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Uniqueness of Solutions for Zakharov Systems

By

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Abstract. We prove that the weak solution of the Cauchy problem for the Klein-Gordon-Zakharov system and for the Zakharov system is unique in the energy space for the former system, and in some larger space for the latter system, in dimensions three or lower. In the three dimensional case, these are the largest Sobolev spaces where the local wellposedness has been proven so far. Our proof uses infinite iteration, where the solution is fixed but the function spaces are converging to the desired ones in the limit.

Key Words and Phrases. Zakharov system, Unconditional uniqueness.

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1. Introduction

In solving nonlinear dispersive equations, various space-time norms play central roles to exploit oscillatory and dispersive nature of the solutions. It becomes more essential when one treats more singular problems or with less regularity for the solutions. Consequently, many wellposedness results only apply to those solutions with some additional properties in space-time norms, besides the time continuity in the initial data space. In other words, the uniqueness is proved only in such function spaces. For instance, for the energy critical wave equation, the solution for the Cauchy problem is unique if the energy and some appropriate space-time L^p norm are finite. But it is not known whether the solution is unique simply in the energy space in the 3D case (see [24, 18]), whereas the uniqueness holds in the energy space in the 4D case [22].

As far as one investigates solutions for a fixed equation, such restrictions for the uniqueness do not usually cause any problem, because one can always regard the solutions as limits of smooth ones, for which one can prove the uniqueness in classical ways. However, when studying relations between different equations and their solutions, for example through some limits or nonlinear transforms, the additional conditions often become bothering, since it is not clear if or how the additional conditions for one equation transfer to another.

On the other hand, uniqueness without any additional condition is very useful for limit problems between equations. For example, if some conserved quantity (such as energy) converges along the limit, and if we have the unconditional uniqueness for the limit equation in the space for that quantity, then one can derive convergence of solutions from that of initial data, without any further information on the solutions before the limit (cf. [14, 17]).

We remark that one should not always expect unconditional uniqueness even if the (conditional) local wellposedness holds. In fact, Christ [5] showed that the unconditional uniqueness breaks down for the nonlinear Schrödinger equation with quadratic nonlinearity u^2 or $\bar{u}^2$ in the Sobolev space $H^s(\boldsymbol{R}/\boldsymbol{Z})$ with $-1/2 < s < 0$, despite of the local wellposedness. Thus in general proving unconditional uniqueness requires some non-trivial extra work.

In this paper, we prove unconditional uniqueness of weak solutions for the Klein-Gordon-Zakharov system (KGZ)

$$\ddot{E} - \Delta E + E = -nE, \qquad E : \boldsymbol{R}^{1+d} \to \boldsymbol{C}^K, \tag{1.1}$$
$$\alpha^{-2}\ddot{n} - \Delta n = \Delta|E|^2, \qquad n : \boldsymbol{R}^{1+d} \to \boldsymbol{R},$$

and the Zakharov system (Z)

$$2i\dot{u} - \Delta u = -nu, \qquad u : \boldsymbol{R}^{1+d} \to \boldsymbol{C}^K, \tag{1.2}$$
$$\alpha^{-2}\ddot{n} - \Delta n = \Delta|u|^2, \qquad n : \boldsymbol{R}^{1+d} \to \boldsymbol{R},$$

where $\alpha > 0$ is a fixed constant, $d = 1, 2, 3$, and $K \in \boldsymbol{N}$ is arbitrary. (KGZ) changes its property depending on the ratio between the propagation speeds of E and n. Here we consider the physically natural case $\alpha < 1$ only. For (Z), we may assume $\alpha < 1$ for simplicity, but without loss of generality, by rescaling

$$(u, n) \mapsto (\lambda u(\lambda^2 t, \lambda x), \lambda^2 n(\lambda^2 t, \lambda x)), \tag{1.3}$$

which changes only α in the equation.

(KGZ) and (Z) are model systems to describe nonlinear interactions in plasma, where E or u denotes the electric field (more precisely it denotes its slowly varying envelope), n denotes the ion density fluctuation, and α is the ion sound speed.

More precisely, these systems can be derived from a coupled Maxwell-Euler system for the electro-magnetic field and two fluids consisting of electrons and ions (cf. [6, 26, 29]). On the other hand, (Z) converges to the nonlinear Schrödinger equation in the limit $\alpha \to \infty$ (cf. [23, 1, 21, 12, 17]):

$$2i\dot{u} - \Delta u = |u|^2 u, \qquad n = -|u|^2. \tag{1.4}$$

It is usual for nonlinear dispersive equations that they are linked to others by various limits, and thus our motivation for proving unconditional uniqueness originates from there.

In order to state our uniqueness result, let us be more precise about the notion of weak solutions.

Definition 1.1. Let $I \subset \boldsymbol{R}$ be an interval and $E, n, u \in C_w(I; L^2(\boldsymbol{R}^d))$, where C_w denotes the space of weakly continuous functions. We say that (E, n) is a weak solution of (1.1) on I if the equations are satisfied in $C(I'; \mathscr{S}'(\boldsymbol{R}^d))$, where I' denotes the interior of I. Similarly, we say that (u, n) is a weak solution of (1.2) on I if the equations are satisfied in $C(I'; \mathscr{S}'(\boldsymbol{R}^d))$.

In the above definition, note that we have $\dot{E}, \dot{n} \in C(I; \mathscr{S}')$ by integrating the equations for $\ddot{E}$ and $\ddot{n}$. The L_x^2 assumption is to make sense of the nonlinearity.

Now we can state our main results.

Theorem 1.2. *Let $I \subset \boldsymbol{R}$ be an interval and $t_0 \in I$. Assume $\alpha < 1$. Then every weak solution (E, n) of* (1.1) *on I satisfying*

$$(E, \dot{E}, n) \in C_w(I; H^1(\boldsymbol{R}^d) \times L^2(\boldsymbol{R}^d) \times L^2(\boldsymbol{R}^d)) \tag{1.5}$$

is uniquely determined by $(E(t_0), \dot{E}(t_0), n(t_0), \dot{n}(t_0))$.

Theorem 1.3. *Let $I \subset \boldsymbol{R}$ be an interval and $t_0 \in I$. Then every weak solution (u, n) of* (1.2) *on I satisfying*

$$(u, n) \in C_w(I; H^{1/2}(\boldsymbol{R}^d) \times L^2(\boldsymbol{R}^d)) \tag{1.6}$$

is uniquely determined by $(u(t_0), n(t_0), \dot{n}(t_0))$.

Remark that we need no extra condition on $\dot{n}$. Actually, we will prove that every weak solution has a space-time norm that ensures the uniqueness criteria in [9, 20]. For that purpose we use the iteration by the integral equation, where the solution is fixed but the function spaces are improved in each step, and we get the desired space-time estimate in the limit of the iteration. This argument seems essential at least in the Zakharov case, where the space regularity cannot be improved by the bilinear estimate if we start with anything below $H^{1/2}$.

This criticality is reflected by its limit equation as $\alpha \to \infty$, for which $\dot{H}^{1/2}$ is the scaling invariant space, and the unconditional uniqueness is an open question due to the failure of the Sobolev embedding $H_6^{1/2}(\boldsymbol{R}^3) \not\subset L^\infty(\boldsymbol{R}^3)$. (See [10, 7, 27] for the uniqueness of NLS in other cases.)

In the above sense, we might say that the Zakharov system is better behaving than the NLS, which is surprising because one usually does not expect better regularity for the wave equation (for n) than the Laplace equation (in the limit $\alpha \to \infty$). The trick comes from the assumption that n is bounded in L_x^2, which is not available in the limit equation (1.4) with $u \in H_x^{1/2}$.

We also point out that there are many works dealing with unconditional uniqueness. We can mention [7, 10, 22, 15, 18, 27, 25, 31, 32] for works related to nonlinear wave equations, [8, 13, 19, 28] for works related to the parabolic case and more precisely to the Navier-Stokes system.

2. Preliminaries

In this section we set up the equations, the function spaces and basic estimates and notations.

