Omega)), \qquad \text{but} \quad \int_0^T (\tau^\alpha \|e^{-\tau A}u_0\|_q)^s \mathrm{d}\tau < \infty, \qquad \frac{2}{s} + \frac{3}{q} = 1 - 2\alpha$$

with some $\alpha > 0$. More precisely, for the case $\alpha = 0$ (classical Serrin class), the condition $e^{-\tau A}u_0 \in L^{s(q,0)}(0, T; L^q(\Omega))$ with $2/s(q, 0) + 3/q = 1$ is equivalent to $u_0 \in \boldsymbol{B}^{-1+3/q}_{q,s(q,0)}(\Omega)$, whereas for α with $0 < \alpha < 1/2$ (weighted Serrin class) the condition $e^{-\tau A}u_0 \in L^{s(q,\alpha)}_\alpha(0, T; L^q(\Omega))$ with $2/s(q, \alpha) + 3/q = 1 - 2\alpha$ is equivalent to $u_0 \in \boldsymbol{B}^{-1+3/q}_{q,s(q,\alpha)}(\Omega)$. Since $s(q, \alpha) > s(q, 0)$, by embedding theorems we know $\boldsymbol{B}^{-1+3/q}_{q,s(q,0)}(\Omega) \subset \boldsymbol{B}^{-1+3/q}_{q,s(q,\alpha)}(\Omega)$. Therefore, the spaces to yield strong solutions are larger than the classical Serrin class discussed in the literature, and the theory of [8, 9] is extended to the scale of Besov spaces $\boldsymbol{B}^{-1+3/q}_{q,s(q,\alpha)}(\Omega)$ filling the gap between $\boldsymbol{B}^{-1+3/q}_{q,s(q,0)}(\Omega)$ and $\boldsymbol{B}^{-1+3/q}_{q,\infty}(\Omega)$.

We mention that there are also some results using weighted Serrin's conditions related to Kato's approach of construction of mild and strong solutions, see [17, 23].

We state our main result in a more precise way as follows.

Theorem 1.2. *Let $\Omega \subseteq \boldsymbol{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2,1}$, and let $0 < T \le \infty$, $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < 1/2$ with $2/s + 3/q = 1 - 2\alpha$ be given. Consider the Navier-Stokes equation with initial value $u_0 \in L^2_\sigma(\Omega)$ and an external force $f = \operatorname{div} F$ where $F \in L^2(0, T; L^2(\Omega)) \cap L^{s/2}_{2\alpha}(0, T; L^{q/2}(\Omega))$. Then there exists a constant $\epsilon_* = \epsilon_*(q, s, \alpha, \Omega) > 0$ with the following property: If*

$$\|e^{-\tau A} u_0\|_{L^s_\alpha(0, T; L^q)} + \|F\|_{L^{s/2}_{2\alpha}(L^{q/2})} \le \epsilon_*, \tag{1.8}$$

then the Navier-Stokes system (1.1) *has a unique $L^s_\alpha(L^q)$-strong solution with data u_0, f on the interval $[0, T)$.*

Theorem 1.3. *Let Ω be as in Theorem* 1.2, *let $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < 1/2$ with $2/s + 3/q = 1 - 2\alpha$ be given, and let $u_0 \in L^2_\sigma(\Omega)$ and an external force $f = \operatorname{div} F$ where $F \in L^2(0, \infty; L^2(\Omega)) \cap L^{s/2}_{2\alpha}(0, \infty; L^{q/2}(\Omega))$.*

(1) *The condition*

$$\int_0^\infty (\tau^\alpha \|e^{-\tau A} u_0\|_q)^s \mathrm{d}\tau < \infty \tag{1.9}$$

is sufficient and necessary for the existence of a unique $L^s_\alpha(L^q)$-strong solution $u \in L^s_\alpha(0, T; L^q)$ of the Navier-Stokes system (1.1), *with data u_0, f in some interval $[0, T)$, $0 < T \le \infty$.*

(2) *Let u be a weak solution of the system* (1.1) *in $[0, \infty) \times \Omega$ with data u_0, f, and let*

$$\int_0^\infty (\tau^\alpha \|e^{-\tau A} u_0\|_q)^s \mathrm{d}\tau = \infty. \tag{1.10}$$

Then the weighted Serrin's condition $u \in L^s_\alpha(0, T; L^q(\Omega))$ does not hold for each $0 < T \le \infty$. Moreover, the system (1.1) *does not have a $L^s_\alpha(L^q)$-strong solution with data u_0, f and weighted Serrin exponents s, q, α in any interval $[0, T)$, $0 < T \le \infty$.*

A weak-strong uniqueness theorem in the sense of the classical Serrin Uniqueness Theorem seems to be out of reach for $L^s_\alpha(L^q)$-strong solutions within the full class of weak solutions satisfying the energy inequality. The reason is based on the algebraic identities and sharp use of norms and Hölder estimates in the proof of Serrin's Theorem, cf. [26, Ch. V, Sect. 1.5]. However,

we prove uniqueness within the *subclass of well-chosen weak solutions* describing weak solutions constructed by concrete approximation procedures. We refer to Assumptions 5.1, 5.4 and Remarks 5.2, 5.3 for precise definitions.

Theorem 1.4. *Let $\Omega \subset \boldsymbol{R}^3$ be a bounded domain with boundary of class $C^{2,1}$ and let $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < 1/2$ with $2/s + 3/q = 1 - 2\alpha$ be given. Moreover, suppose that $u_0 \in L^2_\sigma(\Omega) \cap \boldsymbol{B}^{-1+3/q}_{q,s}$ and an external force $f = \operatorname{div} F$ where $F \in L^2(0,\infty; L^2(\Omega)) \cap L^{s/2}_{2\alpha}(0,\infty; L^{q/2}(\Omega))$ are given. Then the unique $L^s_\alpha(L^q)$-strong solution $u \in L^s_\alpha(0,T; L^q(\Omega))$ is unique on a time interval $[0,T')$, $T' > 0$, in the class of all well-chosen weak solutions.*

The plan of this paper is as follows. In Section 2, to prepare the proof we recall some well-known properties of Stokes operators and some important estimates. In Section 3 we first prove Theorem 1.2 by admitting Lemma 3.1, Lemma 3.2 and Lemma 3.3. Then we prove these Lemmata and finally we give a proof to Theorem 1.3. In Section 4 we discuss these results in terms of Besov spaces, and the final section contains the proof of Theorem 1.4.

