Algebraic Transformations of Gauss Hypergeometric Functions

By

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Abstract. This article gives a classification scheme of algebraic transformations of Gauss hypergeometric functions, or pull-back transformations between hypergeometric differential equations. The classification recovers the classical transformations of degree 2, 3, 4, 6, and finds other transformations of some special classes of the Gauss hypergeometric function. The other transformations are considered more thoroughly in a series of supplementing articles.

Key Words and Phrases. Gauss hypergeometric function, Fuchsian equations, Pull-back transformations.

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1. Introduction

An algebraic transformation of Gauss hypergeometric functions is an identity of the form

$$(1) \qquad {}_2F_1\left(\begin{matrix} \tilde{A}, \tilde{B} \\ \tilde{C} \end{matrix} \,\middle|\, x\right) = \theta(x)\; {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \,\middle|\, \varphi(x)\right).$$

Here $\varphi(x)$ is a rational function of x, and $\theta(x)$ is a radical function, i.e., a product of some powers of rational functions. Examples of algebraic transformations are the following well-known quadratic transformations (see [Erd53, Section 2.11], [Gou81, formulas 38, 45]):

$$(2) \qquad {}_2F_1\left(\begin{matrix} a, b \\ \frac{a+b+1}{2} \end{matrix} \,\middle|\, x\right) = {}_2F_1\left(\begin{matrix} \frac{a}{2}, \frac{b}{2} \\ \frac{a+b+1}{2} \end{matrix} \,\middle|\, 4x(1-x)\right),$$

$$(3) \qquad {}_2F_1\left(\begin{matrix} a, b \\ 2b \end{matrix} \,\middle|\, x\right) = \left(1 - \frac{x}{2}\right)^{-a} {}_2F_1\left(\begin{matrix} \frac{a}{2}, \frac{a+1}{2} \\ b + \frac{1}{2} \end{matrix} \,\middle|\, \frac{x^2}{(2-x)^2}\right).$$

Algebraic transformations of Gauss hypergeometric functions are usually induced by pull-back transformations between their hypergeometric differential equations. General relation between these two kinds of transformations is

given in Lemma 2.1 here below. By that lemma, if a pull-back transformation converts a hypergeometric equation to a hypergeometric equation as well, then there are identities of the form (1) between hypergeometric solutions of the two hypergeometric equations. Conversely, an algebraic transformation (1) is induced by a pull-back transformation of the corresponding hypergeometric equations, unless the hypergeometric series on the left-hand side of (1) satisfies a first order linear differential equation.

This article classifies pull-back transformations between hypergeometric differential equations. At the same time we essentially classify algebraic transformations (1) of Gauss hypergeometric functions. Classical fractional-linear and quadratic transformations are due to Euler, Pfaff, Gauss and Kummer. In [Gou81] Goursat gave a list of transformations of degree 3, 4 and 6. It has been widely assumed that there are no other algebraic transformations, unless hypergeometric functions are algebraic functions. For example, [Erd53, Section 2.1.5] states the following: "Transformations of [degrees other than 2, 3, 4, 6] can exist only if a, b, c are certain rational numbers; in those cases the solutions of the hypergeometric equation are algebraic functions." As our study shows, this assertion is unfortunately not true. This fact is noticed in [AK03] as well. Existence of a few special transformations follows from [Hod18], [Beu02].

Regarding transformations of algebraic hypergeometric functions (or more exactly, pull-back transformations of hypergeometric differential equations with a finite monodromy group), celebrated Klein's theorem [Kle77] ensures that all these hypergeometric equations are pull-backs of a few standard hypergeometric equations. Klein's pull-back transformations do not change the *projective monodromy group*. The possible finite projective monodromy groups are: a cyclic (including the trivial), a finite dihedral, the tetrahedral, the octahedral or the icosahedral groups. Transformations of algebraic hypergeometric functions that reduce the projective monodromy group are compositions of a few "reducing" transformations and Klein's transformation keeping the smaller monodromy group; see Remark 7.1 below.

The ultimate list of pull-back transformations between hypergeometric differential equations (and of algebraic transformations for their hypergeometric solutions) is the following:

- Classical algebraic transformations of degree 2, 3, 4 and 6 due to Gauss, Euler, Kummer, Pfaff and Goursat. We review classical transformations in Section 4, including fractional-linear transformations.
- Transformations of hypergeometric equations with an abelian monodromy group. This is a degenerate case [Vid07]; the hypergeometric equations have 2 (rather than 3) actual singularities. We consider these transformations in Section 5.