2.1. Function spaces and frequency localization

We denote the Lebesgue and the Besov spaces by L^p and $B^s_{p,q}$, respectively, and the L^2 Sobolev space by $H^s := B^s_{2,2}$, where $s \in \boldsymbol{R}$ and $1 \le p, q \le \infty$. For convenience of the Hölder inequality, we also define

$$L_{1/p} := L^p, \qquad B^s_{1/p,1/q} := B^s_{p,q}, \qquad H^s_{1/p} := H^s_p, \tag{2.1}$$

for $1 \le p, q \le \infty$. This does not cause any contradiction because we never use the usual L^p spaces with $p < 1$. For any Banach function space V on $\boldsymbol{R}^d$, we denote the mixed norm by

$$\|u\|_{L_{1/p}V} := \|u\|_{L^pV} := \| \|u(t)\|_V \|_{L^p_t}, \tag{2.2}$$

for $1 \le p \le \infty$.

We denote the Fourier transform on $\boldsymbol{R}^d$ and $\boldsymbol{R}^{1+d}$ by $\mathscr{F}_x$ and $\mathscr{F}_{t,x}$, respectively. For any function $\varphi : \boldsymbol{R}^d \to \boldsymbol{R}$, we define the associated Fourier multiplier by $\varphi(-i\nabla) := \mathscr{F}_x^{-1}\varphi(\xi)\mathscr{F}_x$.

Let $\chi \in C_0^\infty(\boldsymbol{R})$ be a fixed cut-off function satisfying $\chi(\xi) = 1$ for $|\xi| \le 1$ and $\chi(\xi) = 0$ for $|\xi| \ge 2$. For any function $\varphi : \boldsymbol{R}^{1+d} \to \boldsymbol{R}$ and $a, b > 0$, we denote

$$P_{\varphi(\tau,\xi)\le a}u := \begin{cases} \mathscr{F}_{t,x}^{-1}\chi(\varphi(\tau,\xi)/a))\mathscr{F}_{t,x}u, & (a > 1/2), \\ 0, & (a \le 1/2), \end{cases} \tag{2.3}$$

$$P_{\varphi(\tau,\xi)>a}u := u - P_{\varphi(\tau,\xi)\le a}u,$$

$$P_{a<\varphi(\tau,\xi)\le b}u := P_{\varphi(\tau,\xi)>a}P_{\varphi(\tau,\xi)\le b}u.$$

Thus we have the Littlewood-Paley decomposition and the Besov norms

$$u = \sum_{j\in 2^N} P_{j/2<|\xi|\le j}u, \qquad \|u\|_{B^s_{p,q}} = \|j^s P_{j/2<|\xi|\le j}u\|_{\ell^q(2^N;L^p(\boldsymbol{R}^d))}. \tag{2.4}$$

We denote the L^2 coupling on $\boldsymbol{R}^d$ and $\boldsymbol{R}^{1+d}$ respectively by

(2.5) $\langle \varphi | \psi \rangle_x := \Re \int_{\mathbf{R}^d} \varphi(x)\overline{\psi(x)}dx, \qquad \langle u|v \rangle_{t,x} := \Re \int_{\mathbf{R}^{1+d}} u(t,x)\overline{v(t,x)}dtdx.$

2.2. Strichartz estimates

We denote the space-time integrability given by the Strichartz estimate for H^s solutions on finite time intervals to the free Schrödinger equation by $St_S(s)$ and to the free Klein-Gordon equation by $St_K^\varepsilon(s)$:

$$St_S(s) := \bigcap \{ L_t^p B_{q,2}^s(\mathbf{R}^3) \mid 0 \le 1/p \le 1/2, 2/p + 3/q = 3/2 \}, \tag{2.6}$$

$$St_K^\varepsilon(s) := \bigcap \{ L_t^p B_{q,2}^{s-2/p}(\mathbf{R}^3) \mid 0 \le 1/p \le 1/2 - \varepsilon, 1/p + 1/q = 1/2 \},$$

for any $\varepsilon > 0$. The norm is naturally defined by the sup over all the spaces such as

$$\|u\|_{St_S(s)} = \sup_{(p,q) \text{ admissible}} \|u\|_{L^p B_{q,2}^s}. \tag{2.7}$$

We can also define the dual of these spaces

$$St_S(s)^* := \sum \{ L_t^p B_{q,2}^s(\mathbf{R}^3) \mid 1/2 \le 1/p \le 1, 2/p + 3/q = 7/2 \}, \tag{2.8}$$

$$St_K^\varepsilon(s)^* := \sum \{ L_t^p B_{q,2}^{s+1-2/p}(\mathbf{R}^3) \mid 1/2 + \varepsilon \le 1/p \le 1, 1/p + 1/q = 3/2 \},$$

where the norm is naturally defined by the inf of all possible decomposition into those spaces, such as

$$\|f\|_{St_S(s)^*} = \inf_{u = \sum_j u_j} \sum_j \|u_j\|_{L^{p_j} B_{q_j,2}^s}. \tag{2.9}$$

The Strichartz estimate can be written as

$$\|u\|_{St_S(s)} \lesssim \|u(0)\|_{H^s} + \|f\|_{St_S(s)^*}, \tag{2.10}$$

$$\|v\|_{St_K^\varepsilon(s)} \lesssim \|v(0)\|_{H^s} + \|\dot{v}(0)\|_{H^{s-1}} + \|g\|_{St_K^\varepsilon(s)^*},$$

for the solutions u and v of

$$2i\dot{u} - \Delta u = f, \qquad \ddot{v} - \Delta v + v = g. \tag{2.11}$$

2.3. $X^{s,b}$ spaces and extension of local weak solutions

For any self-adjoint operator H on $L^2(\mathbf{R}^d)$, we introduce the $X^{s,b}$ spaces associated to the equation $i\dot{u} + Hu = 0$:

(2.12) $X_H^{s,b} := \{e^{itH}v \mid v \in H_t^b(\boldsymbol{R}; H_x^s(\boldsymbol{R}^d))\}, \qquad \|u\|_{X_H^{s,b}} := \|e^{-itH}u\|_{H_t^b H_x^s}.$

When $H = \omega(-i\nabla)$ is a Fourier multiplier, we can write

$$\|u\|_{X^{s,b}_{\omega(-i\nabla)}} = \|\langle \tau - \omega(\xi)\rangle^b \langle \xi \rangle^s (\mathscr{F}_{t,x}u)(\tau,\xi)\|_{L^2_{\tau,\xi}}. \tag{2.13}$$

For the second order equation of the form $\ddot{u} + H^2 u = 0$, we also define

$$X_{\pm H}^{s,b} := X_H^{s,b} + X_{-H}^{s,b}. \tag{2.14}$$

For any open interval $I \subset \boldsymbol{R}$, we define $X_*^{s,b}(I)$ as the restriction (in the sense of distributions) of $X_*^{s,b}$ onto I. However we should be careful about the precise definition of norms, because our proof will crucially depend on uniformness on the interpolation inequalities for $X^{s,b}$ and the Strichartz estimate. For example, the standard definition by the quotient norm is not appropriate for our purpose, since then the interpolation inequalities would lose uniformness.

On the other hand, we should not use the smooth time cut-off argument as usual in the wellposedness proof, because a weak solution of the original integral equation does not directly give rise to any solution for that with smooth cut-off. Even with the wellposedness result, it is not clear at all whether one can converge to a solution for the latter equation by iteration starting from the given weak solution, because the contraction requires the same (or even stronger) conditions as in the uniqueness for the starting function.