2. Preliminaries

For the reader's convenience, we first explain some well-known properties of the Stokes operator. Let Ω be as in Theorem 1.2, let $[0,T), 0 < T \le \infty$, be a time interval and let $1 < q < \infty$. Then $P_q : L^q(\Omega) \to L^q_\sigma(\Omega)$ denotes the Helmholtz projection, and the Stokes operator $A_q = -P_q\Delta : D(A_q) \to L^q_\sigma(\Omega)$ is defined with domain $D(A_q) = W^{2,q}(\Omega) \cap W^{1,q}_0(\Omega) \cap L^q_\sigma(\Omega)$ and range $R(A_q) = L^q_\sigma(\Omega)$. Since $P_q v = P_\gamma v$ for $v \in L^q(\Omega) \cap L^\gamma(\Omega)$ and $A_q v = A_\gamma v$ for $v \in D(A_q) \cap D(A_\gamma)$, $1 < \gamma < \infty$, we sometimes write $A_q = A$ to simplify the notation if there is no misunderstanding. In particular, if $q = 2$, we always write $P = P_2$ and $A = A_2$. Furthermore, let $A^\alpha_q : D(A^\alpha_q) \to L^q_\sigma(\Omega)$, $-1 \le \alpha \le 1$, denote the fractional powers of A_q. It holds $D(A_q) \subseteq D(A^\alpha_q) \subseteq L^q_\sigma(\Omega)$, $R(A^\alpha_q) = L^q_\sigma(\Omega)$ if $0 \le \alpha \le 1$. We note that $(A^\alpha_q)^{-1} = (A_q^{-\alpha})$ and $(A_q)' = A_{q'}$ where $1/q + 1/q' = 1$.

Now we recall the embedding estimate

$$\|v\|_q \le c\|A^\alpha_\gamma v\|_\gamma, \qquad v \in D(A^\alpha_\gamma),\ 1 < \gamma \le q,\ 2\alpha + \frac{3}{q} = \frac{3}{\gamma},\ 0 \le \alpha \le 1, \tag{2.1}$$

and the estimate

$$\|A^\alpha_q e^{-tA_q} v\|_q \le ct^{-\alpha} e^{-\delta t}\|v\|_q, \qquad v \in L^q_\sigma(\Omega),\ 0 \le \alpha \le 1,\ t > 0, \tag{2.2}$$

with constants $c = c(\Omega, q) > 0$, $\delta = \delta(\Omega, q) > 0$, see [1, 7, 11, 12, 15, 27, 31].

By using the estimates (2.1), (2.2) with $0 < \beta < 3/4$, $2\beta + 3/q = 3/2$ and constants c, $\delta > 0$ not depending on t, we obtain for $u_0 \in L^2_\sigma(\Omega)$ that $A^{-\beta}u_0 \in L^q_\sigma(\Omega)$ and that

$$\|e^{-tA}u_0\|_q = \|A^\beta e^{-tA}A^{-\beta}u_0\|_q = \|A_q^\beta e^{-tA_q}A^{-\beta}u_0\|_q$$
$$\le ct^{-\beta}e^{-\delta t}\|A^{-\beta}u_0\|_q \le ct^{-\beta}e^{-\delta t}\|u_0\|_2$$

for $t > 0$. So $\|e^{-tA}u_0\|_q$ with $u_0 \in L^2_\sigma(\Omega)$ is well-defined at least for $t > 0$, and $\int_\eta^\infty (\tau^\alpha \|e^{-\tau A}u_0\|_q)^s \mathrm{d}\tau < \infty$ for any $\eta > 0$ and $\alpha > 0$. In particular, the assumptions (1.9), (1.10) in Theorem 1.3 may be replaced by the assumption $\int_0^\eta (\tau^\alpha \|e^{-\tau A}u_0\|_q)^s \mathrm{d}\tau < \infty$ or $\int_0^\eta (\tau^\alpha \|e^{-\tau A}u_0\|_q)^s \mathrm{d}\tau = \infty$, respectively, for any $\eta > 0$.

Further note that $D(A_q^{1/2}) = W_0^{1,q}(\Omega) \cap L^q_\sigma(\Omega)$ and that the norms

$$\|A_q^{1/2}v\|_q \approx \|\nabla v\|_q, \qquad v \in D(A_q^{1/2}). \tag{2.3}$$

are equivalent. In particular, if $q = 2$, then

$$\|A^{1/2}v\|_2 = \|\nabla v\|_2, \qquad v \in D(A^{1/2}). \tag{2.4}$$

Another estimate which will be frequently used in Section 3 is as follows. Let $g = \operatorname{div} G$ with $G = (G_{ij})_{i,j=1}^3 \in L^q(\Omega)$. Then an approximation argument, see [26, III Lemma 2.6.1], [6, p. 431], shows that $A_q^{-1/2}P_q \operatorname{div} G \in L^q_\sigma(\Omega)$ is well-defined by the identity

$$\langle A_q^{-1/2}P_q \operatorname{div} G, \varphi\rangle = \langle G, \nabla A_{q'}^{-1/2}\varphi\rangle, \qquad \varphi \in L^{q'}_\sigma(\Omega),$$

$1/q + 1/q' = 1$, and that

$$\|A_q^{-1/2}P_q \operatorname{div} G\|_q \le c\|G\|_q \tag{2.5}$$

holds with $c = c(\Omega, q) > 0$. The estimate (2.5) was first established in [14, Lemma 2.1].

Finally, we recall a weighted version of the Hardy-Littlewood-Sobolev inequality, cf. [28, 29]: For $\alpha \in \boldsymbol{R}$ and $s \ge 1$ we consider the weighted L^s-space

$$L^s_\alpha(\boldsymbol{R}) = \left\{ u : \|u\|_{L^s_\alpha} = \left(\int_{\boldsymbol{R}} (|\tau|^\alpha |u(\tau)|)^s \mathrm{d}\tau \right)^{1/s} < \infty \right\}.$$

Lemma 2.1. *Let* $0 < \lambda < 1$, $1 < s_1 \le s_2 < \infty$, $-1/s_1 < \alpha_1 < 1 - 1/s_1$, $-1/s_2 < \alpha_2 < 1 - 1/s_2$ *and* $1/s_1 + (\lambda + \alpha_1 - \alpha_2) = 1 + 1/s_2$, $\alpha_2 \le \alpha_1$. *Then the integral operator*

$$I_\lambda f(t) = \int_{\boldsymbol{R}} |t - \tau|^{-\lambda} f(\tau)\mathrm{d}\tau$$

is bounded as operator $I_\lambda : L^{s_1}_{\alpha_1}(\boldsymbol{R}) \to L^{s_2}_{\alpha_2}(\boldsymbol{R})$.

3. Proof of Theorems 1.2 and 1.3

Now we are in the position to prove the main theorem.

