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Blow-Up at Space Infinity for Solutions of Cooperative Reaction-Diffusion Systems

By

Masahiko Shimojo and Noriaki Umeda
(National Taiwan Normal University, R.O. China and The University of Tokyo, Japan)

Abstract. We consider the Cauchy problem of cooperative reaction-diffusion systems with nonnegative initial data. Here we discuss the blow-up of a solution that occurs only at space infinity. We give sufficient conditions for such phenomena, and study an asymptotic behaviour at space infinity of the solutions at the blow-up time. In general, relatively little is proved on the locations of blow-up point for semilinear parabolic systems. However, our results can be applied to a large class of nonlinearity for some class of initial value. The reason of this is that, when the blow-up occurs only at space infinity, the effect of reaction is much stronger than that of diffusion, and the behaviour at space infinity is well approximated by the flat solution.

Key Words and Phrases. Blow-up at space infinity, Cooperative reaction-diffusion systems.

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1. Introduction and main theorems

We consider the initial value problem for a reaction-diffusion system:

$$
\begin{cases} u_t = d_1 \Delta u + f(u,v), & x \in \boldsymbol{R}^n,\ t > 0, \\ v_t = d_2 \Delta v + g(u,v), & x \in \boldsymbol{R}^n,\ t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \boldsymbol{R}^n, \end{cases} \tag{1}
$$

where $n \geq 1$, d_1 and d_2 are positive constants, u_0 and v_0 are bounded nonnegative continuous functions in $\boldsymbol{R}^n$, and we discuss the blow-up of solutions that occurs only at space infinity. In this paper we also assume that nonlinear terms f and g are continuous functions in $[0,\infty) \times [0,\infty)$. Our results below include the case $f(u,v) = v^q$, $g(u,v) = u^p$ with $p, q > 1$ for example.

The problem (1) has a unique bounded classical solution at least locally in time, provided that f and g are locally Lipschitz continuous in the range of the solutions. However, the solution may cease to exist in finite time. For given initial data (u_0, v_0) and nonlinear terms f and g, let

$$
T = T(u_0, v_0) = T(u_0, v_0, f, g)
$$

be the maximal existence time of the solution. If $T = \infty$, the solution is said to exist globally in time. If $T < \infty$ and

$$\limsup_{t \to T} \{ \|u(\cdot, t)\|_{L^\infty(\boldsymbol{R}^n)} + \|v(\cdot, t)\|_{L^\infty(\boldsymbol{R}^n)} \} = \infty, \tag{2}$$

we say that the solution *blows up* in finite time. For the case $T < \infty$, a point $x_{BU} \in \boldsymbol{R}^n$ is called a *blow-up point* of the solution to the problem (1) if there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ such that

$$t_m \uparrow T, \qquad x_m \to x_{BU} \qquad \text{and} \qquad (u+v)(x_m, t_m) \to \infty \qquad \text{as } m \to \infty.$$

A set of all blow-up points is called a blow-up set. If there exists a sequence $\{(x_m, t_m)\}_{m=1}^\infty$ such that

$$t_m \uparrow T, \qquad |x_m| \to \infty \qquad \text{and} \qquad (u+v)(x_m, t_m) \to \infty \qquad \text{as } m \to \infty,$$

then we say that the solution *blows up at space infinity*.

There is a huge amount of literature about location of blow-up points for single equations. In the works of Weissler [36] and Friedman-McLeod [9], it was shown that the single point blow-up occurs when the solution on the ball is radially symmetric positive and monotone decreasing for $r = |x|$. The solution whose blow-up set is sphere was constructed by Giga-Kohn [10] (See also [24, 38, 19, 20, 1, 2]). On the other hand, problems about blow-up set for parabolic systems is widely open. Friedman-Giga [8] considered the system $u_t = \Delta u + v^q$, $v_t = \Delta v + u^p$ with $p = q > 1$ and construct a radially symmetric solution that blows up only at the origin. The generalization of this result was recently obtained by Souplet [31] for $p, q > 1$. For other blow-up problems of semilinear parabolic systems, we also refer the reader to, e.g., [3, 4, 6, 15, 16, 17, 18, 26, 37, 28, 34, 35].

Let us recall some known results about blow-up at space infinity. Lacey [21] considered the Dirichlet problem in a half line and constructed solutions that blow up only at space infinity. Giga-Umeda [11] proved that blow-up only at space infinity occurs under the condition $\lim_{|x| \to \infty} u_0(x) = M$ and $u_0 \not\equiv M$ for nonnegative solutions of $u_t = \Delta u + u^p$ in $\boldsymbol{R}^n$. Later, Shimojo [33] extended their results by relaxing the assumptions of initial data u_0 which are similar to (5) and (6) appeared in Theorem 1. After that Giga-Umeda [12] obtained the same results for semilinear heat equations of the form $u_t = \Delta u + f(u)$ with more general nonlinearity f. Later Shimojo [32] also calculate the shape of blow-up profile $u(x, T) := \lim_{t \to T} u(x, t)$ for $x \in \boldsymbol{R}^n$ precisely. See also Seki-Suzuki-Umeda [30] and Seki [29] for quasilinear parabolic equations, which generalize the result of [12].

The aim of the present paper is to show the existence of solutions that blow up only at space infinity for large class of semilinear parabolic system. This is regarded as a generalization of the results for single equation of the above to parabolic system (1).

Now let us state our main theorems. In the following, let $(U(t), V(t))$ be a solution of the system of ordinary differential equations:

$$\begin{cases} U_t = f(U, V), & t > 0, \\ V_t = g(U, V), & t > 0, \\ U(0) = M, \ V(0) = N, \end{cases} \tag{3}$$

where $M, N > 0$ are positive constants. Here we denote the maximum existence time of (3) by $T(M, N, f, g)$. We write $T(M, N) := T(M, N, f, g)$ for simplicity. The first theorem provides us the information about the exact maximal existence time of the solution.

Theorem 1. *Let $M, N > 0$ be constants, and assume that there exist constants $b_1 \in [0, M)$ and, $b_2 \in [0, N)$ satisfying the following*:

1. *The nonlinear terms f and g satisfy*

$$\begin{cases} f, g \in C^2([b_1, \infty) \times [b_2, \infty)), \\ f_v \geq 0, \ g_u \geq 0, \ f > 0, \ g > 0 \quad \text{for } u > b_1, \ v > b_2. \end{cases} \tag{4}$$

2. *The initial data (u_0, v_0) satisfies*

$$u_0 \in [b_1, M], \qquad v_0 \in [b_2, N], \qquad u_0 \not\equiv M, \qquad v_0 \not\equiv N \qquad \text{for a.e. } x \in \boldsymbol{R}^n. \tag{5}$$

Furthermore, suppose that there exist sequences $\{r_m\}_{m=1}^{\infty} \subset \boldsymbol{R}$ of radii and $\{a_m\}_{m=1}^{\infty} \subset \boldsymbol{R}^n$ satisfying $\lim_{m\to\infty} r_m = \infty$ such that

$$\limsup_{m\to\infty} \{\|M - u_0\|_{L^\infty(B(a_m, r_m))} + \|N - v_0\|_{L^\infty(B(a_m, r_m))}\} = 0, \tag{6}$$

where $B(a, r)$ denotes the open ball of radius $r > 0$ centered at $a \in \boldsymbol{R}^n$. Then $T(u_0, v_0) = T(M, N)$.