Therefore we will use instead an explicit extension operator which gives

(i) uniform embeddings and interpolations,

(ii) weak solution for the extended integral equation,

and will estimate only in $X^{s,b}$ spaces on $\boldsymbol{R}$, not on finite intervals. For simplicity, we will consider only the solutions on $[0,T]$ with arbitrary $T \in (0,1)$. We extend functions on $[0,T]$ to those on $\boldsymbol{R}$ right before the integration by the following operator $\mathscr{E}_T$:

$$(\mathscr{E}_T f)(t) = \begin{cases} f(t) & (0 \le t \le T), \\ -f(2T-t) & (T < t \le 2T), \\ 0 & (t < 0, 2T < t) \end{cases} \tag{2.15}$$

Then an extended integration from functions on $[0,T]$ to those on $\boldsymbol{R}$ is defined by composition of $\mathscr{E}_T$ and integration from 0:

$$(\mathscr{I}_T f)(t) := \int_0^t (\mathscr{E}_T f)(s)ds = \int_{\substack{0<s<t \\ 2T-t<s<T}} f(s)ds. \tag{2.16}$$

Notice that $\mathscr{E}_T f$ is in general discontinuous at $t = 0, T, 2T$. Discontinuity at T seems inevitable in order to satisfy both of the above requirements, and anyway the discontinuity at 0 is necessary to solve the initial value problem.

For each integral equation of the form

$$u(t) = e^{itH}\varphi + \int_0^t e^{i(t-s)H} f(s)ds, \tag{2.17}$$

on the functions u and f defined on $[0, T]$, we associate the following extended integral equation

$$\hat{u}(t) = \chi(t)e^{itH}\varphi + e^{itH}\mathscr{I}_T[e^{-itH}f(t)], \tag{2.18}$$

where $\chi \in C_0^\infty(\boldsymbol{R})$ is a fixed cut-off function satisfying $\chi(t) = 1$ for $|t| < 4$.

It is clear from the definition that $\hat{u}(t) = u(t)$ for $0 \le t \le T$. Also it is easy to prove

$$\|e^{itH}\mathscr{I}_T[e^{-itH}f(t)]\|_{X_H^{s,b+\theta}} \lesssim T^{1-\theta}\|f\|_{X_H^{s,b}}, \tag{2.19}$$

for $|b| < 1/2$, $\theta \in [0, 1]$ and $f \in X_H^{s,b}$, where the implicit constant is uniform for f, s, H and θ, but not for $|b| \to 1/2$. The threshold $|b| = 1/2$ is due to the discontinuity of $\mathscr{E}_T$.

Thus for each weak solution $u = (u_{(1)}, \dots u_{(N)}) \in C(I; \mathscr{S}'(\boldsymbol{R}^d))^N$ for any system of nonlinear integral equations of the form

$$u_{(j)}(t) = e^{itH_j(\mathscr{D})}\varphi_{(j)} + \int_0^t e^{i(t-s)H_j(\mathscr{D})}F_{(j)}(u(s))ds, \qquad (j = 1, 2, \dots N) \tag{2.20}$$

on $[0, T]$, we associate the extension $\hat{u}$ given by

$$\hat{u}_{(j)}(t) = \chi(t)e^{itH_j(\mathscr{D})}\varphi_{(j)} + e^{itH_j(\mathscr{D})}\mathscr{I}_T[e^{-itH_j(\mathscr{D})}F_{(j)}(u)]. \tag{2.21}$$

Then we have $\hat{u}(t) = u(t)$ for $0 \le t \le T$ and moreover $\hat{u}$ is the solution of

$$\hat{u}_{(j)}(t) = \chi(t)e^{itH_j(\mathscr{D})}\varphi_{(j)} + e^{itH_j(\mathscr{D})}\mathscr{I}_T[e^{-itH_j(\mathscr{D})}F_{(j)}(\hat{u})]. \tag{2.22}$$

The second order equations are also extended in the above way, after decomposing $\cos(Ht)$ and $\sin(Ht)$ into $e^{\pm itH}$.

For simplicity, we will deliberately confuse the local solution u and the extended solution $\hat{u}$, and also the free part u^0 with the first term in the above, and the nonlinear part u^1 with the second term.

2.4. Interpolation and embeddings for $X^{s,b}$ with the Strichartz estimate

Interpolation of $X^{s,b}$ spaces and embeddings between the Strichartz estimate have been proved to be very useful already in the case of Maxwell-Klein-Gordon and Maxwell-Dirac systems [15]. Here we state the general interpolation and embeddings in their sharp forms, just for the sake of simple numerology in their use. In fact, the non-sharp version used in [15] would be sufficient for our purpose. We denote by $(\cdot, \cdot)_{\theta,r}$ the real interpolation functor.

Lemma 2.1. *Let H be self-adjoint on L^2, and $V \subset \mathscr{S}'(\boldsymbol{R}^d)$ be a Banach space containing $\mathscr{S}(\boldsymbol{R}^d)$ as a dense subset. Assume that we have*

$$\|e^{itHt}\varphi\|_{V^*} \lesssim |t|^{-1}\|\varphi\|_V. \tag{2.23}$$

Then for any $b \in [0, 1/2)$, we have

$$X_H^{0,b} \subset L_t^2(L^2, V^*)_{2b,2}, \qquad L_t^2(L^2, V)_{2b,2} \subset X_H^{0,-b}, \tag{2.24}$$

The above holds actually for the homogeneous version of the $X^{0,b}$ spaces. When we have the endpoint Strichartz estimate, it follows immediately by interpolation. However, the above remains valid even if the endpoint estimate is false, which is the case for the three dimensional wave (Klein-Gordon) equation.

Proof. The proof is almost the same as that for the endpoint estimate [11]. Denote

$$V_\theta := (L^2, V)_{\theta,2}, \qquad V_\theta^* := (L^2, V^*)_{\theta,2}, \tag{2.25}$$

with the convention $V_0 = V_0^* = L^2$, $V_1 = V$, and $V_1^* = V^*$. By the duality, it suffices to prove the latter embedding, which will follow from

$$\|e^{-itH}f(t)\|_{\dot{H}_t^{-b}L_x^2} = \|\,|\partial_t|^{-b}e^{-itH}f(t)\|_{L_{t,x}^2} \lesssim \|f\|_{L^2(V_{2b})}. \tag{2.26}$$

Expanding the square, the last inequality is equivalent to

$$\iint |t-s|^{2b-1}\langle e^{-itH}f(t) \,|\, e^{-isH}f(s)\rangle_x dsdt \lesssim \|f\|_{L^2(V_{2b})}^2. \tag{2.27}$$

Decomposing the left hand side dyadically in $|t-s|$,

$$T_j^b(f,g) := \iint_{2^j<|t-s|<2^{j+1}} |t-s|^{2b-1}\langle e^{-itH}f(t) \,|\, e^{-isH}g(s)\rangle_x dsdt, \tag{2.28}$$

we are going to prove

$$\|T_j^b(f,g)\|_{\ell_1^0(j\in\boldsymbol{Z})} \lesssim \|f\|_{L^2(V_{2b})}\|g\|_{L^2(V_{2b})}, \tag{2.29}$$

by the bilinear real interpolation, where ℓ_p^s denotes the weighted ℓ^p space on $\boldsymbol{Z}$ with the weight 2^{js}.

By the real interpolation between the decay assumption and the trivial L^2 conservation, we have

$$\|e^{itH}\varphi\|_{V_\theta^*} \lesssim |t|^{-\theta}\|\varphi\|_{V_\theta}, \tag{2.30}$$

for $0 \le \theta \le 1$. Using the Young and the Hölder inequalities in time, we get

$$(2.31)\qquad |T_j^b(f,g)| \lesssim \iint_{|t-s|\sim 2^j} |t-s|^{2b-1-\theta}\|f(s)\|_{V_\theta}\|g(t)\|_{V_\theta}dsdt \lesssim 2^{(2b-\theta)j}\|f\|_{L^2V_\theta}\|g\|_{L^2V_\theta},$$

for $0\le\theta\le 1$.