Proof of Theorem 1.2. Let u be a weak solution of (1.1) with initial value $u_0 \in L^2_\sigma$ and external force $f = \operatorname{div} F$ where $F \in L^2(L^2) \cap L^{s/2}_{2\alpha}(L^{q/2})$. Furthermore, let E_{f,u_0} denote the solution of the Stokes problem

$$\partial_t v - \Delta v + \nabla p = f, \qquad \operatorname{div} v = 0$$
$$v|_{\partial\Omega} = 0, \qquad v(0) = u_0,$$

i.e.,

$$E_{f,u_0}(t) = e^{-tA}u_0 + \int_0^t A^{1/2}e^{-(t-\tau)A}A^{-1/2}P \operatorname{div} F(\tau)\mathrm{d}\tau$$
$$=: E_{0,u_0}(t) + E_{f,0}(t).$$

Assume $E_{0,u_0} \in L^s_\alpha(L^q)$, i.e., $\int_0^t \|\tau^\alpha e^{-\tau A}u_0\|_q^s \mathrm{d}\tau < \infty$. Since $u_0 \in L^2_\sigma$ and $F \in L^2(L^2)$, we know that $E_{f,u_0} \in C^0([0,T];L^2) \cap L^2(H^1)$, satisfying the energy equality. Moreover, by using the estimates (2.1) and (2.2) with $2\beta + 3/q = 3/(q/2)$ with $q > 3$, i.e., $\beta = 3/(2q) < 1/2$,

$$\|E_{f,0}(t)\|_q \le c\int_0^t \|A^{1/2+\beta}e^{-(t-\tau)A}(A^{-1/2}P \operatorname{div})F(\tau)\|_{q/2}\mathrm{d}\tau$$
$$\le c\int_0^t (t-\tau)^{-\beta-1/2}\|F(\tau)\|_{q/2}\mathrm{d}\tau.$$

By applying the weighted Hardy-Littlewood-Sobolev inequality (see Lemma 2.1) with the exponents $s_2 = s$, $\alpha_2 = \alpha$, $s_1 = s/2$, $\alpha_1 = 2\alpha$, $\lambda = \beta + 1/2 \in (0,1)$, $-2/s < 2\alpha < 1 - 2/s$ and $-1/s < \alpha < 1 - 1/s$, we have

$$\|E_{f,0}\|_{L^s_\alpha(L^q)} \le c\|F\|_{L^{s/2}_{2\alpha}(L^{q/2})} \tag{3.1}$$

provided $2/s + (3/(2q) + 1/2 + 2\alpha - \alpha) = 1 + 1/s$ (which is equivalent to $2/s + 3/q = 1 - 2\alpha$). We then set $\tilde{u} = u - E_{f,u_0}$ which solves the (Navier-)Stokes system

$$\partial_t \tilde{u} - \Delta\tilde{u} + u\cdot\nabla u + \nabla p = 0, \qquad \operatorname{div}\tilde{u} = 0$$
$$\tilde{u}|_{\partial\Omega} = 0, \qquad \tilde{u}(0) = 0.$$

So we can write at least formally

$$\tilde{u}(t) = -\int_0^t e^{-(t-\tau)A}P \operatorname{div}(u\otimes u)(\tau)\mathrm{d}\tau \tag{3.2}$$
$$= -\int_0^t A^{1/2}e^{-(t-\tau)A}(A^{-1/2}P \operatorname{div})(u\otimes u)(\tau)\mathrm{d}\tau.$$

With $\beta = 3/(2q)$ as above we get

$$\|\tilde{u}(t)\|_q \le c\int_0^t \|A^{1/2+\beta}e^{-(t-\tau)A}\|\,\|A^{-1/2}P\,\mathrm{div}\|\,\|(u\otimes u)\|_{q/2}\mathrm{d}\tau \tag{3.3}$$

$$\le c\int_0^t (t-\tau)^{-1/2-\beta}\|u\|_q^2\mathrm{d}\tau$$

Then the Hardy-Littlewood-Sobolev inequality as above implies that

$$\|\tilde{u}(t)\|_{L^s_\alpha(L^q)} \le c\|(\|u\|_q^2)\|_{L^{s/2}_{2\alpha}} = c\|u\|^2_{L^s_\alpha(L^q)}. \tag{3.4}$$

Since $u = \tilde{u} + E_{f,u_0}$ we have

$$\|\tilde{u}\|_{L^s_\alpha(0,T;L^q)} \le c(\|\tilde{u}\|_{L^s_\alpha(0,T;L^q)} + \|F\|_{L^{s/2}_{2\alpha}(0,T;L^{q/2})} + \|e^{-\tau A}u_0\|_{L^s_\alpha(0,T;L^q)})^2. \tag{3.5}$$

As in [9, p. 99] there exists by Banach's Fixed Point Theorem an $\epsilon_* = \epsilon_*(q,s,\alpha,\Omega) > 0$ such that we get the existence of a unique fixed point $\tilde{u} \in L^s_\alpha(0,T;L^q)$ solving

$$\partial_t\tilde{u} - \Delta\tilde{u} + (\tilde{u} + E_{f,u_0})\cdot\nabla(\tilde{u} + E_{f,u_0}) + \nabla p = 0, \qquad \mathrm{div}\,\tilde{u} = 0$$

$$\tilde{u}|_{\partial\Omega} = 0, \qquad \tilde{u}(0) = 0$$

provided (1.8) is satisfied, i.e., $\|e^{-\tau A}u_0\|_{L^s_\alpha(0,T;L^q)} + \|F\|_{L^{s/2}_{2\alpha}(L^{q/2})} \le \epsilon_*$. Hence $u = \tilde{u} + E_{f,u_0} \in L^s_\alpha(0,T;L^q)$.

Now we need to prove that this constructed mild solution u is indeed a weak solution under the following conditions, cf. the assumptions in Theorem 1.2 and some facts already proved:

$$u, \tilde{u} \in L^s_\alpha(L^q), \qquad u_0 \in L^2_\sigma, \qquad e^{-\tau A}u_0 \in L^s_\alpha(L^q), \qquad F \in L^2(L^2)\cap L^{s/2}_{2\alpha}(L^{q/2}).$$

To this aim we need the following lemmata which will be proved later.

Lemma 3.1. *The mild solution u constructed in the above procedure satisfies* $\nabla u \in L^2(0,T;L^2(\Omega))$.

Lemma 3.2. *Under the assumptions of Lemma* 3.1 *we have that* $u \in L^{s_2}(0,T;L^{q_2}(\Omega))$ *for all* $2/s_2 + 3/q_2 = 3/2$, $2 \le s_2 \le \infty$, $2 \le q_2 \le 6$. *Moreover,* $\|\tilde{u}(t)\|_2 \to 0$ *and* $u(t) \to u_0$ *in* $L^2(\Omega)$ *as* $t \to 0+$.

Lemma 3.3. *Under the assumptions of Lemma* 3.1 $u \in L^4_{\alpha/(2+8\alpha)}(0,T;L^4(\Omega))$.