Next we state our main result on a sufficient condition for blow-up only at space infinity.

Theorem 2. *Assume the same hypotheses as in Theorem* 1. *Let the solution (U, V) of* (3) *blow up at $t = T(M, N) < \infty$. If f, g and the solution (U, V) of* (3) *satisfy*

$$\limsup_{t\to T} \frac{f(\theta U(t), \theta V(t))}{\theta f(U(t), V(t))} < 1, \qquad \limsup_{t\to T} \frac{g(\theta U(t), \theta V(t))}{\theta g(U(t), V(t))} < 1 \tag{7}$$

for any $\theta \in (0,1)$, *and*

$$(8) \qquad \limsup_{t\to T}\left\{(T-t)\left|\frac{\ddot{U}(t)}{\dot{U}(t)}\right|\right\} < \infty, \qquad \limsup_{t\to T}\left\{(T-t)\left|\frac{\ddot{V}(t)}{\dot{V}(t)}\right|\right\} < \infty,$$

then the solution of (1) *has no blow-up points in* $\boldsymbol{R}^n$ (*It blows up only at space infinity*).

Remark 1.1. Assume that the nonlinear terms f and g can be written in either one of the following forms:

(i) $f(u,v) = u^{p_1}v^{q_1}$ and $g(u,v) = u^{p_2}v^{q_2}$ with $p_1, p_2, q_1, q_2 \geq 0$, $p_1 + q_1 > 1$, $p_2 + q_2 > 1$, $p_2 - p_1 + 1 > 0$ and $q_1 - q_2 + 1 > 0$.

(ii) $f(u,v) = e^{\alpha_1 u + \beta_1 v}$ and $g(u,v) = e^{\alpha_2 u + \beta_2 v}$ with $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $\alpha_1 + \beta_1 > 0$, $\alpha_2 + \beta_2 > 0$, $\alpha_1 < \alpha_2$ and $\beta_1 > \beta_2$.

Then the nonlinear terms f, g and the solution (U, V) of (3) satisfy (7) and (8).

Remark 1.2. Theorem 1 with their proof contain that

$$\limsup_{t\to T(M,N)}\left[\limsup_{m\to\infty}\left\{\operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} u(x,t) + \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} v(x,t)\right\}\right] = \infty$$

with the hypotheses of Theorem 1.

Since the above blow-up occurs only at space infinity, the pointwise limit $(u(x,T(M,N)), v(x,T(M,N))) = \lim_{t\to T(M,N)}(u(x,t), v(x,t))$ exists for every $x \in \boldsymbol{R}^N$ (see Lemma 4.1). The next theorem describes the behaviour at infinity of the functions $(u(x,T(M,N)), v(x,T(M,N)))$.

Theorem 3. *Assume the same hypotheses as in Theorem* 2. *Then*

$$\lim_{m\to\infty}(u(a_m, T(M,N)) + v(a_m, T(M,N))) = \infty,$$

where the sequence $\{a_m\}_{m=1}^{\infty} \subset \boldsymbol{R}^n$ *is the same as in Theorem* 1.

This paper is organized as follows. In Section 2 we prove Theorem 1. Section 3 is devoted to the proof of Theorem 2. The proof of Theorem 3 will be given in Section 4. In Section 5 we discuss the conditions about the nonlinear terms f and g satisfying (7) in Theorem 2, and give some examples of f and g.

2. The maximal existence time

In this section we consider the pairs of the continuous functions (u_0, v_0) and $(\bar{u}_0, \bar{v}_0)$. Recall that the functions $f(u,v)$ and $g(u,v)$ satisfy (4).

Let (u, v) and $(\bar{u}, \bar{v})$ be solutions of the system (1) with initial data (u_0, v_0) and $(\bar{u}_0, \bar{v}_0)$ respectively. Put $\tilde{u} = u - \bar{u}$ and $\tilde{v} = v - \bar{v}$. Then $(\tilde{u}, \tilde{v})$ is the solution of the following problem:

$$
(9) \qquad \begin{cases} \tilde{u}_t = d_1 \Delta \tilde{u} + f(u, v) - f(\bar{u}, \bar{v}), & x \in \mathbf{R}^n,\ 0 < t < \tilde{T}, \\ \tilde{v}_t = d_2 \Delta \tilde{v} + g(u, v) - g(\bar{u}, \bar{v}), & x \in \mathbf{R}^n,\ 0 < t < \tilde{T}, \\ \tilde{u}(x, 0) = u_0(x) - \bar{u}_0(x), & x \in \mathbf{R}^n, \\ \tilde{v}(x, 0) = v_0(x) - \bar{v}_0(x), & x \in \mathbf{R}^n, \end{cases}
$$

where $\tilde{T} = \min\{T(u_0, v_0), T(\bar{u}_0, \bar{v}_0)\}$.

The following proposition is some kind of generalization of results in the papers [13, 14] by Gladkov and [12] by Giga-Umeda.

Proposition 2.1. *Assume that u_0, v_0, $\bar{u}_0$ and $\bar{v}_0$ are nonnegative, continuous and bounded functions. Let $f(u, v)$ and $g(u, v)$ satisfy* (4) *with* $b_1 = \inf_{x \in \mathbf{R}^n} \min\{u_0, \bar{u}_0\}(x)$, $b_2 = \inf_{x \in \mathbf{R}^n} \min\{v_0, \bar{v}_0\}(x)$. *If there exist sequences $\{a_m\}_{m=1}^{\infty} \subset \mathbf{R}^n$ and $\{r_m\}_{m=1}^{\infty}$ satisfying $0 < r_1 < r_2 < \cdots \to \infty$ such that*

$$
(10) \qquad \limsup_{m \to \infty} \{\|u_0 - \bar{u}_0\|_{L^\infty(B(a_m, r_m))} + \|v_0 - \bar{v}_0\|_{L^\infty(B(a_m, r_m))}\} = 0,
$$

then solutions (u, v) and $(\bar{u}, \bar{v})$ of (1) *with initial data (u_0, v_0) and $(\bar{u}_0, \bar{v}_0)$ satisfy*

$$
\limsup_{m \to \infty} \{\|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^\infty(B(a_m, r_m/2))} + \|v(\cdot, t) - \bar{v}(\cdot, t)\|_{L^\infty(B(a_m, r_m/2))}\} = 0
$$

for any $t \in (0, \tilde{T})$.