On the other hand, we have for $0<\theta<1$,

$$(2.32)\qquad \left\|\int_{\boldsymbol{R}} e^{-itH}f(t)dt\right\|_{L^2}^2 = \iint \langle e^{i(t-s)H}f(s)\mid f(t)\rangle_x dsdt \lesssim \iint |t-s|^{-\theta}\|f(s)\|_{V_\theta}\|f(t)\|_{V_\theta}dsdt \lesssim \|f\|_{L^{2/(2-\theta)}V_\theta}^2,$$

where we used the Hardy-Littlewood-Sobolev in the last step. Applying it, we get

$$(2.33)\qquad \begin{aligned} |T_j^b(f,g)| &\lesssim \int \|\,|t-s|^{2b-1}f(s)\|_{L^{2/(2-\theta)}(|t-s|\sim 2^j;\,V_\theta)}\|g(t)\|_{L_x^2}dt \\ &\lesssim \sum_{k\in\boldsymbol{Z}}\int_{|2^{-j}t-k|\le 2} 2^{j(2b-1)}2^{j(1/2-\theta/2)}\|f(s)\|_{L^2(|2^{-j}s-k|\le 2;\,V_\theta)}\|g(t)\|_{L_x^2}dt \\ &\lesssim 2^{j(2b-1/2-\theta/2)}\sum_{k\in\boldsymbol{Z}}\|f(s)\|_{L^2(|2^{-j}s-k|\le 2;\,V_\theta)}2^{j/2}\|g(t)\|_{L^2(|2^{-j}t-k|\le 2;\,L^2)} \\ &\lesssim 2^{j(2b-\theta/2)}\|f\|_{L^2V_\theta}\|g\|_{L^2L^2}. \end{aligned}$$

Applying the bilinear real interpolation to (2.31) and (2.33), we get

$$(2.34)\qquad \|T_j^b(f,g)\|_{(\ell_\infty^{\theta/2-2b},\,\ell_\infty^{\theta-2b})_{\alpha+\beta-1,1}} \lesssim \|f\|_{(L^2L^2,\,L^2V_\theta)_{\alpha,2}}\|g\|_{(L^2L^2,\,L^2V_\theta)_{\beta,2}}$$

for $0<\theta<1$ and $0<\alpha,\beta<1<\alpha+\beta$. By the interpolation property of ℓ_p^s and L^p, together with the reiteration theorem, the above is equivalent to

$$(2.35)\qquad \|T_j(f,g)\|_{\ell_1^{(\alpha\theta+\beta\theta)/2-2b}} \lesssim \|f\|_{L^2V_{\theta\alpha}}\|g\|_{L^2V_{\theta\beta}},$$

and by choosing $\alpha=\beta$ and θ such that $\alpha\theta=2b$, we get ℓ_1^0 bound on T_j, as desired. □

Applying the above to the Schrödinger and the Klein-Gordon equations, we will use in particular the following embeddings:

$$(2.36)\qquad L_t^2H_{1/2+2b/3}^s \subset X_{-\Delta/2}^{s,-b}, \qquad L_t^2H_{1/2+b}^{s+2b} \subset X_{\pm\langle\nabla\rangle}^{s,-b}$$

for $0\le b<1/2$.

For the other direction of embedding, we will use the following interpolation:

Lemma 2.2. *Let H be a real Fourier multiplier and V be a Besov space on $\boldsymbol{R}^d$, satisfying*

$$\|e^{iHt}\varphi\|_{L_t^p V} \lesssim \|\varphi\|_{H^s} \tag{2.37}$$

for some $s \in \boldsymbol{R}$ and $p \geq 2$. Then we have

$$(X_H^{s_0,b_0}, X_H^{s_1,b_1})_{\theta,1} \subset L_t^p V. \tag{2.38}$$

if $0 < \theta < 1$, $s = (1-\theta)s_0 + \theta s_1$, $1/2 = (1-\theta)b_0 + \theta b_1$ and $b_0 \neq b_1$.

For a proof, see [17, Lemma 2.4], which was written for $p = 2$, but applies to $p > 2$ as well, by using the Minkowski inequality. Notice that unless $s_0 = s_1$ the above interpolation space is different from the Besov-type $X^{s,b}$ space.

2.5. Nonresonance property

The most important feature of the nonlinear terms in the Zakharov systems is the following nonresonance property with respect to the linear space-time oscillation, which has played the central role in those wellposedness results [4, 9, 20].

For the Klein-Gordon-Zakharov, let $\alpha \leq \gamma < 1$. Then there exists $\varepsilon \in (0, 1/10)$, depending only on γ, such that the following holds. For any functions n, u, v on $\boldsymbol{R}^{1+d}$, let $j \geq 10$, $0 < k \leq \varepsilon j$, $0 < \delta \leq \varepsilon j$, and

$$\begin{aligned} &n^C := P_{j/2<|\xi|\leq j} P_{||\tau|-\alpha|\xi||\leq\delta} n, \\ &u^C := P_{|\xi|\leq k} P_{||\tau|-\langle\xi\rangle|\leq\delta} u, \qquad v^C := P_{||\tau|-\langle\xi\rangle|\leq\delta} v. \end{aligned} \tag{2.39}$$

Then we have

$$\langle n^C u^C \mid v^C \rangle_{t,x} = 0. \tag{2.40}$$

Similarly for the Zakharov, there exists $\varepsilon \in (0, 1/10)$ such that if $j \geq 10$, $0 < k \leq \varepsilon j$, $0 < \delta \leq \varepsilon j^2$, then for

$$\begin{aligned} &n^C := P_{j/2<|\xi|\leq j} P_{||\tau|-\alpha|\xi||\leq\delta} n, \\ &u^C := P_{|\xi|\leq k} P_{|\tau-|\xi|^2/2|\leq\delta} u, \qquad v^C := P_{|\tau-|\xi|^2/2|\leq\delta} v, \end{aligned} \tag{2.41}$$

we have

$$\langle n^C u^C \mid v^C \rangle_{t,x} = 0. \tag{2.42}$$

The above property can be easily checked by writing the inner product in the space-time Fourier space. It implies that in the interaction of high-low frequencies in the bilinear term nu, either one of the functions or the product is away from the characteristic surface of the linearized equation, and thus we gain δ^{-b} by using the $X^{s,b}$ norms. We have the same gain in the term $|u|^2$, though it will be much less important in our later argument.

3. Zakharov $H^{1/2} \times L^2$

In this section we prove the unconditional uniqueness for the Zakharov system (Z). The following lemma is the main part of our proof:

Lemma 3.1. *Let $I \subset \boldsymbol{R}$ be an interval, $(u,n) \in C_w(I; H^{1/2} \times L^2)$ be a weak solution of* (Z), *$t_0 \in I$ and n^0 be the linear solution*

$$\alpha^{-2}\ddot{n}^0 - \Delta n^0 = 0, \qquad n^0(t_0) = n(t_0), \qquad \dot{n}^0(t_0) = \dot{n}(t_0). \tag{3.1}$$

Then we have $u \in X^{1/2,3/4-\varepsilon}_{-\Delta/2}(I)$ and $n - n^0 \in X^{0,3/4}_{\pm\alpha|\nabla|}(I)$ for any $\varepsilon > 0$.

Before entering the proof, we give the outline. By time translation and inversion invariance of the equation, it suffices to prove the theorem in the case

$$I = [0,T], \qquad t_0 = 0, \tag{3.2}$$

for small $T > 0$. We will iterate the integral equations for $0 < t < T$,

$$u = u^0 + u^1,\ u^1 := I_u(nu), \qquad I_u(f) := \frac{1}{2i}\int_0^t e^{i\Delta(t-s)/2} f(s)\,ds, \tag{3.3}$$

$$n = n^0 + n^1,\ n^1 := I_n(|u|^2), \qquad I_n(g) := \int_0^t \sin(|\alpha\nabla|(t-s))|\alpha\nabla| f(s)\,ds,$$

where u^0 and n^0 are the free solutions with the same initial data as u and n. More precisely, we extend the weak solution on $[0,T]$ to $\boldsymbol{R}$ by the procedure in (2.21), and iterate the extended integral equation of the form (2.22). For simplicity, we do not distinguish the extended solutions and the original local solutions.

First we prove that $u \in St_S(1/2) \cap X^{1/2,3/4-\varepsilon}_{-\Delta/2}$ for any $\varepsilon > 0$, by iterating the integral equation infinite times and taking the limit. More precisely, we prove

(i) $u \in St_S(0)$.