By Lemma 3.3 we may use that $u \in L^4_{\alpha/(2+8\alpha)}(L^4)$. Hence $u \in L^4(\epsilon,T;L^4)$ for all $0 < \epsilon < T$. So, by [26, IV. Thm. 2.3.1, Lemma 2.4.2] and for a.a. $\epsilon \in (0,T)$, u is the unique weak solution in $L^4(\epsilon,T;L^4)$ on (ϵ,T) of the linear Stokes problem

$$\partial_t u - \Delta u + \nabla p = \operatorname{div} \tilde{F}, \qquad \operatorname{div} u = 0$$

$$u|_{\partial\Omega} = 0, \qquad u|_{t=\epsilon} = u(\epsilon)$$

with external force $\operatorname{div} \tilde{F}$, $\tilde{F} = F - u \otimes u \in L^2(\epsilon, T; L^2)$ and initial value $u(\epsilon) \in L^4(\Omega) \subset L^2(\Omega)$. Therefore, u satisfies the energy equality on (ϵ, T), i.e.

$$\frac{1}{2}\|u(t)\|_2^2 + \int_\epsilon^t \|\nabla u\|_2^2 \mathrm{d}\tau = \frac{1}{2}\|u(\epsilon)\|_2^2 - \int_\epsilon^t (F, \nabla u)\mathrm{d}\tau$$

for all $t \in (\epsilon, T)$ and a.a. $\epsilon \in (0, T)$. Moreover, $u \in C^0([\epsilon, T); L^2)$ and hence $u \in C^0((0, T); L^2)$, see [26, IV 2.1–2.3]. Furthermore, since by Lemma 3.2 $u \in L^\infty((0, T); L^2)$, it also satisfies the energy equality on $[0, T)$. Hence u is a weak solution; this completes the proof of Theorem 1.2. □

Now we prove the above Lemmata which are used in the proof of Theorem 1.2.

Proof of Lemma 3.1. We use a modification of the proof described in [9]. Since for the moment we have no differentiability property for the mild solution u, we apply the Yosida operator $J_n = (I + n^{-1}A^{1/2})^{-1}$, $n \in \boldsymbol{N}$, to (3.2) and write $J_n P \operatorname{div} u \otimes u$ in the form $J_n P \operatorname{div}(u \otimes (\tilde{u} + E_{f,u_0}))$, $\tilde{u} = (I + n^{-1}A^{1/2})\tilde{u}_n$, where $\tilde{u}_n = J_n \tilde{u}$. Then we have

$$\begin{aligned} J_n P \operatorname{div} u \otimes u &= J_n P(u \cdot \nabla E_{f,u_0}) + J_n P(u \cdot \nabla \tilde{u}_n) + \frac{1}{n} J_n P \operatorname{div}(u \otimes A^{1/2}\tilde{u}_n) \\ &= J_n P(u \cdot \nabla E_{f,u_0}) + J_n P(u \cdot \nabla \tilde{u}_n) \\ &\quad + \frac{1}{n} A^{1/2} J_n (A^{-1/2} P \operatorname{div})(u \otimes A^{1/2}\tilde{u}_n). \end{aligned}$$

We use Hölder's inequality with $1/\gamma = 1/2 + 1/q$ to obtain the estimate

$$\begin{aligned} \|J_n P \operatorname{div}(u \otimes u)\|_\gamma &\le c\|u\|_q(\|\nabla E_{f,u_0}\|_2 + \|\nabla \tilde{u}_n\|_2 + \|A^{1/2}\tilde{u}_n\|_2) \\ &= c\|u\|_q(\|\nabla E_{f,u_0}\|_2 + 2\|A^{1/2}\tilde{u}_n\|_2) \end{aligned}$$

since $\|J_n\| \le c$ and $\|n^{-1}A^{1/2}J_n\| \le c$ uniformly in $n \in \boldsymbol{N}$.

From (3.2) we get that

$$A^{1/2}\tilde{u}_n(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} J_n P \operatorname{div}(u \otimes u)(\tau)\mathrm{d}\tau.$$

By the embedding estimate (2.1) with $2\beta + 3/2 = 3/\gamma$ (i.e. $\beta = 3/(2q)$ since $1/\gamma = 1/2 + 1/q$) we see that

$$\|A^{1/2}\tilde{u}_n(t)\|_2 \le c\int_0^t \|A^{1/2+\beta} e^{-(t-\tau)A}\| \, \|J_n P \operatorname{div}(u \otimes u)(\tau)\|_\gamma \mathrm{d}\tau.$$

Applying Lemma 2.1 we have for $0 < T_1 < T$

$$\|A^{1/2}\tilde{\boldsymbol{u}}_n(t)\|_{L^2(0,T_1;L^2)} \le c\left(\int_0^{T_1} (\tau^\alpha \|u\|_q (\|\nabla E_{f,u_0}\|_2 + \|A^{1/2}\tilde{\boldsymbol{u}}_n\|_2))^{s_1} \mathrm{d}\tau\right)^{1/s_1}$$

where $s_1 = (1/2 + 1/s)^{-1}$, $\alpha_1 = \alpha$, $s_2 = 2$, $\alpha_2 = 0$, and $(1/2 + 1/s) + 3/(2q) + 1/2 + \alpha - 0 = 1 + 1/2$, which is equivalent to $2/s + 3/q = 1 - 2\alpha$. Thus, by Hölder's inequality,

$$\|A^{1/2}\tilde{\boldsymbol{u}}_n\|_{L^2(0,T_1;L^2)} \le c\|u\|_{L^s_\alpha(0,T_1;L^q)}(\|\nabla E_{f,u_0}\|_{L^2(0,T_1;L^2)} + \|A^{1/2}\tilde{\boldsymbol{u}}_n\|_{L^2(0,T_1;L^2)}). \tag{3.6}$$

Assume $0 < T_1 < T$ so small such that $c\|u\|_{L^s_\alpha(0,T_1;L^q)} \le 1/2$ is satisfied. Then the absorption argument easily leads from (3.6) to the estimate

$$\|A^{1/2}\tilde{\boldsymbol{u}}_n\|_{L^2(0,T_1;L^2)} \le 2c\|u\|_{L^s_\alpha(0,T_1;L^q)}\|\nabla E_{f,u_0}\|_{L^2(0,T_1;L^2)} < \infty$$

independent of $n \in \boldsymbol{N}$. Consequently, $A^{1/2}\tilde{\boldsymbol{u}}, \nabla\tilde{\boldsymbol{u}} \in L^2(0,T_1;L^2)$ and $\nabla u \in L^2(0,T_1;L^2)$. By the same procedure we obtain a new constant $c = c(T) > 0$, a new length T_2 and consecutive intervals $(T_1, T_1+T_2), (T_1+T_2, T_1+2T_2), \ldots$, that $\nabla\tilde{\boldsymbol{u}} \in L^2(T_1, T_1+T_2; L^2), \ldots$, and consequently that $\nabla\tilde{\boldsymbol{u}}, \nabla u \in L^2(0,T;L^2)$. This completes the proof. □