Proof. It is clear that the proposition holds for the case $u_0 \equiv \bar{u}_0$ and $v_0 \equiv \bar{v}_0$. Hence we consider the case

$$
(11) \qquad u_0 \not\equiv \bar{u}_0 \qquad \text{or} \qquad v_0 \not\equiv \bar{v}_0.
$$

First we consider vector valued functions:

$$
W(x, t) = \begin{pmatrix} u - \bar{u} \\ v - \bar{v} \end{pmatrix}(x, t) \qquad \text{and} \qquad W_0(x) = \begin{pmatrix} u_0 - \bar{u}_0 \\ v_0 - \bar{v}_0 \end{pmatrix}(x).
$$

For any domain $\Omega \subset \mathbf{R}^n$, we define $\|W(\cdot, t)\|_{L^\infty(\Omega)} := \|u(\cdot, t) - \bar{u}(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}(\cdot, t)\|_{L^\infty(\Omega)}$.

From (10), for any $\varepsilon > 0$ there exists $m_0 \in \mathbf{N}$ large enough such that for any $m \geq m_0$

$$
(12) \qquad \|W_0\|_{L^\infty(B(a_m, r_m))} \leq \frac{\varepsilon}{4}.
$$

Take $t_0 \in (0, \tilde{T})$. Put

$$K = \sup_{(u_1, v_1), (u_2, v_1) \in A} \max\left\{ \left| \frac{f(u_1, v_1) - f(u_2, v_1)}{u_1 - u_2} \right|, \left| \frac{f(u_2, v_1) - f(u_2, v_2)}{v_1 - v_2} \right|, \right.$$
$$\left. \left| \frac{g(u_1, v_1) - g(u_2, v_1)}{u_1 - u_2} \right|, \left| \frac{g(u_2, v_1) - g(u_2, v_2)}{v_1 - v_2} \right| \right\},$$

where $A = [b_1, b] \times [b_2, b]$ with $b = \sup_{(x, t) \in \boldsymbol{R}^n \times [0, t_0]} \max\{u, v, \bar{u}, \bar{v}\}(x, t)$. From (4), the solution of the problem

$$\text{(13)} \qquad \begin{cases} Z_t = \begin{pmatrix} d_1 \Delta & 0 \\ 0 & d_2 \Delta \end{pmatrix} Z + K \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} Z, & x \in \boldsymbol{R}^n,\ t \in (0, t_0), \\ Z(x, 0) = W_0(x), & x \in \boldsymbol{R}^n \end{cases}$$

is a supersolution of (9). Let $G = (G_{i,j})_{1 \le i,j \le 2}$ be the Green matrix for the equation (13). Then the solution Z of (13) is given by

$$\begin{aligned} Z(x, t) &= \int_{\boldsymbol{R}^n} G(x - y, t) W_0(y) dy \\ &= \int_{\boldsymbol{R}^n \setminus B(x, r_m/2)} G(x - y, t) W_0(y) dy + \int_{B(x, r_m/2)} G(x - y, t) W_0(y) dy \\ &= I + II. \end{aligned}$$

From the estimates of the fundamental solution (see [7, CHAPTER 9, Theorem 1]), we have, for any $x, y \in \boldsymbol{R}^n$ and $t \in (0, t_0)$,

$$\text{(14)} \qquad |G(x - y, t)| \le C t^{-n/2} e^{-c\rho}$$

with $C = C(d_1, d_2, K, t_0)$ and $c = c(d_1, d_2, K, t_0)$, where $\rho = |x - y|^2 / t$. Let $t \in (0, t_0)$. Then by (11) and (14) we have

$$\int_{\boldsymbol{R}^n \setminus B(x, r_m/2)} |G(x - y, t)| dy \le \int_{\boldsymbol{R}^n \setminus B(x, r_m/2)} C t^{-n/2} e^{-c\rho}\, dy \le \frac{\varepsilon}{4 \| W_0 \|_{L^\infty(\boldsymbol{R}^n)}}$$

for any $m \in \boldsymbol{N}$ large enough. Thus, for any $t \in (0, t_0)$

$$\text{(15)} \qquad \begin{aligned} |I| &\le \left| \int_{\boldsymbol{R}^n \setminus B(x, r_m/2)} G(x - y, t) W_0(y) dy \right| \\ &\le \| W_0 \|_{L^\infty(\boldsymbol{R}^n)} \int_{\boldsymbol{R}^n \setminus B(x, r_m/2)} C t^{-n/2} e^{-c\rho}\, dy \le \frac{\varepsilon}{2}. \end{aligned}$$

Note that $x \in B(a_m, r_m/2)$ and $y \in B(x, r_m/2)$ imply $y \in B(a_m, r_m)$. Combining this with (12), we have

$$|II| \leq \left| \|W_0\|_{L^\infty(B(a_m, r_m/2))} \int_{\boldsymbol{R}^n} G(x-y,t)dy \right| \leq \frac{\varepsilon}{2}.$$

These two estimates yield $\|Z(x,t)\|_{L^\infty(B(a_m,r_m/2))} \leq \varepsilon$ for $t \in (0, t_0)$. Hence, by the comparison principle, for any $t \in (0, t_0)$

$$\limsup_{m\to\infty} \|W(\cdot,t)\|_{L^\infty(B(a_m,r_m/2))} = 0.$$

Since $t_0 \in (0, \tilde{T})$ is arbitrary, we complete the proof. □

Proof of Theorem 1. It is clear that if $T(M,N) = \infty$, then $T(u_0, v_0) = \infty$ by comparison. It is enough for proving the theorem to show that

$$\limsup_{t\to T(M,N)} \left[\sup_{m\in\boldsymbol{N}} \left\{ \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} u(x,t) + \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} v(x,t) \right\} \right] = \infty$$

in the case $T(M,N) < \infty$. To the contrary, we assume that there exists a positive constant $L < \infty$ such that

$$\sup_{t\in(0,T(M,N))} \left[\sup_{m\in\boldsymbol{N}} \left\{ \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} u(x,t) + \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} v(x,t) \right\} \right] \leq L. \tag{16}$$