(ii) We have a sequence $0 = \sigma_0 < \sigma_1 < \sigma_2 < \cdots \to 1/2$ such that

$$\|u^1\|_{St_S(\sigma_j)} \le CT^a(1 + \|u^1\|_{St_S(\sigma_{j-1})})^b \qquad (j \in \boldsymbol{N}), \tag{3.4}$$

for some constants $C, a, b > 0$ which do not depend on j.

Then by induction we get $u^1 \in St_S(\sigma)$ for any $\sigma < 1/2$, and moreover the norm is uniformly bounded if $T > 0$ is sufficiently small. Hence we can take the limit

$\sigma \to 1/2 - 0$, deducing the same bound in $St_S(1/2)$ by the monotone convergence theorem. $X^{1/2,3/4-\varepsilon}_{-\Delta/2}$ will be obtained as a byproduct in this procedure. The ε loss is due to the failure of the Sobolev embedding $H^{1/2}_6 \subset L^\infty$, which naturally appears if one uses the endpoint Strichartz estimate. We should avoid it because it would break the uniformness in the above argument. The estimate for n comes in the end and with only one iteration.

Proof. First we prove $u \in St_S(\sigma)$ for $0 \le \sigma < 1/2$ by induction on σ. The bound should be uniform so that we can take the limit $\sigma \to 1/2 - 0$. We fix $\varepsilon \in (0, 1/9)$, which is a parameter to avoid the endpoint Strichartz, which would cause a loss of uniformness due to $H^{1/2}_6(\boldsymbol{R}^3) \not\subset L^\infty(\boldsymbol{R}^3)$. So the iterative estimates will depend on ε and α, as well as the $H^{1/2} \times L^2$ bound.

For the first step of induction, we have by the Strichartz

$$\|u^1\|_{St_S(0)} \lesssim \|nu\|_{L^2L^{6/5}} \lesssim T^{1/2}\|n\|_{L^\infty L^2}\|u\|_{L^\infty H^{1/2}} < \infty, \tag{3.5}$$

where we used the Sobolev embedding $H^{1/2} \subset L^3$.

Now we assume $u \in St_S(\sigma)$ with $0 \le \sigma < 1/2$ to get the same bound for a larger $\sigma' < 1/2$. We will assume that $0 < T < 1$. The following estimates are uniform in $\sigma \in [0, 1/2]$, as long as $\varepsilon \in (0, 1/9)$ is fixed.

We get $X^{s,b}$ type bounds with $1/2$ loss of derivatives in x. For u we have

$$\|u^1\|_{X^{\sigma-1/2,1-\varepsilon/2}_{-\Delta/2}} \lesssim T^{\varepsilon/2}\|nu\|_{L_{1/2-\varepsilon/2}H^{\sigma-1/2}_{1/2+\varepsilon/3}} \lesssim \|n\|_{L^\infty L^2}\|u\|_{L_{1/2-\varepsilon/2}H^{\sigma}_{1/6+\varepsilon/3}}, \tag{3.6}$$

where we used the Sobolev embeddings

$$H^{\sigma}_{1/6+\varepsilon/3} \subset L_{1/6+\varepsilon/3-\sigma/3}, \qquad L_{2/3+\varepsilon/3-\sigma/3} \subset H^{\sigma-1/2}_{1/2+\varepsilon/3}. \tag{3.7}$$

For n we have

$$\|n^1\|_{X^{\sigma-3/2,1}_{\pm\alpha|\nabla|}} \lesssim T^{1/4}\alpha\| |u|^2\|_{L^4H^{\sigma-1/2}} \lesssim \|u\|_{L^\infty H^{1/2}}\|u\|_{L^4H^{\sigma}_3}. \tag{3.8}$$

Hence we have $n^1 \in L^\infty_{loc}(H^{\sigma-3/2}_x)$. Since $n \in L^\infty_{loc}(L^2_x)$, we get

$$|\alpha\nabla|^{-1}\sin(|\alpha\nabla|t)\dot{n}(0) \in L^\infty_{loc}(H^{\sigma-3/2}_x), \tag{3.9}$$

and hence $\dot{n}(0) \in H^{\sigma-5/2}$.

Next we improve the estimate on u by using the nonresonance property of the bilinear term (see Section 2.5). For that purpose we decompose nu in the space-time frequency:

$$nu = n_{\lesssim}u + n_{\gg}u, \qquad n_{\lesssim}u := \sum_{j\lesssim k} n_j u_k, \qquad n_{\gg}u := \sum_{j\gg k} n_j u_k, \tag{3.10}$$

$$n_{\gg}u = n^F_{\gg}u + n_{\gg}u^F + (n_{\gg}u)^F := \sum_{j\gg k} n^F_j u_k + n_j u^F_k + (n_j u_k)^F,$$

where we denote $n_j := P_{j/2<|\xi|\le j}n$, $u_k := P_{k/2<|\xi|\le k}u$, and

$$(3.11)\quad n_j^F = P_{||\tau|-\alpha|\xi||>\delta} n_j, \quad u_k^F = P_{|\tau-|\xi|^2/2|>\delta} u_k, \quad (n_j u_k)^F = P_{|\tau-|\xi|^2/2|>\delta}(n_j u_k),$$

with $\delta := \varepsilon j^2$, where $\varepsilon > 0$ is as in (2.42). Then we estimate its contribution to u, namely $\|I_u(nu)\|_{X^{\sigma,3/4-\varepsilon}}$, piece by piece:

$$(3.12)\quad \|n_{\lesssim} u\|_{L^2 H^\sigma_{2/3+\varepsilon/3}} \lesssim T^{\varepsilon/2} \|n\|_{L^\infty L^2} \|u\|_{L_{1/2-\varepsilon/2} H^\sigma_{1/6+\varepsilon/3}},$$

$$\|n^F_{\gg} u\|_{L^2 H^\sigma_{2/3}} \lesssim \|\delta^{-1} \|n^F_j\|_{L^2 H^{\sigma-3/2}} \|u\|_{L^\infty L^3}\|_{\ell^2_{j\gg 1}} \lesssim \|n^F_{\gg 1}\|_{X^{\sigma-3/2,1}_{\pm\alpha|\nabla|}} \|u\|_{L^\infty H^{1/2}},$$

$$\|n_{\gg} u^F\|_{L^2 H^\sigma_{2/3}} \lesssim \sup_{j \gg k} \delta^{-1+\varepsilon} \|n_j\|_{L^\infty L^2} \|u^F_k\|_{L^2 H^{\sigma-1/2}} \lesssim \|n\|_{L^\infty L^2} \|u\|_{X^{\sigma-1/2,1-\varepsilon}_{-\Delta/2}},$$

$$\|(n_{\gg} u)^F\|_{X^{\sigma,-1/4-\varepsilon}} \lesssim \sup_{j \gg k} j^\varepsilon \delta^{-1/4-\varepsilon} \|n_j u_k\|_{L^2 H^\sigma}$$

$$\lesssim T^{\varepsilon/2} \|n\|_{L^\infty L^2} \|u\|_{L_{1/2-\varepsilon/2} H^\sigma_{1/6+\varepsilon/3}} \lesssim \|n\|_{L^\infty L^2} \|u\|_{St_S(\sigma)}.$$

For the free part of n^F, we have

$$(3.13)\quad \|n_j^{0F}\|_{X^{a,1}_{\pm\alpha|\nabla|}} \sim \|\chi_{|\tau|>\delta}(t)\|_{H^1_t} (\|n_j(0)\|_{H^a_x} + \|\dot{n}_j(0)\|_{H^a_x})$$

$$\lesssim \delta^{-a-3} j^a (\|n_j(0)\|_{L^2} + j^{5/2} \|\dot{n}_j(0)\|_{H^{-5/2}}),$$

which is bounded for $\varepsilon j^2 > 1$. Hence $n^{0F}_{>1} \in X^{a,1}_{\pm\alpha|\nabla|}$ for any $a > 0$.