Proof of Lemma 3.2. Let $1/q_1 = 1/2 + 1/q$, $1/s_1 = 1/2 + 1/s$ and choose β by $2\beta + 3/q_2 = 3/q_1 = 3/2 + 3/q$. From (3.2) and (2.1) we conclude that

$$\begin{aligned}\|\tilde{\boldsymbol{u}}(t)\|_{q_2} &\le c\int_0^t \|A^\beta e^{-(t-\tau)A}\| \, \|P(u\cdot\nabla u)\|_{q_1} \mathrm{d}\tau \\ &\le c\int_0^t (t-\tau)^{-\beta}\|u\|_q\|\nabla u\|_2 \mathrm{d}\tau.\end{aligned}$$

By the Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned}\|\tilde{\boldsymbol{u}}\|_{L^{s_2}(L^{q_2})} &\le c\| \, \|u\|_q\|\nabla u\|_2\|_{L^{s_1}_\alpha} \\ &\le c\|u\|_{L^s_\alpha(L^q)}\|\nabla u\|_{L^2(L^2)} < \infty\end{aligned}$$

for $\alpha_2 = 0$, $\alpha_1 = \alpha < 1 - 1/s_1 = 1/2 - 1/s$ and $s_2 \ge 2 \ge s_1$ with $(1/2 + 1/s) + (3/4 + 3/(2q) - 3/(2q_2)) + \alpha = 1 + 1/s_2$, i.e., $2/s + 3/q + 2\alpha - 1 = 2/s_2 + 3/q_2 - 3/2 = 0$. The case $s_2 = 2$, $q_2 = 6$ also follows from Lemma 3.1. As for the case $s_2 = \infty$, $q_2 = 2$, where $\beta = 3/(2q)$, Hölder's inequality directly implies that

$$\|\tilde{u}(t)\|_2 \le c\int_0^t (t-\tau)^{-3/(2q)}\tau^{-\alpha}(\tau^\alpha\|u\|_q)\|\nabla u\|_2 \mathrm{d}\tau \tag{3.7}$$

$$\le C\|u\|_{L^s_\alpha(L^q)}\|\nabla u\|_{L^2(L^2)}$$

where the integral $\int_0^t((t-\tau)^{-3/(2q)}\tau^{-\alpha})^{(1/2-1/s)^{-1}}\mathrm{d}\tau$ is finite and independent of t; we note that here $\alpha > 0$ is necessary.

To be more precise, with a constant $C > 0$ independent of t,

$$\|\tilde{u}\|_{L^\infty(0,t;L^2)} \le C\|u\|_{L^s_\alpha(0,t;L^q)}\|\nabla u\|_{L^2(0,t;L^2)} \to 0 \qquad \text{as } t \to 0+.$$

So $\|\tilde{u}(t)\|_2 \to 0$ as $t \to 0+$. Hence $u(t) = \tilde{u}(t) + E_{f,u_0}(t) \to u_0$ in $L^2(\Omega)$ as $t \to 0+$. The proof is now complete. □

Remark 3.4. From $\nabla u \in L^2(L^2)$ which implies $u \in L^2(L^6)$ and from $u \in L^\infty(L^2)$, cf. (3.7), it also follows immediately via Hölder's inequality that $u \in L^{s_2}(L^{q_2})$ for all $2/s_2 + 3/q_2 = 3/2,\ 2 \le s_2 \le \infty,\ 2 \le q_2 \le 6$.

Proof of Lemma 3.3. Given q, s, α and $\beta = (2+8\alpha)^{-1}$ we define q_1, s_1 by $1/4 = \beta/q + (1-\beta)/q_1$ and $1/4 = \beta/s + (1-\beta)/s_1$. From Hölder's inequality we know that $\|u(t)\|_4 \le \|u\|_q^\beta \|u\|_{q_1}^{1-\beta}$. Hence

$$\int_0^T \tau^{4\alpha\beta}\|u\|_4^4 \mathrm{d}\tau \le \int_0^T (\tau^\alpha\|u\|_q)^{4\beta}\|u\|_{q_1}^{4(1-\beta)}\mathrm{d}\tau$$

$$\le \|u\|^{4\beta}_{L^s_\alpha(L^q)}\|u\|^{4(1-\beta)}_{L^{s_1}(L^{q_1})} < \infty$$

since $2/s_1 + 3/q_1 = 3/2$. The proof is now complete. □

Finally, we give a proof to Theorem 1.3.

Proof of Theorem 1.3. (1) Using (1.9) and the assumption on F we can choose $0 < T \le \infty$ in such a way that (1.8) is satisfied. Then Theorem 1.2 yields the existence of a unique $L^s_\alpha(L^q)$-strong solution $u \in L^s_\alpha(0,T;L^q(\Omega))$ of (1.1).

Conversely, assume that $u \in L^s_\alpha(0,T;L^q(\Omega))$, $0 < T \le \infty$, is an $L^s_\alpha(L^q)$-strong solution of (1.1). Recall that $E_{0,u_0} = u - \tilde{u} - E_{f,0}$ where by (3.4) $\tilde{u} \in L^s_\alpha(L^q)$, and by (3.1) $E_{f,0} \in L^s_\alpha(L^q)$. Hence $E_{0,u_0} \in L^s_\alpha(L^q)$ as well, and (1.9) is satisfied. This proves part (1) of Theorem 1.3.

(2) Let u be a weak solution as in Theorem 1.3 (2), and suppose that $u \in L^s_\alpha(0,T;L^q)$ holds for some $T > 0$. Then we conclude from (1) that $\int_0^\infty (\tau^\alpha\|e^{-\tau A}u_0\|_q)^s \mathrm{d}\tau < \infty$ which is a contradiction to (1.10). This completes the proof. □