From the facts $U' \geq 0$, $V' \geq 0$ and $\lim_{t\to T(M,N)}(U(t)+V(t)) = \infty$, there exists $T_0 \in [0, T(M,N))$ such that

$$U(T_0) + V(T_0) \geq 3L.$$

By Proposition 2.1 there exists a constant $m_0 \geq 0$ such that

$$\sup_{m\geq m_0} \{\|U(T_0) - u(\cdot,T_0)\|_{L^\infty(B(a_m,r_m/2))} + \|V(T_0) - v(\cdot,T_0)\|_{L^\infty(B(a_m,r_m/2))}\} \leq L.$$

From (16) we see that

$$\begin{aligned}
&\sup_{m\geq m_0} \{\|U(T_0) - u(\cdot,T_0)\|_{L^\infty(B(a_m,r_m/2))} + \|V(T_0) - v(\cdot,T_0)\|_{L^\infty(B(a_m,r_m/2))}\} \\
&= \sup_{m\geq m_0} \left\{ U(T_0) + V(T_0) - \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} u(x,t) - \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} v(x,t) \right\} \\
&\geq 3L - L = 2L > L.
\end{aligned}$$

This is a contradiction. Thus we conclude that

$$\sup_{t\in[0,T(M,N))} \left[\sup_{m\in\boldsymbol{N}} \left\{ \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} u(x,t) + \operatorname*{ess\,inf}_{x\in B(a_m,r_m/2)} v(x,t) \right\} \right] = \infty.$$

By comparing $(u(x,t), v(x,t))$ with the solution $(U(t), V(t))$ of (3), we see that the solution $(u(x,t), v(x,t))$ does not blow up for all $t \in [0, T(M,N))$. Then we obtain

$$\limsup_{t \to T(M,N)} \left[\sup_{m \in N} \left\{ \operatorname*{ess\,inf}_{x \in B(a_m, r_m/2)} u(x,t) + \operatorname*{ess\,inf}_{x \in B(a_m, r_m/2)} v(x,t) \right\} \right] = \infty.$$

We thus have $T(u_0, v_0) = T(M,N)$. □

Remark 2.2. In [12] the authors used the sequence of the Green kernel of the Dirichlet problem of the heat equation $u_t = \Delta u$ in the domains $B(a_m, r_m)$ for $m = 1, 2, \ldots$. In this paper we use the Green kernel for the total space, and we do not need limiting argument for Green kernel. Hence our proof is much simpler than that of [12].

3. Blow-up only at space infinity

In this section, we prove Theorem 2 by using the same argument that was employed by [29] and [33].

Assume that the solution (U, V) of (3) blows up at $T = T(M,N) \in (0, \infty)$. Put $\varphi(s) = U(T-s)$ and $\psi(s) = V(T-s)$ with $s \in (0, T]$. Thus φ and ψ satisfy the following ordinary differential equation.

$$\frac{d\varphi}{ds} = -f(\varphi, \psi), \qquad \frac{d\psi}{ds} = -g(\varphi, \psi), \qquad \varphi(T) = M, \qquad \psi(T) = N. \tag{17}$$

From the assumptions of Theorem 2, for any $\theta \in (0,1)$, the functions f, g, φ and ψ satisfy

$$\limsup_{s \to 0} \frac{f(\theta\varphi(s), \theta\psi(s))}{\theta f(\varphi(s), \psi(s))} < 1, \tag{18}$$

$$\limsup_{s \to 0} \frac{g(\theta\varphi(s), \theta\psi(s))}{\theta g(\varphi(s), \psi(s))} < 1, \tag{19}$$

$$\limsup_{s \to 0} \left| \frac{s\ddot{\varphi}(s)}{\dot{\varphi}(s)} \right| < \infty, \qquad \limsup_{s \to 0} \left| \frac{s\ddot{\psi}(s)}{\dot{\psi}(s)} \right| < \infty. \tag{20}$$

The following lemma originally appeared in [23, Lemma 2.3] for single semilinear parabolic equations. Here we generalize it to parabolic systems by modifying their argument. See also [29] for the fast diffusion single equations.

Lemma 3.1. *Assume* (18), (19) *and* (20). *Let* $(u(x,t), v(x,t))$ *be a solution of* (1) *in* $\boldsymbol{R}^n \times [0,T)$. *Suppose that there exist* $t_0 \in (0,T)$, $a \in \boldsymbol{R}^n$, $r_0 > 0$ *and* $\theta \in (0,1)$ *such that*

$$u(x,t) \le \theta\varphi(T-t), \qquad v(x,t) \le \theta\psi(T-t)$$

in $|x-a|<r_0,\ t_0 \le t < T$. *Then* (u,v) *does not blow up at* $t=T$ *in a neighborhood of a.*

Proof. We shall construct a suitable supersolution. It is possible to let domains of φ and ψ be extended to $s \in (T, T+t_0)$ with some $t_0 > 0$. We thus may define the functions

$$w(x,t) = \theta\varphi(T-t+h(r)), \qquad z(x,t) = \theta\psi(T-t+h(r)),$$

where $r = |x-a|$ and

$$h(r) = \varepsilon\left(\frac{1+\cos(\pi r/r_0)}{2}\right) = \varepsilon\left\{\cos\left(\frac{\pi r}{2r_0}\right)\right\}^2$$

for $\varepsilon > 0$ small enough. From (17) we have

$$\begin{aligned} w_t - d_1\Delta w - f(w,z) &= -\theta\dot{\varphi} - d_1\theta\dot{\varphi}\Delta h - d_1\theta\ddot{\varphi}|\nabla h|^2 - f(\theta\varphi,\theta\psi) \\ &= \theta f(\varphi,\psi)\left\{1 + d_1\Delta h + d_1\frac{\ddot{\varphi}}{\dot{\varphi}}|\nabla h|^2 - \frac{f(\theta\varphi,\theta\psi)}{\theta f(\varphi,\psi)}\right\}. \end{aligned}$$

Define $\mu_1 = \limsup_{t\to 0} f(\theta\varphi(t),\theta\psi(t))/\theta f(\varphi(t),\psi(t))$. Then there exist t_1 near T and $C = C(T,t_1)$ such that for $t \in (t_1, T)$

$$\begin{aligned} &\left\{1 + d_1\Delta h + d_1\frac{\ddot{\varphi}}{\dot{\varphi}}|\nabla h|^2 - \frac{f(\theta\varphi,\theta\psi)}{\theta f(\varphi,\psi)}\right\} \\ &\qquad \ge (1-\mu_1) + d_1\left(h_{rr} + \frac{N-1}{r}h_r\right) - C\frac{|h_r|^2}{h} \end{aligned} \tag{21}$$

with $\varepsilon > 0$ small enough. Since $\mu_1 \in (0,1)$, and h_{rr}, h_r/r and $|h_r|^2/h$ are bounded on $r \ge r_0$ with order ε, we conclude that

$$\begin{cases} w_t \ge d_1\Delta w + f(w,z), & |x-a| < r_0,\ t_1 < t < T, \\ w(x,t_0) \ge u(x,t_0), & |x-a| < r_0, \\ w(x,t) \ge u(x,t), & |x-a| = r_0,\ t_1 \le t < T \end{cases} \tag{22}$$

for any $\varepsilon > 0$ sufficient small. Applying the same argument to v, g and ψ, we obtain that

$$\begin{cases} z_t \ge d_2\Delta z + g(w,z), & |x-a| < r_0,\ t_1 < t < T, \\ z(x,t_0) \ge v(x,t_0), & |x-a| < r_0, \\ z(x,t) \ge v(x,t), & |x-a| = r_0,\ t_1 \le t < T \end{cases} \tag{23}$$

for any $\varepsilon > 0$ small enough.