- Transformations of hypergeometric equations with a dihedral monodromy group. These transformations are considered here in Section 6, or more thoroughly in [Vid08a, Sections 6 and 7].

- Transformations of hypergeometric equations with a finite monodromy group. The hypergeometric solutions are algebraic functions. Transformations of hypergeometric equations with finite cyclic or dihedral monodromy groups can be included in the previous two cases. Transformations of hypergeometric equations with the tetrahedral, octahedral or icosahedral projective monodromy groups are considered here in Section 7, or more thoroughly in [Vid08b].

- Transformations of hypergeometric functions which are incomplete elliptic integrals. These transformations correspond to endomorphisms of certain elliptic curves. They are considered in Section 8, or more thoroughly in [Vid08c].

- Finitely many transformations of so-called *hyperbolic hypergeometric functions*. Hypergeometric equations for these functions have local exponent differences $1/k_1$, $1/k_2$, $1/k_3$, where k_1, k_2, k_3 are positive integers such that $1/k_1 + 1/k_2 + 1/k_3 < 1$. These transformations are described in Section 9, or more thoroughly in [Vid05].

The classification scheme is presented in Section 3. Sections 4 through 9 characterize various cases of algebraic transformations of hypergeometric functions. We mention some three-term identities with Gauss hypergeometric functions as well. The non-classical cases are considered more thoroughly in separate articles [Vid08a], [Vid08b], [Vid08c], [Vid05].

Recently, Kato [Kat08] classified algebraic transformations of the $_3F_2$ hypergeometric series. The rational transformations for the argument z in that list form a strict subset of the transformations considered here.

2. Preliminaries

The hypergeometric differential equation is [AAR99, Formula (2.3.5)]:

$$(4) \qquad z(1-z)\frac{d^2y(z)}{dz^2} + (C - (A+B+1)z)\frac{dy(z)}{dz} - ABy(z) = 0.$$

This is a Fuchsian equation with 3 regular singular points $z = 0, 1$ and ∞. The local exponents are:

$$0, 1 - C \text{ at } z = 0; \qquad 0, C - A - B \text{ at } z = 1; \qquad \text{and } A, B \text{ at } z = \infty.$$

A basis of solutions for general equation (4) is

$$(5) \qquad {}_2F_1\left(\begin{matrix} A, B \\ C \end{matrix} \middle| z\right), \qquad z^{1-C} \, {}_2F_1\left(\begin{matrix} 1 + A - C, 1 + B - C \\ 2 - C \end{matrix} \middle| z\right).$$

For basic theory of hypergeometric functions and Fuchsian equations see [Beu02], [vdW02, Chapters 1, 2] or [Tem96, Chapters 4, 5]. We use the approach of Riemann and Papperitz [AAR99, Sections 2.3, 3.9].

A (*rational*) pull-back transformation of an ordinary linear differential equation has the form

$$(6) \qquad z \mapsto \varphi(x), \qquad y(z) \mapsto Y(x) = \theta(x)y(\varphi(x)),$$

where $\varphi(x)$ and $\theta(x)$ have the same meaning as in formula (1). Geometrically, this transformation *pull-backs* a differential equation on the projective line $\boldsymbol{P}_z^1$ to a differential equation on the projective line $\boldsymbol{P}_x^1$, with respect to the finite covering $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$ determined by the rational function $\varphi(x)$. Here and throughout the paper, we let $\boldsymbol{P}_x^1$, $\boldsymbol{P}_z^1$ denote the projective lines with rational parameters x, z respectively. A pull-back transformation of a Fuchsian equations gives a Fuchsian equation again. In [AK03] pull-back transformations are called *RS-transformations*.

We introduce the following definition: an *irrelevant singularity* for an ordinary differential equation is a regular singularity which is not logarithmic, and where the local exponent difference is equal to 1. An irrelevant singularity can be turned into a non-singular point after a suitable pull-back transformation (6) with $\varphi(x) = x$. (For comparison, an *apparent singularity* is a regular singularity which is not logarithmic, and where the local exponents are integers. Recall that at a *logarithmic point* is a singular point where there is only one local solution of the form $x^\lambda(1 + \alpha_1 x + \alpha_2 x^2 + \cdots)$, where x is a local parameter there.) For us, a *relevant singularity* is a singular point which is not an irrelevant singularity.