4. Interpretation in terms of Besov spaces

For $1 < q < 3/2$ and $0 < t < 1/q$ let $B^t_{q,r}(\Omega)^3$ denote the usual Besov space of vector fields, and let $\boldsymbol{B}^t_{q,r}(\Omega) = B^t_{q,r}(\Omega)^3 \cap L^q_\sigma(\Omega)$, see [3, (0.5), (0.6)]. Then, by [3, (0.4), (3.18)] with $\boldsymbol{H}^2_q(\Omega) = D(A_q)$,

$$\boldsymbol{B}^t_{q,r}(\Omega) = (L^q_\sigma(\Omega), D(A_q))_{\theta,r}, \qquad 0 < \theta < 1,\ 1 < r < \infty,\ t = 2\theta,$$

and $C^\infty_{0,\sigma}(\Omega)$ is dense in $\boldsymbol{B}^t_{q,r}(\Omega)$. Further, let $\boldsymbol{B}^{-t}_{q,r}(\Omega) := (\boldsymbol{B}^t_{q',r'}(\Omega))'$, cf. [3, (0.6)]. Hence, with $t = 2\alpha + 2/s = 1 - 3/q$ and the duality theorem for real interpolation, cf. [30, Thm. 1.11.2],

$$\begin{aligned}\boldsymbol{B}^{-1+3/q}_{q,s} = \boldsymbol{B}^{-2\alpha-2/s}_{q,s} &= (\boldsymbol{B}^{2\alpha+2/s}_{q',s'})' = (L^{q'}_\sigma, D(A_{q'}))'_{\alpha+1/s,s'} \\ &= (D(A_{q'}), L^{q'}_\sigma)'_{1-\alpha-1/s,s'} = (D(A_{q'})', L^q_\sigma)_{1-\alpha-1/s,s}.\end{aligned}$$

Using the identity $(A^{-1}u_0, A\varphi) = (u_0, \varphi)$ for $\varphi \in D(A)$ we get that

$$\begin{aligned}\|u_0\|_{\boldsymbol{B}^{-1+3/q}_{q,s}} &\approx \|u_0\|_{(D(A_{q'})', L^q_\sigma)_{1-\alpha-1/s,s}} \approx \|A^{-1}u_0\|_{(L^q_\sigma, D(A_q))_{1-\alpha-1/s,s}} \\ &\approx \|A^{-1}u_0\|_q + \left(\int_0^\infty (\tau^{\alpha+1/s}\|Ae^{-\tau A}A^{-1}u_0\|_q)^s \frac{\mathrm{d}\tau}{\tau}\right)^{1/s} \\ &\approx \|A^{-1}u_0\|_q + \left(\int_0^\infty (\tau^\alpha \|e^{-\tau A}u_0\|_q)^s \mathrm{d}\tau\right)^{1/s}.\end{aligned}$$

Since the semigroup $e^{-\tau A}$ is exponentially decreasing, we may omit the term $\|A^{-1}u_0\|_q$ in the last norm above, see [30, Thm. 1.14.5]. Fixing $q \in (3, \infty)$ and considering s, α as related by $2(1/s + \alpha) = 1 - 3/q$, we conclude that the norms

$$\|u_0\|_{\boldsymbol{B}^{-1+3/q}_{q,s}} \text{ and } \|e^{-\tau A}u_0\|_{L^s_\alpha(L^q)} \text{ are equivalent.}$$

For later use, we introduce the notation

$$\|u_0\|_{\boldsymbol{B}^{-1+3/q}_{q,s}(T)} = \|e^{-\tau A}u_0\|_{L^s_\alpha((0,T;L^q)}, \qquad 0 < T \le \infty.$$

In the limit $\alpha \to 0$ we approach the case $\boldsymbol{B}^{-1+3/q}_{q,s(q,0)}$ with $2/s(q,0) + 3/q = 1$ of the classical Serrin condition considered in [8, 9], whereas for $s \to \infty$ we approach the limit space $\boldsymbol{B}^{-1+3/q}_{q,\infty}$.

5. Restricted Serrin's uniqueness theorem

Assumption 5.1. Let $\Omega \subset \boldsymbol{R}^3$ be a bounded domain with boundary of class $C^{2,1}$.

(1) Given $u_0 \in L^2_\sigma(\Omega)$ and an external force $f = \operatorname{div} F$ where $F \in L^2(0,\infty; L^2(\Omega))$ we assume the existence of approximating sequences $(u_{0n}) \subset L^2_\sigma(\Omega)$ of u_0 such that

$$u_{0n} \to u_0 \qquad \text{in } L^2_\sigma(\Omega)$$

and $(F_n) \subset L^2(0,\infty; L^2(\Omega))$ of F such that

$$F_n \to F \qquad \text{in } L^2(0,\infty; L^2(\Omega)) \text{ as } n \to \infty.$$

(2) Let (J_n) denote a family of bounded operators in $\mathscr{L}(L^q_\sigma(\Omega), D(A_q^{1/2}))$ such that for each $1 < q < \infty$ there exists a constant $C_q > 0$ such that

$$\|J_n\|_{\mathscr{L}(L^q_\sigma)} + \left\| \frac{1}{n} A_q^{1/2} J_n \right\|_{\mathscr{L}(L^q_\sigma)} \le C_q \qquad \text{and} \qquad J_n u \to u \text{ in } L^q_\sigma(\Omega) \text{ as } n \to \infty.$$

(3) For each $n \in \boldsymbol{N}$ let u_n denote the weak solution of the approximate Navier-Stokes system

$$\partial_t u_n - \Delta u_n + (J_n u_n)\cdot \nabla u_n + \nabla p_n = \operatorname{div} F_n, \qquad \operatorname{div} u_n = 0 \text{ in } (0,T)\times\Omega \tag{5.1}$$

$$u_n|_{\partial\Omega} = 0, \qquad u_n(0) = u_{n0}$$

Remark 5.2. A typical example of operators (J_n) in Assumption 5.1 is given by the family of Yosida operators $J_n = (I + n^{-1}A_q^{1/2})^{-1}$. It is well known that this family of operators is uniformly bounded on $L^q_\sigma(\Omega)$ as well as on $D(A_q^{1/2})$ for each $1 < q < \infty$. Moreover, $J_n u \to u$ in $L^q_\sigma(\Omega)$ as $n \to \infty$. By analogy, the operators $J_n = e^{-A_q^{1/2}/n}$ have the same properties.

We know from [26, Ch. V, Thm. 2.5.1] (with a minor modification in the case of $J_n = e^{-A_q^{1/2}/n}$) that there exists a unique weak solution $u_n \in \mathscr{L}H_T := L^\infty(L^2) \cap L^2(H^1_0)$ of (5.1) satisfying the uniform estimate

$$\begin{aligned} \|u_n\|_{L^\infty(L^2)} + \|u_n\|_{L^2(H^1)} &\le C(\|u_{0n}\|_2 + \|F_n\|_{L^2(L^2)}) \\ &\le C(\|u_0\|_2 + \|F\|_{L^2(L^2)} + 1) \end{aligned}$$

for all sufficiently large $n \in \boldsymbol{N}$. Therefore, there exists $v \in \mathscr{L}H_T$ and a subsequence (u_{n_k}) of (u_n) such that

$$u_{n_k} \rightharpoonup v \quad \text{in } L^2(H^1_0), \qquad u_{n_k} \overset{*}{\rightharpoonup} v \quad \text{in } L^\infty(L^2), \qquad u_{n_k} \to v \quad \text{in } L^2(L^2).$$

From the last convergence we also conclude that $u_{n_k}(t_0) \to v(t_0)$ in $L^2(\Omega)$ for a.a. $t_0 \in (0,T)$. Actually, $v \in \mathscr{L}H_T$ is a weak solution of (1.1).