By the comparison principle, for $x \in B(a, r_0)$ and $t \in [t_1, T)$, we have $u(x,t) \le w(x,t)$. Since φ is a decreasing function, we obtain

$$u(x,t) \le \theta\varphi\left(T - t + h\left(\frac{r_0}{2}\right)\right) = \theta\varphi\left(T - t + \frac{\varepsilon}{2}\right),$$

$$v(x,t) \le \theta\psi\left(T - t + h\left(\frac{r_0}{2}\right)\right) = \theta\psi\left(T - t + \frac{\varepsilon}{2}\right)$$

for $(x,t) \in B(a, r_0/2) \times [t_1, T)$. □

Remark 3.2. In [10, Theorem 4.2] the corresponding assertion of Lemma 3.1 is shown for the single equation $u_t = \Delta u + |u|^{p-1}u$ with $1 < p < (n+2)/(n-2)$ or $n \le 2$ (see also [25, Corollary 1.3]).

Finally, we shall prove that the blow-up occurs only at space infinity by using Lemma 3.1.

Proof of Theorem 2. We need to show that for any $a \in \boldsymbol{R}^n$ there exist $t_0 \in (0, T)$, $r_0 > 0$ and $\theta \in (0,1)$ such that for $x \in B(a, r_0)$ and $t \in [t_0, T)$

$$|u(x,t)| \le \theta\varphi(T-t), \qquad |v(x,t)| \le \theta\psi(T-t).$$

We may assume that $d_1 > d_2$ without loss of generality. From the strong maximum principle, the solution (u, v) of (1) satisfies

$$u(x,t) < U(t), \qquad v(x,t) < V(t) \qquad \text{for } (x,t) \in D \times (0,T)$$

for any compact set $D \subset \boldsymbol{R}^n$. Thus we may let $u_0(x) < M$ and $v_0(x) < N$ for $x \in B(a, r_0)$ without loss of generality. Let $w(x,t)$ be a solution of

$$\begin{cases} w_t = d_1 \Delta w, & x \in B(a, r_0),\ t \ge 0, \\ w(x,t) = 1, & x \in \partial B(a, r_0),\ t \ge 0, \\ 1 \ge w(x,0) \ge \max\{u_0(x)/M, v_0(x)/N\}, & x \in B(a, r_0), \\ \Delta w(x,0) \ge 0, & x \in B(a, r_0), \\ w(x,0) \not\equiv 1, & x \in B(a, r_0). \end{cases} \tag{24}$$

We shall prove that for any $a \in \boldsymbol{R}^n$ and any $r_0 > 0$, (Uw, Vw) is the super-solution of (1) in $B(a, r_0)$. By (18) we derive

$$(Uw)_t = f(U,V)w + U d_1 \Delta w \ge f(Uw, Vw) + d_1 \Delta(Uw).$$

Using the same reasoning as above and (19) we also derive

$$(Vw)_t = g(U,V)w + V d_1 \Delta w \ge (Uw, Vw) + d_1 \Delta(Vw).$$

By the maximum principle, $\Delta w(x,t) \geq 0$ for any $(x,t) \in B(a,r_0) \times (0,T)$, we obtain

$$(Vw)_t \geq g(Uw, Vw) + d_2 \Delta(Vw).$$

Also $Uw > u$, $Vw > v$ on $\partial B(a,r_0) \times (0,T)$ and $B(a,r_0) \times \{0\}$. Again, by the comparison principle, we conclude

$$u \leq Uw, \qquad v \leq Vw \qquad \text{in } B(a,r_0) \times (0,T). \tag{25}$$

Besides, by the strong maximum principle, we have that $0 \leq w(x,t) < 1$ for any $(x,t) \in B(a,r_0) \times (0,T)$. In particular, for any $\tilde{r}_0 \in (0,r_0)$ there exist $0 < \theta < 1$ and $t_0 \in (0,T)$ such that

$$0 \leq w(x,t) \leq \theta, \qquad |x-a| < \tilde{r}_0,\ t_0 \leq t < T.$$

This and (25) imply

$$0 \leq u(x,t) \leq \theta\varphi(T-t), \qquad |x-a| < \tilde{r}_0,\ t_0 \leq t < T.$$

Analogously, we obtain that

$$0 \leq v(x,t) \leq \theta\psi(T-t), \qquad |x-a| < \tilde{r}_0,\ t_0 \leq t < T.$$

Combining this with Lemma 3.1, we obtain the desired result. □

4. Behavior at blow-up time

In this section we prove Theorem 3.

Lemma 4.1. *Assume the same hypotheses as in Theorem* 2. *Then* $(u,v)(x,T) = \lim_{t \to T}(u,v)(x,t)$ *exists for any* $x \in \boldsymbol{R}^n$ *with* $T = T(M,N)$.

Proof. From Theorem 2, for any $a \in \boldsymbol{R}^n$, there exists $\epsilon > 0$ such that

$$b_1 \leq \sup_{t \in (0,T)} u(x,t) \leq L, \qquad b_2 \leq \sup_{t \in (0,T)} v(x,t) \leq L \qquad \text{for } x \in B(a,\varepsilon)$$

with some $L = L(a,\varepsilon) < \infty$. By the standard parabolic estimates [5, CHAPTER II, THEOREM 2.2], we have $(u(\cdot,t), v(\cdot,t)) \in BC^{2+\alpha}_{loc}(\boldsymbol{R}^n)$ for $t \in (0,T)$ and $\alpha \in (0,1)$. Moreover, we see $(u_t(\cdot,t), v_t(\cdot,t)) \in BC^{\alpha}_{loc}(\boldsymbol{R}^n)$ for $t \in (0,T)$ and $\alpha \in (0,1)$. By integrating $(u_t(\cdot,t), v_t(\cdot,t))$ from 0 to T and subtracting $(u_0(\cdot), v_0(\cdot))$, we obtain $(u(\cdot,T), v(\cdot,T))$. Thus we see that $(u(\cdot,T), v(\cdot,T))$ exists. □