Thus we obtain $u^1 \in X^{\sigma,3/4-\varepsilon}_{-\Delta/2}$, and also $u^1 \in L^\infty(0,T;H^{1/2}) \subset T^{1/2} X^{1/2,0}$. Interpolating them, we get

$$(3.14)\quad u^1 \in (T^{1/2} X^{1/2,0}_{-\Delta/2}, X^{\sigma,3/4-\varepsilon}_{-\Delta/2})_{\theta,1} \subset T^{(1-\theta)/2} St_S(\sigma'),$$

$$(3/4-\varepsilon)\theta = 1/2, \qquad \sigma'[\sigma] = (1-\theta)/2 + \theta\sigma.$$

By the convex combination it is clear that the unique solution for $\sigma'[\sigma] = \sigma$ is $\sigma = 1/2$, and $1/2 > \sigma' > \sigma$ as far as $\sigma < 1/2$. Hence if we iterate $\sigma_{j+1} = \sigma'[s_j]$, then we have $\sigma_j \to 1/2 - 0$ as $j \to \infty$.

Summarizing the above estimates, we have obtained

$$(3.15)\quad \|u^1\|_{St_S(\sigma')} \le CT^a (1 + \|u^1\|_{St_S(\sigma)})^b,$$

where the constant C depends only on $\|u\|_{L^\infty H^{1/2}} + \|n\|_{L^\infty L^2}$, and $a, b > 0$ are determined only by $\varepsilon > 0$.

If $T > 0$ is sufficiently small, there exists a minimum positive solution $B > 0$ for

$$(3.16)\quad B = CT^a (1 + B)^b.$$

Hence for sufficiently small interval $\subset [0, T]$, or if we start the induction with $\|u^1\|_{St_S(0)} < B$, then in the above iteration there is a uniform bound

$$\|u^1\|_{St_S(\sigma)} < B. \tag{3.17}$$

Since $\sigma \to 1/2 - 0$, this holds for any $\sigma < 1/2$. Therefore by the limit $\sigma \to 1/2 - 0$ with the monotone convergence theorem in the Fourier space, we get

$$\|u^1\|_{St_S(1/2)} \le B. \tag{3.18}$$

Iterating once more, we get $u \in X^{1/2,3/4-\varepsilon}$ as well, hence $u \in St_S(1/2)$.

Finally we improve n by decomposing $|u|^2$:

$$|u|^2 = u_{\sim}\bar{u} + 2\Re(u_{\gg}\bar{u}), \qquad u_{\sim}\bar{u} = \sum_{j\sim k} u_j\overline{u_k}, \qquad u_{\gg}\bar{u} = \sum_{j\gg k} u_j\overline{u_k}, \tag{3.19}$$

$$u_{\gg}\bar{u} = u^F_{\gg}\bar{u} + u_{\gg}\overline{u^F} + (u_{\gg}\bar{u})^F = \sum_{j\gg k} u^F_j\overline{u_k} + u_j\overline{u^F_k} + (u_j\overline{u_k})^F,$$

where u_j and u^F_j are as before, and

$$(u_j\overline{u_k})^F = P_{||\tau|-\alpha|\xi||>\delta}(u_j\overline{u_k}), \tag{3.20}$$

with $\delta = \varepsilon j^2$. Then $\|I_n|u|^2\|_{Y^{0,3/4}}$ is bounded by the sum of

$$\|u_{\sim}u\|_{L^{4/3}H^1} \lesssim \|u\|^2_{L_t^{8/3}H_4^{1/2}} \lesssim \|u\|^2_{St_S(1/2)}, \tag{3.21}$$

$$\|u^F_{\gg}u\|_{L^{4/3}H^1} \lesssim \sup_{l\gg k} \delta^{-5/8}\|u^F_l\|_{L^2H^{1/2}}\|u_k\|_{L^4H_3^{1/2}} \lesssim \|u\|_{X^{1/2,5/8}}\|u\|_{St_S(1/2)},$$

$$\|u_{\gg}u^F\|_{L^{4/3}H^1} \lesssim \sup_{l\gg k} \delta^{-5/8}\|u_l\|_{L^4H_3^{1/2}}\|u^F_k\|_{L^2H^{1/2}} \lesssim \|u\|_{St_S(1/2)}\|u\|_{X^{1/2,5/8}},$$

$$\|(u_{\gg}u)^F\|_{X^{1,-1/4}_{\pm\alpha|\nabla|}} \lesssim \|\delta^{-1/4}l\|u_l\|_{L_x^3}\|u\|_{L_x^6}\|_{\ell_l^2L_t^2}$$

$$\lesssim \|u\|_{L^4B^{1/2}_{3,2}}\|u\|_{L^4L^6} \lesssim \|u\|^2_{St_S(1/2)}.$$

Thus we obtain $n \in X^{0,3/4}_{\pm\alpha|\nabla|}$. □

Remark 3.2. We have the same result in $\boldsymbol{R}^2$ and $\boldsymbol{R}$, because there the Sobolev and the Strichartz are better, and the resonance distance is just the same.

Now the unconditional uniqueness follows, since the above lemma shows that every weak solution falls into the wellposedness class in [9]. For the sake of completeness, we give a proof.

Proof of Theorem 1.3. Let $(u_j, n_j) \in C_w(\boldsymbol{R}; H^{1/2} \times L^2)$ be weak solutions for $j = 0, 1$ satisfying

$$u_0(0) = u_1(0), \qquad n_0(0) = n_1(0), \qquad \dot{n}_0(0) = \dot{n}_1(0). \tag{3.22}$$

Let $(u', n') := (u_1, n_1) - (u_0, n_0)$. By the above lemma, we have

$$u_j,\ u' \in X^{1/2, 3/4-\varepsilon}_{-\Delta/2} \cap X^{0, 1-\varepsilon}_{-\Delta/2}, \qquad n^1_j,\ n' \in X^{0, 3/4}_{\pm\alpha|\nabla|}, \tag{3.23}$$

for any $\varepsilon > 0$. Fix $\varepsilon \in (0, 1/9)$. The difference (u', n') satisfies $u'(0) = n'(0) = 0$ and

$$\begin{aligned} &2i\dot{u}' - \Delta u' = -n_0 u' - n' u_1, \\ &\alpha^{-2}\ddot{n}' - \Delta n' = -\Delta \Re[\overline{u'}(u_0 + u_1)], \end{aligned} \tag{3.24}$$

in the sense of integral equations in $C_t(\mathscr{S}')$. We want to use the same arguments as in (3.12) and (3.21) to u' and n', but we have to replace the estimates for $n^F u$ and $n u^F$ by

$$\begin{aligned} \|I_u(n^F_{\gg} u)\|_{X^{1/2,3/4}_{-\Delta/2}} &\lesssim T^{1/4} \|n^F_{\gg} u\|_{L^2 H^1_{3/2}} \lesssim T^{1/4} \|\delta^{-3/4} \|n^F_j\|_{L^2 L^2} \|u\|_{L^\infty L^3}\|_{\ell^2_{j\gg 1}} \\ &\lesssim T^{1/4} \|n^F_{\gg 1}\|_{X^{0,3/4}_{\pm\alpha|\nabla|}} \|u\|_{L^\infty H^{1/2}}, \\ \|I_u(n_{\gg} u^F)\|_{X^{1/2,3/4}_{-\Delta/2}} &\lesssim T^{1/4} \|n_{\gg} u^F\|_{L^2 H^{1-2\varepsilon}_{3/2}} \\ &\lesssim T^{1/4} \sup_{j \gg k} \|n_j\|_{L^\infty L^2} \delta^{-3/4+\varepsilon} \|u_k\|_{L^2 H^{1/2}} \\ &\lesssim T^{1/4} \|n\|_{L^\infty L^2} \|u\|_{X^{1/2,3/4-\varepsilon}_{-\Delta/2}}. \end{aligned} \tag{3.25}$$

Thus we get

$$\begin{aligned} \|u'\|_{\tilde{X}} &\lesssim T^{\varepsilon/2} \|n_0\|_{\tilde{Y}} \|u'\|_{\tilde{X}} + T^{\varepsilon/2} \|n'\|_{\tilde{Y}} \|u_1\|_{\tilde{X}}, \\ \|n'\|_{\tilde{Y}} &\lesssim \|u_0 + u_1\|_{\tilde{X}} \|u'\|_{\tilde{X}}, \end{aligned} \tag{3.26}$$

where

$$\tilde{X} := X^{1/2, 3/4-\varepsilon}_{-\Delta/2}, \qquad \|n\|_{\tilde{Y}} := \|n\|_{L^\infty L^2} + \|n^F_{>1}\|_{X^{0,3/4}_{\pm\alpha|\nabla|}}. \tag{3.27}$$

Hence we obtain

$$\|u'\|_{\tilde{X}} + T^{-\varepsilon/4} \|n'\|_{\tilde{Y}} = 0 \tag{3.28}$$

when $T > 0$ is sufficiently small. Since we can repeat the above argument from $t = T$, these solutions are the same as long as both exist. □

4. Klein-Gordon-Zakharov

In this section we prove the unconditional uniqueness for the KGZ system. Unlike the Zakharov system, it is also possible to prove the uniqueness for the KGZ by estimating the difference of solutions in rougher spaces. That is because we have much more regularity room for the KGZ, which can be observed for example from the fact that the Strichartz estimate by itself can give wellposedness for KGZ in $H^{1+\varepsilon}$ regularity for any $\varepsilon > 0$, but not for the Z system in any regularity because of derivative loss.