Remark 5.3. (1) Since we do not know whether weak solutions of (1.1) are unique, v may depend on the subsequence (u_{n_k}) chosen above. In this case, we say that

(5.2) $\qquad v$ is a well-chosen weak solution of (1.1).

Note that a well-chosen weak solution v is always related to a concrete approximation procedure as in Assumption 5.1 and the choice of an adequate (weakly–*) convergent subsequence of a sequence of approximate solutions (u_n).

(2) The question whether solutions constructed by the Galerkin method fall into the scope of a modified Assumption 5.1 and yield uniqueness in the sense of Theorem 1.4 has not been settled. A similar question concerning the property to be a suitable weak solution, cf. H. Beirão da Veiga [4, p. 321], has been answered in the affirmative, see J.-L. Guermond [16].

Assumption 5.4. Under the assumptions of Assumption 5.1 additionally let $2 < s < \infty$, $3 < q < \infty$, $0 < \alpha < 1/2$ with $2/s + 3/q = 1 - 2\alpha$ be given. Suppose that even $u_0, u_{0n} \in \boldsymbol{B}^{-1+3/q}_{q,s}$ and $F, F_n \in L^{s/2}_{2\alpha}(0,\infty; L^{q/2}(\Omega))$ such that also

$$u_{0n} \to u_0 \quad \text{in } \boldsymbol{B}^{-1+3/q}_{q,s}, \qquad F_n \to F \quad \text{in } L^{s/2}_{2\alpha}(0,\infty; L^{q/2}(\Omega)) \text{ as } n \to \infty.$$

From now on by a well-chosen weak solution of (1.1) we also assume that the approximation satisfies Assumption 5.4 as well as Assumption 5.1.

Proof of Theorem 1.4. As in Sect. 3, we set $u_n(t) = \tilde{u}_n(t) + E_{f_n, u_{0n}}(t)$ where, cf. (3.2),

$$\tilde{u}_n(t) = -\int_0^t A^{1/2} e^{-(t-\tau)A} (A^{-1/2} P \operatorname{div})(J_n u_n \otimes u_n)(\tau) \mathrm{d}\tau.$$

By the assumptions on u_{0n}, F_n and a similar argument as in Sect. 3, $(E_{f_n, u_{0n}}) \subset L^s_\alpha(L^q)$ is uniformly bounded and converges to E_{f, u_0}; to be more precisely, due to the estimate for E_{0, u_0} and (3.1),

$$\|E_{f_n, u_{0n}} - E_{f, u_0}\|_{L^s_\alpha(0, T'; L^q)} \le c(\|u_{0n} - u_0\|_{\boldsymbol{B}^{-1+3/q}_{q,s}(T')} + \|F_n - F\|_{L^s_\alpha(0, T'; L^q)}) \tag{5.3}$$

where $c = c(q, s, \alpha, \Omega) > 0$ is independent of the interval $(0, T')$, $0 < T' \le T$, on which (5.3) is considered.

We also observe that as in (3.3)–(3.5)

$$\begin{aligned} \|\tilde{u}_n\|_{L^s_\alpha(0, T'; L^q)} &\le C_q \|J_n u_n\|_{L^s_\alpha(0, T'; L^q)} \|u_n\|_{L^s_\alpha(0, T'; L^q)} \le C \|u_n\|^2_{L^s_\alpha(0, T'; L^q)} \\ &\le C(\|\tilde{u}_n\|_{L^s_\alpha(0, T'; L^q)} + \|E_{f_n, u_{0n}}\|_{L^s_\alpha(0, T'; L^q)})^2 \\ &\le C(\|\tilde{u}_n\|_{L^s_\alpha(0, T'; L^q)} + \|u_{0n}\|_{\boldsymbol{B}^{-1+3/q}_{q,s}(T')} + \|F_n\|_{L^{s/2}_{\alpha/2}(0, T'; L^{q/2})})^2 \end{aligned} \tag{5.4}$$

with a constant $C > 0$ independent of $0 < T' \le T$. Actually, as in the proof of Theorem 1.2 in Sect. 3, cf. [9, p. 99], there exists an $\varepsilon_* > 0$ and $T' \in (0, T)$ independent of $n \in \boldsymbol{N}$ such that we find a unique solution u_n of (5.1) on $(0, T')$ in $L^s_\alpha(0, T'; L^q)$ for all sufficiently large $n \in \boldsymbol{N}$. Moreover, (u_n) is uniformly bounded in $L^s_\alpha(L^q)$ with bound $\|u_n\|_{L^s_\alpha(0,T';L^q)} \le C\varepsilon_*$ where C is independent of $N \in \boldsymbol{N}$ and T'. Hence we may assume that $u_{n_k} \rightharpoonup U$ in $L^s_\alpha(L^q)$ as $k \to \infty$, using without loss of generality the same subsequence as the sequence (u_{n_k}) considered in the L^2-theory of Remark 5.2. Consequently, $U = v$.

It remains to show that U equals the given strong $L^s_\alpha(L^q)$-solution $u \in L^s_\alpha(0, T'; L^q)$ with data u_0, F. Due to (3.2)

$$u_n(t) - u(t) = E_{f_n, u_{0n}}(t) - E_{f, u_0}(t) - \int_0^t A^{1/2} e^{-(t-\tau)A} (A^{-1/2} P \operatorname{div})((J_n u_n - u) \otimes u_n + u \otimes (u_n - u))(\tau) \mathrm{d}\tau$$

yielding the estimate

$$\begin{aligned} (5.5) \qquad & \|u_n - u\|_{L^s_\alpha(L^q)} \\ & \le \|E_{f_n, u_{0n}} - E_{f, u_0}\|_{L^s_\alpha(L^q)} \\ & \quad + C(\|J_n u_n - u\|_{L^s_\alpha(L^q)} + \|u_n - u\|_{L^s_\alpha(L^q)})(\|u_n\|_{L^s_\alpha(L^q)} + \|u\|_{L^s_\alpha(L^q)}). \end{aligned}$$