Proof of Theorem 3. Let $\varepsilon \in (0, \min\{M-b_1, N-b_2\})$. We consider the following elements of initial data

$$(26)\quad u_0^{m,\varepsilon}(x) = \begin{cases} M-\varepsilon, & |x-a_m| < r_m - 1, \\ -(M-\varepsilon-b_1)(|x-a_m|-r_m)+b_1, & r_m - 1 \le |x-a_m| < r_m, \\ b_1, & |x-a_m| \ge r_m, \end{cases}$$

$$(27)\quad v_0^{m,\varepsilon}(x) = \begin{cases} N-\varepsilon, & |x-a_m| < r_m - 1, \\ -(N-\varepsilon-b_2)(|x-a_m|-r_m)+b_2, & r_m - 1 \le |x-a_m| < r_m, \\ b_2, & |x-a_m| \ge r_m, \end{cases}$$

where b_1, b_2 are the same as in Theorem 1. Let $(u^{m,\varepsilon}, v^{m,\varepsilon})$ be the solutions of the equation (1) with the initial data $(u_0^{m,\varepsilon}, v_0^{m,\varepsilon})$. Let $(U^\varepsilon, V^\varepsilon)$ be the solutions of (3) with initial value $(M-\varepsilon, N-\varepsilon)$. We shall write $T^\varepsilon = T(M-\varepsilon, N-\varepsilon)$ for simplicity.

By Proposition 2.1 for any $\varepsilon > 0$ there exists a natural number $m_0 \in \boldsymbol{N}$ such that for any $m > m_0$,

$$(28)\qquad U^\varepsilon(t) - \varepsilon \le u^{m,\varepsilon}(x,t), \qquad V^\varepsilon(t) - \varepsilon \le v^{m,\varepsilon}(x,t)$$

for $x \in B(a_m, r_m/2)$, $t \in (0, T^\varepsilon)$. By the comparison principle, due also to (6) we could find $m_0 \in \boldsymbol{N}$ such that

$$(29)\qquad u^{m,\varepsilon}(x,t) \le u(x,t), \qquad v^{m,\varepsilon}(x,t) \le v(x,t)$$

for any $x \in \boldsymbol{R}^n$, $t \in (0, T^\varepsilon)$ provided that $m \ge m_0$. By using the fact $T = T(M,N) < T^\varepsilon$, Lemma 4.1, (28) and (29), for any ε there exist an index still denoted by $m_0 = m_0(\varepsilon) \in \boldsymbol{N}$ such that for any $m > m_0$

$$(30)\qquad U^\varepsilon(T) - \varepsilon \le u(a_m, T), \qquad V^\varepsilon(T) - \varepsilon \le v(a_m, T) \qquad \text{for } m \ge m_0.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small and $\lim_{\varepsilon\to 0}(U^\varepsilon(T) + V^\varepsilon(T)) = \infty$, (30) implies

$$\lim_{m\to\infty}(u(a_m, T) + v(a_m, T)) = \infty. \qquad \square$$

5. Examples of nonlinearities

In this section we exhibit several examples of nonlinearities that satisfy the conditions of the theorems in the previous sections. More precisely, suppose that there exist nonnegative functions $f_1, g_1 \in C^2([b_1, \infty))$, $f_2, g_2 \in C^2([b_2, \infty))$ with b_1, b_2 used in Theorem 1 such that

$$(31)\qquad f(u,v) = f_1(u)f_2(v), \qquad g(u,v) = g_1(u)g_2(v),$$

$$(32)\qquad f_2'(v) > 0 \quad \text{and} \quad g_1'(u) > 0 \qquad \text{for } u > b_1,\ v > b_2.$$

Then the corresponding system of ordinary differential equations (3) becomes

$$\left\{\begin{array}{ll} U_t = f_1(U)f_2(V), & t > 0, \\ V_t = g_1(U)g_2(V), & t > 0, \\ U(0) = M,\ V(0) = N. \end{array}\right. \tag{33}$$

Define

$$E(t) = E(U,V)(t) = F(U(t)) - G(V(t)), \tag{34}$$

where

$$F(w) = \int_M^w \frac{g_1(s)}{f_1(s)}\,ds, \qquad G(w) = \int_N^w \frac{f_2(s)}{g_2(s)}\,ds.$$

We have that $E(t) = E(U(t), V(t))$ is independent for t, since, by using (33),

$$\begin{aligned} E'(t) &= F'(U(t))U'(t) - G'(V(t))V'(t) \\ &= \frac{g_1(U(t))}{f_1(U(t))} f_1(U(t))f_2(V(t)) - \frac{f_2(V(t))}{g_2(V(t))} g_1(V(t))g_2(V(t)) \\ &= g_1(U(t))f_2(V(t)) - f_2(V(t))g_1(U(t)) = 0. \end{aligned}$$

Thus there exists a constant $E_0 \in \boldsymbol{R}$ such that

$$F(U(t)) - G(V(t)) = F(M) - G(N) = E_0$$

with $t \in [0, T(M,N))$. Next we define the following two functions:

$$\Phi(\xi) = f_1(\xi)f_2(G^{-1}(F(\xi) - E_0)), \qquad \Gamma(\xi) = g_1(F^{-1}(E_0 + G(\xi)))g_2(\xi). \tag{35}$$

The main proposition of this section is the following.

Proposition 5.1. *Assume the same hypotheses as in Theorem* 1. *Let f, g have the forms of* (31) *and* (32). *Let $(\Phi(\xi))^\gamma$ and $(\Gamma(\xi))^\gamma$ are convex functions for some $\gamma \in (0,1)$ and any $\xi > \xi_1$ with some $\xi_1 > 0$, and satisfy*

$$\int_{\xi_1}^\infty \frac{d\xi}{\Phi(\xi)} < \infty, \qquad \int_{\xi_1}^\infty \frac{d\xi}{\Gamma(\xi)} < \infty. \tag{36}$$

Then the condition (8) *in Theorems* 2 *and* 3 *holds.*

Before proving this proposition, we need a lemma.