However we give a proof by using the infinite iteration as in the previous section, since it seems to give stronger estimates on the weak solution and apply to more general situations.

Thus the main ingredient of our proof is the following.

Lemma 4.1. *Let $I \subset \boldsymbol{R}$ be an interval, $(E, \dot{E}, n) \in C_w(I; H^1 \times L^2 \times L^2)$ be a weak solution of* (KGZ), *$t_0 \in I$ and n^0 be the free solution*

$$\alpha^{-2}\ddot{n}^0 - \Delta n^0 = 0, \qquad n^0(t_0) = n(t_0), \qquad \dot{n}^0(t_0) = \dot{n}(t_0). \tag{4.1}$$

Then we have $E \in X^{1,1-\varepsilon}_{\pm\langle\nabla\rangle}$ and $n - n^0 \in X^{0,1-\varepsilon}_{\pm\alpha|\nabla|}$ for any $\varepsilon > 0$.

It seems necessary to assume that $\dot{E}$ is in L^2, since $E^1 \in L^\infty H^1$ is essential to regularize it in the $X^*_{\pm\langle\nabla\rangle}$ space.

As before, we may assume that $I = [0, T]$ and $t_0 = 0$ with small $T > 0$. We are going to iterate the integral equation

$$\begin{aligned} &E = E^0 + E^1,\ E^1 := I_E(nE), \qquad I_E(f) := \int_0^t \sin(\langle\nabla\rangle(t-s))\langle\nabla\rangle^{-1} f(s)\,ds, \\ &n = n^0 + n^1,\ n^1 := I_n(|E|^2), \end{aligned} \tag{4.2}$$

where E^0 and n^0 are the free solutions with the same initial data as E and n. More precisely, we should extend the solution on $[0, T]$ to $\boldsymbol{R}$ by the procedure in (2.21) and iterate the extended integral equation of the form (2.22). However for simplicity, we identify the extended solutions with the original local solutions.

Proof. We will prove $E^1 \in St_K(\sigma)$ on $(0, T)$ for $0 \le \sigma < 1$ by induction on σ. The bound will be uniform such that we can take the limit $\sigma \to 1 - 0$. We assume that $0 < T < 1$.

We will use the iteration twice. Fixing $\varepsilon \in (0, 1/9)$ arbitrary, we show that $E^1 \in X^{1-\varepsilon, 1-\varepsilon/2}_{\pm\langle\nabla\rangle}$ by the first iteration, then $E^1 \in X^{1,1-\varepsilon}_{\pm\langle\nabla\rangle}$ by the second one[1]

[1] Instead, by choosing the induction parameters nonlinearly, we can get the desired regularity by one time iteration limit. We chose the twice limit because the parameters are simpler and the first step is valid even for $0 < s < 1$.

For $a \in (0,1)$ and $b \in \boldsymbol{R}$, we denote

$$W^b(a) := L_t^{2/a} H_{2/(1-a)}^b \supset St_K^{1-a}(a+b), \qquad D^b(a) := L_t^2 H_{2/(2-a)}^b, \tag{4.3}$$

where we omit b when $b = 0$. Then we have on $(0,T)$

$$L^\infty L^2 \times W(1-a) \subset T^{a/2} D(1-a) \subset \begin{cases} T^{1/2} L_t^{2/(1+a)} L_x^{2/(2-a)} \to X_{\pm\langle\nabla\rangle}^{1-a,1-a/2} \\ T^{1/2} L^2 H^{-3a/2} \to X_{\pm\langle\nabla\rangle}^{1-3a/2,1}, \end{cases} \tag{4.4}$$

where $\to$ means the mapping property of I_E.

We will prove $E^1 \in W(\sigma)$ for $\sigma \le 1-\varepsilon$ by induction on σ starting with $\sigma = 2/3$, which is obvious by the Sobolev. Assume $E^1 \in W(\sigma)$ with $\sigma \in [2/3, 1-\varepsilon]$. By the above estimate,

$$\|E^1\|_{X_{\pm\langle\nabla\rangle}^{\sigma,1/2+\sigma/2}} \lesssim \|nE\|_{D(\sigma)} \lesssim T^{(1-\sigma)/2} \|n\|_{L^\infty L^2} \|E\|_{W(\sigma)}, \tag{4.5}$$

and therefore

$$\begin{gathered} E^1 \in (T^{1/2} X_{\pm\langle\nabla\rangle}^{1-\varepsilon,0}, X_{\pm\langle\nabla\rangle}^{\sigma,1/2+\sigma/2})_{\theta,1} \subset St_K^\varepsilon(\sigma') \subset W(\sigma'), \\ (1/2+\sigma/2)\theta = 1/2, \qquad \sigma' = (1-\theta)(1-\varepsilon) + \theta\sigma. \end{gathered} \tag{4.6}$$

Thus by the iteration limit we get $\sigma \to 1-\varepsilon-0$ and

$$E \in X_{\pm\langle\nabla\rangle}^{1-\varepsilon,1-\varepsilon/2}. \tag{4.7}$$

The above estimate is very simple, but cannot reach H^1, because the estimates blow up by the failure of endpoint Strichartz estimate. To get H^1, we need to use the $X^{s,b}$ spaces as in the Zakharov case. Before the induction, we notice

$$\begin{aligned} \|E^1\|_{X_{\pm\langle\nabla\rangle}^{1-3\varepsilon/2,1}} &\le \|nE\|_{L^2 H^{-3\varepsilon/2}} \lesssim T^{\varepsilon/2} \|n\|_{L^\infty L^2} \|E\|_{W(1-\varepsilon)}, \\ \|n^1\|_{X_{\pm\alpha|\nabla|}^{-3\varepsilon/2,1}} &\le \||E|^2\|_{L^2 H^{1-3\varepsilon/2}} \lesssim T^{\varepsilon/2} \|E\|_{L^\infty H^1} \|E\|_{W(1-\varepsilon)}. \end{aligned} \tag{4.8}$$

Now we restart the induction. Assume that $E^1 \in St_K^\varepsilon(1-a) \subset W^{\varepsilon-a}(1-\varepsilon)$ with $a \in (0,\varepsilon]$. We improve it by decomposing nE:

$$nE = n_{\lesssim} E + n_{\gg} E, \qquad n_{\gg} E = n_{\gg}^F E + n_{\gg} E^F + (n_{\gg} E)^F, \tag{4.9}$$

as in (3.10), where $n_j = P_{j/2<|\xi|\le j} n$, $E_k = P_{k/2<|\xi|\le k} E$, and

$$n_j^F = P_{||\tau|-\alpha|\xi||>\delta} n_j, \quad E_k^F = P_{||\tau|-\langle\xi\rangle|>\delta} E_k, \quad (n_j E_k)^F = P_{||\tau|-\langle\xi\rangle|>\delta}(n_j E_k), \tag{4.10}$$

with $\delta := \varepsilon j$ as in (2.40). Then $\|E^1\|_{X_{\pm\langle\nabla\rangle}^{1-a,1-\varepsilon}}$ is bounded by the sum of