Since

$$\|J_n u_n - u\|_q \le \|J_n(u_n - u)\|_q + \|J_n u - u\|_q \le C_q \|u_n - u\|_q + o(1) \qquad \text{as } n \to \infty$$

and

$$\|u_n\|_{L^s_\alpha(0,T';L^q)} + \|u\|_{L^s_\alpha(0,T';L^q)} \le C\varepsilon_*,$$

we conclude from (5.5) and Lebesgue's Theorem on Dominated Convergence that

$$\|u_n - u\|_{L^s_\alpha(0,T';L^q)} \le \|E_{f_n, u_{0n}} - E_{f, u_0}\|_{L^s_\alpha(0,T;L^q)} + C\varepsilon_* \|u_n - u\|_{L^s_\alpha(0,T';L^q)} + o(1)$$

for all $0 < T' \le T$ and $n \in \boldsymbol{N}$, but with $C > 0$ independent of T'. Choosing $\varepsilon_* > 0$ so small that even $C\varepsilon_* \le 1/2$, we get that

$$\|u_n - u\|_{L^s_\alpha(0,T'';L^q)} \le 2\|E_{f_n, u_{0n}} - E_{f, u_0}\|_{L^s_\alpha(0,T'';L^q)} + o(1) \qquad \text{as } n \to \infty.$$

In order to fulfill the inequality $C\varepsilon_* \le 1/2$ and (1.8) for u_{0n}, u_0 and F_n, F this step possibly required to replace T' by a sufficiently small $T'' \in (0, T']$. Since the first term on the right-hand side converges to 0 by Assumption 5.4, we obtain that $\|u_n - u\|_{L^s_\alpha(0,T'';L^q)} \to 0$ as $n \to \infty$ and consequently that $u = U = v$ on $[0, T'')$. □

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References

[1] Amann, H., *Linear and Quasilinear Parabolic Problems, Vol. I*, Monographs in Mathematics, 89, Birkhäuser Boston, Inc., Boston, MA, 1995.

[2] Amann, H., On the strong solvability of the Navier-Stokes equations, J. Math. Fluid Mech., **2** (2000), 16–98.

[3] Amann, H., Nonhomogeneous Navier-Stokes equations with integrable low-regularity data, Int. Math. Ser. (N.Y.), 2, Kluwer/Plenum New York, 2002, pp. 1–28.

[4] Beirão da Veiga, H., On the construction of suitable weak solutions to the Navier-Stokes equations via a general approximation theorem, J. Math. Pures Appl. (9), **64** (1985), 321–334.

[5] Farwig, R., On regularity of weak solutions to the instationary Navier-Stokes system—a review on recent results, Ann. Univ. Ferrara Sez. VII Sci. Mat., **60** (2014), 91–122.

[6] Farwig, R., Galdi, G. P. and Sohr, H., A new class of weak solutions of the Navier-Stokes equations with nonhomogeneous data, J. Math. Fluid Mech., **8** (2006), 423–444.

[7] Farwig, R. and Sohr, H., Generalized resolvent estimates for the Stokes system in bounded and unbounded domains, J. Math. Soc. Japan, **46** (1994), 607–643.

[8] Farwig, R. and Sohr, H., Optimal initial value conditions for the existence of local strong solutions of the Navier-Stokes equations, Math. Ann., **345** (2009), 631–642.

[9] Farwig, R., Sohr, H. and Varnhorn, W., On optimal initial value conditions for local strong solutions of the Navier-Stokes equations, Ann. Univ. Ferrara Sez. VII Sci. Mat., **55** (2009), 89–110.

[10] Fujita, H. and Kato, T., On the Navier-Stokes initial value problem, Arch. Rational Mech. Anal., **16** (1964), 269–315.

[11] Galdi, G. P., An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Vol. II, Nonlinear Steady Problems, Springer Tracts in Natural Philosophy, 39, Springer-Verlag, New York, 1994.

[12] Giga, Y., Analyticity of the semigroup generated by the Stokes operator in L^r-spaces, Math. Z., **178** (1981), 297–329.

[13] Giga, Y., Solution for semilinear parabolic equations in L^p and regularity of weak solutions for the Navier-Stokes system, J. Differential Equations, **62** (1986), 186–212.

[14] Giga, Y. and Miyakawa, T., Solutions in L_r of the Navier-Stokes initial value problem, Arch. Rational Mech. Anal., **89** (1985), 267–281.

[15] Giga, Y. and Sohr, H., Abstract L^q-estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains, J. Funct. Anal., **102** (1991), 72–94.

[16] Guermond, J.-L., Faedo-Galerkin weak solutions of the Navier-Stokes equations with Dirichlet boundary conditions are suitable, J. Math. Pures Appl. (9), **88** (2007), 87–106.
[17] Haak, B. H. and Kunstmann, P. C., On Kato's method for Navier-Stokes equations, J. Math. Fluid Mech., **11** (2009), 492–535.
[18] Heywood, J. G., The Navier-Stokes equations: on the existence, regularity and decay of solutions, Indiana Univ. Math. J., **29** (1980), 639–681.
[19] Hopf, E., Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, Math. Nachr., **4** (1951), 213–231.
[20] Kato, T., Strong L^p-solutions of the Navier-Stokes equation in $\boldsymbol{R}^m$, with applications to weak solutions, Math. Z., **187** (1984), 471–480.
[21] Kiselev, A. A. and Ladyzhenskaya, O. A., On the existence and uniqueness of solutions of the non-stationary problems for flows of non-compressible fluids, Amer. Math. Soc. Transl. (2), **24** (1963), 79–106.
[22] Kozono, H. and Yamazaki, M., Local and global unique solvability of the Navier-Stokes exterior problem with Cauchy data in the space $L^{n,\infty}$, Houston J. Math., **21** (1995), 755–799.
[23] Kunstmann, P. C., Navier-Stokes equations on unbounded domains with rough initial data, Czechoslovak Math. J., **60(135)** (2010), 297–313.
[24] Leray, J., Sur le mouvement d'un liquide visqueux emplissant l'espace, Acta Math., **63** (1934), 193–248.
[25] Miyakawa, T., On the initial value problem for the Navier-Stokes equations in L^p-spaces, Hiroshima Math. J., **11** (1981), 9–20.
[26] Sohr, H., *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser Advanced Texts: Basler Lehrbücher, Birkhäuser Verlag, Basel, 2001.
[27] Solonnikov, V. A., Estimates for solutions of nonstationary Navier-Stokes equations, J. Soviet Math., **8** (1977), 467–529.
[28] Stein, E. M. and Weiss, G., Fractional integrals on n-dimensional Euclidean space, J. Math. Mech., **7** (1958), 503–514.
[29] Strichartz, R. S., L^p estimates for integral transforms, Trans. Amer. Math. Soc., **136** (1969), 33–50.
[30] Triebel, H., Interpolation Theory, Function Spaces, Differential Operators, North-Holland Mathematical Library, 18, North-Holland Publishing Co., Amsterdam-New York, 1978.
[31] Varnhorn, W., The Stokes Equations, Mathematical Research, 76, Akademie-Verlag, Berlin, 1994.

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