Lemma 5.2. *Let $U \in C([0,T)) \cap C^2(0,T)$ be a solution to*

$$\dot{U} = \Phi(U), \qquad U(0) = M \qquad \text{with } \Phi \in C^2(b,\infty), \tag{37}$$

where $T > 0$, $b \geq 0$, $\Phi(s) > 0$ *for* $s > b$ *and* $M > b$ *satisfy* $\int_M^\infty ds/\Phi(s) \in [T, \infty)$. *Assume that* $\Phi^\gamma(\xi)$ *is convex for* $\xi > \xi_0$ *with some* $\xi_0 > b$ *and some* $\gamma \in (0,1)$. *Then*

$$\limsup_{t\to T}\left|\frac{(T-t)\ddot{U}(t)}{\dot{U}(t)}\right| < \infty.$$

Proof. Define $\varphi(s) = U(T-s)$ with $s \in (0, T]$. Note that $\dot{\varphi}(s) = -\Phi(\varphi(s))$. Since Φ^γ is convex, we have

$$(\Phi^\gamma)'' = \gamma\Phi^{\gamma-2}\{\Phi\Phi'' - (1-\gamma)(\Phi')^2\} \geq 0.$$

Put $\delta = 1 - \gamma$. Then, we obtain

$$\Phi\Phi'' \geq \delta(\Phi')^2. \tag{38}$$

From (37) we have

$$s = \lim_{\eta\to+0}\int_{\varphi(s)}^{\varphi(\eta)} \frac{d\xi}{\Phi(\xi)}.$$

Since (38) implies

$$\frac{1}{\Phi} \leq \frac{\Phi''}{\delta(\Phi')^2},$$

derive

$$s \leq \lim_{\eta\to+0}\int_{\varphi(s)}^{\varphi(\eta)} \frac{\Phi''(\xi)}{\delta(\Phi'(\xi))^2}\,d\xi = \lim_{\eta\to+0}\int_{\varphi(s)}^{\varphi(\eta)} \frac{1}{\delta}\left(-\frac{1}{\Phi'(\xi)}\right)' d\xi$$
$$= \lim_{\eta\to+0}\frac{1}{\delta}\left\{\frac{1}{\Phi'(\varphi(s))} - \frac{1}{\Phi'(\varphi(\eta))}\right\} = \frac{1}{\delta}\,\frac{1}{\Phi'(\varphi(s))}.$$

On the other hand, we have

$$\frac{\ddot{\varphi}}{\dot{\varphi}} = \frac{-\Phi'(\varphi)\dot{\varphi}}{\dot{\varphi}} = -\Phi'(\varphi).$$

Then we obtain

$$\left|\frac{\ddot{\varphi}(s)}{\dot{\varphi}(s)}\right| = -\frac{\ddot{\varphi}(s)}{\dot{\varphi}(s)} = \Phi'(\varphi(s)) \leq \frac{1}{\delta s}$$

and

$$\left|\frac{s\ddot{\varphi}(s)}{\dot{\varphi}(s)}\right| \leq \frac{1}{\delta}. \qquad \square$$

Proof of Proposition 5.1. From (34) we have

$$F(U) = E_0 + G(V), \qquad G(V) = F(U) - E_0$$

and

$$U = F^{-1}(E_0 + G(V)), \qquad V = G^{-1}(F(U) - E_0). \tag{39}$$

Substituting (39) into (33), we obtain

$$U_t = f_1(U) f_2(G^{-1}(F(U) - E_0)) = \Phi(U),$$

$$V_t = g_1(F^{-1}(E_0 + G(V))) g_2(V) = \Gamma(V).$$

Since Φ^γ and Γ^γ are convex with some $\gamma \in (0,1)$, (8) holds by Lemma 5.2. □

Proposition 5.3. *Assume the same hypotheses as in Theorem* 1. *Let* $f(u,v) = u^{p_1} v^{q_1}$ *and* $g(u,v) = u^{p_2} v^{q_2}$ *with* $p_1, p_2, q_1, q_2 \geq 0$, $p_1 + q_1 > 1$ *and* $p_2 + q_2 > 1$. *Let* $b_1, b_2 > 0$. *Furthermore, suppose that one of the following four conditions hold*;
(i) $p_2 - p_1 + 1 > 0$ *and* $q_1 - q_2 + 1 > 0$,
(ii) $p_2 - p_1 + 1 = 0$ *and* $q_1 - q_2 + 1 = 0$,
(iii) $p_2 - p_1 + 1 > 0$, $q_1 - q_2 + 1 = 0$ *and* $p_1 > 0$,
(iv) $p_2 - p_1 + 1 = 0$, $q_1 - q_2 + 1 > 0$ *and* $q_2 > 0$.
Then Theorems 2 *and* 3 *hold.*

Remark 5.4. In Proposition 5.3, from (4) and (5), if $b_1 > 0$ (or $b_2 > 0$), then p_1, p_2 (or q_1, q_2) may be taken $0 \leq p_1 < 1$ and/or $0 \leq p_2 < 1$ (or $0 \leq q_1 < 1$ and/or $0 \leq q_2 < 1$). On the other hand, if $p_1, p_2 \geq 1$ (or $q_1, q_2 \geq 1$), then it is possible that $b_1 = 0$ (or $b_2 = 0$).

Proof of Proposition 5.3. We consider the problem:

$$\begin{cases} U_t = U^{p_1} V^{q_1}, & t > 0, \\ V_t = U^{p_2} V^{q_2}, & t > 0, \\ U(0) = M > 0, \ V(0) = N > 0, \end{cases} \tag{40}$$

which is the associated ordinary differential equation (3) in this case. Since $p_1 + q_1 > 1$ and $p_2 + q_2 > 1$, we can easily check that $T(M,N) < \infty$ and (7) holds.

Next we shall check validity of the condition (8) by using proposition 5.1. In this example (34) becomes

$$E(U,V)(t) = h_{p_2 - p_1 + 1}(U(t)) - h_{q_1 - q_2 + 1}(V(t)),$$

where

$$h_p(s) = \begin{cases} s^p/p, & p \neq 0, \\ \log s, & p = 0. \end{cases}$$

Thus from (40) we have $dE/dt = 0$. This implies that there exists a constant E_0 such that

$$E_0 = E(U, V) = E(M, N).$$

First we consider the case (i). In this case we have

$$\Phi(U) = U^{p_1}\left\{\left(\frac{U^{p_2-p_1+1}}{p_2-p_1+1} - E_0\right)(q_1-q_2+1)\right\}^{q_1/(q_1-q_2+1)}$$
$$\sim U^{p_1+q_1(p_2-p_1+1)/(q_1-q_2+1)} \qquad \text{as } U \to \infty$$

and

$$\Gamma(V) = \left\{\left(E_0 + \frac{V^{q_1-q_2+1}}{q_1-q_2+1}\right)(p_2-p_1+1)\right\}^{p_2/(p_2-p_1+1)} V^{q_2}$$
$$\sim V^{p_2(q_1-q_2+1)/(p_2-p_1+1)+q_2} \qquad \text{as } V \to \infty$$

with

$$E_0 = \frac{M^{p_2-p_1+1}}{p_2-p_1+1} - \frac{N^{q_1-q_2+1}}{q_1-q_2+1},$$

where "$f(s) \sim g(s)$ as $s \to \infty$" means that there exist positive constants C_1 and C_2 such that $C_1 \leq \lim_{s\to\infty} f(s)/g(s) \leq C_2$. Since $p_1 + q_1 > 1$, $p_2 + q_2 > 1$, $p_2 - p_1 + 1 > 0$ and $q_1 - q_2 + 1 > 0$, we have

$$p_1 + \frac{q_1(p_2-p_1+1)}{q_1-q_2+1} > 1 \qquad \text{and} \qquad \frac{p_2(q_1-q_2+1)}{p_2-p_1+1} + q_2 > 1,$$

i.e. Φ^γ, Γ^γ are convex functions and satisfy (36). Thus by Proposition 5.1 we obtain (8).