(4.11)
$$\begin{aligned}\|n_{\lesssim}E\|_{D^{\varepsilon-a}(1-\varepsilon)} &\lesssim T^{\varepsilon/2}\|n\|_{L^\infty L^2}\|E\|_{W^{\varepsilon-a}(1-a)},\\ \|n^F_{\gg}E\|_{D^\varepsilon(1-\varepsilon)} &\lesssim T^{\varepsilon/2}\sup_{j\gg k}\delta^{-1+2\varepsilon}\|n_j^F\|_{L^2H^{-3\varepsilon/2}}\|E_k\|_{L^\infty L^6}\\ &\lesssim T^{\varepsilon/2}\|n^F_{>1}\|_{X^{-3\varepsilon/2,1-2\varepsilon}_{\pm\alpha|\nabla|}}\|E\|_{L^\infty H^1},\\ \|n_{\gg}E^F\|_{D^\varepsilon(1-\varepsilon)} &\lesssim T^{\varepsilon/2}\sup_{j\gg k}\delta^{-1+2\varepsilon}\|n_j\|_{L^\infty L^2}\|E_k^F\|_{L^2H^{1-3\varepsilon/2}}\\ &\lesssim T^{\varepsilon/2}\|n\|_{L^\infty L^2}\|E\|_{X^{1-3\varepsilon/2,1-2\varepsilon}_{\pm\langle\nabla\rangle}},\\ \|(n_{\gg}E)^F\|_{X^{1-a,-\varepsilon}_{\pm\langle\nabla\rangle}} &\lesssim \sup_{j\gg k}\frac{k^{a+\varepsilon/2}}{j^a\delta^{\varepsilon/2}}\|n_j\|_{L^\infty L^2}\|k^{-a-\varepsilon/2}E_k\|_{L^2L^\infty}\\ &\lesssim \|n\|_{L^\infty L^2}T^{\varepsilon/2}\|E\|_{W^{\varepsilon-a}(1-\varepsilon)}.\end{aligned}$$

For the free part of $n^F_{>1}$, we can use (3.13). Hence we have $E\in X^{1-a,1-\varepsilon}_{\pm\langle\nabla\rangle}$, and by interpolation we get

(4.12)
$$E^1\in(T^{1/2}X^{1,0}_{\pm\langle\nabla\rangle},X^{1-a,1-\varepsilon}_{\pm\langle\nabla\rangle})_{\theta,1}\subset St^\varepsilon_K(1-a'),$$
$$\theta(1-\varepsilon)=1/2,\qquad a'=a\theta.$$

Hence by iterating the above argument we get $a\to+0$, and $E^1\in St^\varepsilon_K(1)\cap X^{1,1-\varepsilon}_{\pm\langle\nabla\rangle}$.

Finally we improve N by decomposing as before,

(4.13)
$$|E|^2=E_{\sim}\bar E+2\Re(E_{\gg}\bar E),\qquad E_{\gg}\bar E=E^F_{\gg}\bar E+E_{\gg}\overline{E^F}+(E_{\gg}\bar E)^F,$$

where

(4.14)
$$(E_j\overline{E_k})^F=P_{||\tau|-\alpha|\xi||>\delta}(E_j\overline{E_k}),$$

with $\delta=\varepsilon j$. Then $\|n^1\|_{X^{0,1-2\varepsilon}_{\pm\alpha|\nabla|}}$ is bounded by the sum of

(4.15)
$$\begin{aligned}&\|E_{\sim}\bar E\|_{L^2H^1}\lesssim\|E\|^2_{L^4H^{1/2}_4},\\ &\|E^F_{\gg}E\|_{L^2H^1}\lesssim\sup_{l\gg k}\delta^{-2/3}\|E^F_l\|_{L^2H^1}\|E_k\|_{L^\infty H^1}\lesssim\|E\|_{X^{1,2/3}_{\pm\langle\nabla\rangle}}\|E\|_{L^\infty H^1},\\ &\|E_{\gg}E^F\|_{L^2H^1}\lesssim\sup_{l\gg k}\delta^{-2/3}\|E_l\|_{L^\infty H^1}\|E^F_k\|_{L^2H^1}\lesssim\|E\|_{L^\infty H^1}\|E\|_{X^{1,2/3}_{\pm\langle\nabla\rangle}},\\ &\|(E_{\gg}E)^F\|_{X^{1,-2\varepsilon}_{\pm\alpha|\nabla|}}\lesssim\sup_l\delta^{-2\varepsilon}l^\varepsilon\|E_l\|_{L^\infty H^1}\|E\|_{W(1-\varepsilon/3)}\lesssim\|E\|_{L^\infty H^1}\|E\|_{St_K^{\varepsilon/3}(1)}.\end{aligned}$$

Thus we obtain $N\in X^{0,1-2\varepsilon}_{\pm\alpha|\nabla|}$. □

The first step in the above proof works even with less regularity. In fact, we can prove that

Lemma 4.2. *Let* $0 \le s < 1$ *and* $(E, \dot{E}, n) \in C_w(H^s \times H^{s-1} \times L^2)$ *be a weak solution of the Klein-Gordon-Zakharov system. Then we have*

$$E \in X^{s,1/2+s/2}_{\pm\langle\nabla\rangle}. \tag{4.16}$$

However, the above argument does not give desired bound for n such as $n^1 \in X^{0,1/2+}_{\pm\alpha|\nabla|}$. The problem is in the high-high interaction, where we have resonance.

Returning to the energy space, the uniqueness readily follows from the local wellposedness in [20], now that we have much better regularity by Lemma 4.1 than that required in that paper. For the sake of completeness, we give a proof:

Proof of Theorem 1.2. Let $(E_j, \dot{E}_j, n_j) \in C_w(H^1 \times L^2 \times L^2)$ for $j = 0, 1$ be weak solutions satisfying

$$E_0(0) = E_1(0), \qquad n_0(0) = n_1(0), \qquad \dot{n}_0(0) = \dot{n}_1(0). \tag{4.17}$$

Let $(E', n') := (E_1, n_1) - (E_0, n_0)$. By the above lemma, we have

$$E_j,\ E' \in X^{1,1-\varepsilon}_{\pm\langle\nabla\rangle}, \qquad n_j,\ n' \in X^{0,1-\varepsilon}_{\pm\alpha|\nabla|}, \tag{4.18}$$

for any $\varepsilon > 0$. Fix $\varepsilon \in (0, 1/9)$. The difference satisfies $0 = E'(0) = E'_t(0) = n'(0) = n'_t(0)$ and

$$\begin{aligned} &\ddot{E}' - \Delta E' + E' = -n_0 E' - n' E_1, \\ &\alpha^{-2}\ddot{n}' - \Delta n' = \Delta \Re[\overline{E'}(E_0 + E_1)], \end{aligned} \tag{4.19}$$

in $C_t(\mathscr{S}')$. Applying the same estimates as in (4.11) and (4.15) with $a = 0$ to E' and n', we get

$$\begin{aligned} &\|E'\|_{\tilde{X}} \lesssim T^{\varepsilon/2}\|n_0\|_{\tilde{Y}}\|E'\|_{\tilde{X}} + T^{\varepsilon/2}\|n'\|_{\tilde{Y}}\|E_1\|_{\tilde{X}}, \\ &\|n'\|_{\tilde{Y}} \lesssim \|E_0 + E_1\|_{\tilde{X}}\|E'\|_{\tilde{X}}, \end{aligned} \tag{4.20}$$

with

$$\tilde{X} := X^{1,1-\varepsilon}_{\pm\langle\nabla\rangle}, \qquad \|n\|_{\tilde{Y}} := \|n\|_{L^\infty L^2} + \|n^F_{>1}\|_{X^{1,1-2\varepsilon}_{\pm\alpha|\nabla|}}. \tag{4.21}$$

Thus we get $E' = n' = 0$ on $(0, T)$ if $T > 0$ is sufficiently small, and repeating this argument, $(E_0, n_0) = (E_1, n_1)$ as long as both exist. □

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