Next, we consider the case (ii). Under this condition,

$$\Phi(U) = U^{p_1}(e^{-E_0}U)^{q_1} \sim U^{p_1+q_1} \qquad \text{as } U \to \infty,$$
$$\Gamma(V) = (e^{E_0}V)^{p_2}V^{q_2} \sim V^{p_2+q_2} \qquad \text{as } V \to \infty$$

with

$$E_0 = \log M - \log N.$$

Since $p_1 + q_1 > 1$ and $p_2 + q_2 > 1$, we have (8) by Proposition 5.1.

Finally, we consider the cases (iii) and (iv). In the case (iii) we obtain

$$\Phi(U) = U^{p_1} e^{-q_1 E_0} \exp\left\{\frac{q_1 U^{p_2-p_1+1}}{p_2 - p_1 + 1}\right\},$$

$$\Gamma(V) = \{(E_0 + \log V)(p_2 - p_1 + 1)\}^{p_2/(p_2-p_1+1)} V^{q_2}$$

with

$$E_0 = \frac{M^{p_2-p_1+1}}{p_2 - p_1 + 1} - \log N.$$

Since $q_1 > 0$ and $q_1 - q_2 + 1 = 0$, we have $q_2 > 1$. Hence again there exist γ such that Φ^γ, Γ^γ are convex functions. Thus by Proposition 5.1 we obtain (8). For the case (iv) we can use the same argument to show the validity of (8). □

Proposition 5.5. *Under the same assumptions as in Theorem* 1, *let* $f(u,v) = e^{\alpha_1 u + \beta_1 v}$ *and* $g(u,v) = e^{\alpha_2 u + \beta_2 v}$ *with* $\alpha_1, \alpha_2, \beta_1, \beta_2 \geq 0$, $\alpha_1 + \beta_1 > 0$ *and* $\alpha_2 + \beta_2 > 0$. *Furthermore, one of the following four conditions hold*;

(i) $\alpha_1 < \alpha_2$ *and* $\beta_1 > \beta_2$,
(ii) $\alpha_1 = \alpha_2$ *and* $\beta_1 = \beta_2$,
(iii) $\alpha_1 < \alpha_2$, $\beta_1 = \beta_2$ *and* $\alpha_1 + \beta_2 > 0$,
(iv) $\alpha_1 = \alpha_2$, $\beta_1 > \beta_2$ *and* $\alpha_1 + \beta_2 > 0$.

Then Theorems 2 *and* 3 *hold.*

Proof. We consider the following system of equations:

$$\begin{cases} U_t = e^{\alpha_1 U + \beta_1 V}, & t > 0, \\ V_t = e^{\alpha_2 U + \beta_2 V}, & t > 0, \\ U(0) = M > 0, \ V(0) = N > 0, \end{cases} \tag{41}$$

which is the associated ordinary differential system (3) in this case. Since $\alpha_1 + \beta_1 > 0$ and $\alpha_2 + \beta_2 > 0$, we can easily check that $T(M,N) < \infty$ and (7) holds.

Next, we shall verify the validity of the condition (8) by using Proposition 5.1. Now (34) becomes

$$E(U,V)(t) = h_{\alpha_2-\alpha_1}(U(t)) - h_{\beta_2-\beta_1}(V(t)),$$

where

$$h_\alpha(s) = \begin{cases} e^{\alpha s}/\alpha, & \alpha \neq 0, \\ s, & \alpha = 0. \end{cases}$$

Thus from (41) we have $dE/dt = 0$ and there exists a constant E_0 such that

$$E_0 = E(U,V) = E(M,N).$$

First we consider the case (i). In this case we have

$$\Phi(U) = e^{\alpha_1 U}\left[\left(\frac{e^{(\alpha_2-\alpha_1)U}}{\alpha_2-\alpha_1} - E_0\right)(\beta_1-\beta_2)\right]^{\beta_1/(\beta_1-\beta_2)},$$

$$\Gamma(V) = \left[\left(\frac{e^{(\beta_1-\beta_2)V}}{\beta_1-\beta_2} + E_0\right)(\alpha_2-\alpha_1)\right]^{\alpha_2/(\alpha_2-\alpha_1)} e^{\beta_2 V}$$

with

$$E_0 = \frac{e^{(\alpha_2-\alpha_1)M}}{\alpha_2-\alpha_1} - \frac{e^{(\beta_1-\beta_2)N}}{\beta_1-\beta_2}.$$

Hence (36) is satisfied. By Proposition 5.1, we obtain (8).

Next, we consider the case (ii). We can easily check that

$$\Phi(U) = e^{(\alpha_1+\beta_1)U-\beta_1 E_0},$$

$$\Gamma(V) = e^{\alpha_2 E_0+(\alpha_2+\beta_2)V}$$

with

$$E_0 = M - N.$$

Thus (36) holds, and we obtain (8) by Proposition 5.1.

Finally, we consider the cases (iii) and (iv). In the case (iii) we obtain

$$\Phi(U) = \exp\left[\alpha_1 U + \frac{\beta_1 e^{(\alpha_2-\alpha_1)U}}{\alpha_2-\alpha_1} - E_0\beta_1\right],$$

$$\Gamma(V) = \{(E_0+V)(\alpha_2-\alpha_1)\}^{\alpha_2/(\alpha_2-\alpha_1)} e^{\beta_2 V}$$

with

$$E_0 = \frac{e^{(\alpha_2-\alpha_1)M}}{\alpha_2-\alpha_1} - N.$$

Since $\alpha_1 + \beta_2 > 0$ we obtain (8) by Proposition 5.1 as before. For the case (iv) we can obtain (8) by using the same argument. □

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nuna adreso:
Masahiko Shimojo
Department of Mathematics
National Taiwan Normal University
88, Section 4, Ting Chou Road
Taipei 11677
R. O. China
E-mail: shimojotw@gmail.com

Noriaki Umeda
Graduate School of Mathematical Sciences
The University of Tokyo
3-8-1, Komaba, Meguro-ku
Tokyo, 153-8914
Japan
E-mail: dor@dh.mbn.or.jp

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