We are interested in pull-back transformations of one hypergeometric equation to other hypergeometric equation, possibly with different parameters A, B, C. These pull-back transformations are related to algebraic transformations of Gauss hypergeometric functions as follows.

Lemma 2.1. 1. *Suppose that pull-back transformation* (6) *of hypergeometric equation* (4) *is a hypergeometric equation as well* (*with the new indeterminate* x). *Then, possibly after fractional-linear transformations on* $\boldsymbol{P}_x^1$ *and* $\boldsymbol{P}_z^1$, *there is an identity of the form* (1) *between hypergeometric solutions of the two hypergeometric equations.*

2. *Suppose that hypergeometric identity* (1) *holds in some region of the complex plane. Let* $Y(x)$ *denote the left-hand side of the identity. If* $Y'(x)/Y(x)$ *is not a rational function of* x, *then the transformation* (6) *converts the hypergeometric equation* (4) *into a hypergeometric equation for* $Y(x)$.

Proof. We have a two-term identity whenever we have a singular point $S \in \boldsymbol{P}_x^1$ of the transformed equation above a singular point $Q \in \boldsymbol{P}_z^1$ of the

starting equation. Using fractional-linear transformations on $\boldsymbol{P}_x^1$ and $\boldsymbol{P}_z^1$ we can achieve S is the point $x = 0$ and that Q is the point $z = 0$. Then identification of two hypergeometric solutions with the local exponent 0 and the value 1 at (respectively) $x = 0$ and $z = 0$ gives a two-term identity as in (1). If all three singularities of the transformed equation do not lie above $\{0, 1, \infty\} \subset \boldsymbol{P}_z^1$, they are apparent singularities. Then the transformed equation has trivial monodromy, while the starting hypergeometric equation has a finite monodromy group. As we will consider explicitly in Sections 7 and 5, 6, the pull-back transformations reducing the monodromy group and the pull-back transformations keeping the trivial monodromy group can be realized by two-term hypergeometric identities. This is recaped in Remark 7.1 below.

For the second statement, we have two second-order linear differential equations for the left-hand side of (1): the hypergeometric equation for $Y(x)$, and a pull-back transformation (6) of the hypergeometric equation (4). If these two equations are not $\boldsymbol{C}(x)$-proportional, then we can combine them linearly to a first-order differential equation $Y'(x) = r(x)Y(x)$ with $r(x) \in \boldsymbol{C}(x)$, contradicting the condition on $Y'(x)/Y(x)$. ∎

If we have an identity (1) without a pull-back transformation between corresponding hypergeometric equations, the left-hand side of the identity can be expressed as terminating hypergeometric series up to a radical factor; see Kovacic algorithm [Kov86], [vdPS03, Section 4.3.4]. In a formal sense, any pair of terminating hypergeometric series is algebraically related. We do not consider these degenerations.

Remark 2.2. We also do not consider transformations of the type $_2\mathrm{F}_1(\varphi_1(z)) = \theta(z)\,_2\mathrm{F}_1(\varphi_2(z))$, where $\varphi_1(z)$, $\varphi_2(z)$ are rational functions (of degree at least 2). Therefore we miss transformations of some complete elliptic integrals, such as

$$(7) \qquad \mathbf{K}(x) = \frac{2}{1+y}\mathbf{K}\left(\frac{1-y}{1+y}\right),$$

where $x^2 + y^2 = 1$ and

$$\mathbf{K}(x) = \frac{\pi}{2}\,_2\mathrm{F}_1\left(\begin{matrix} 1/2, 1/2 \\ 1 \end{matrix} \middle| x^2\right) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-x^2t^2)}}.$$

Identity (7) plays a key role in the theory of arithmetic-geometric mean; see [AAR99, Chapter 3.2]. Other similar example is the following formula, proved in [BBG95, Theorem 2.3]:

$$(8) \qquad _2\mathrm{F}_1\left(\begin{matrix} c, c + \frac{1}{3} \\ \frac{3c+5}{6} \end{matrix} \middle| x^3\right) = (1 + 2x)^{-3c}\,_2\mathrm{F}_1\left(\begin{matrix} c, c + \frac{1}{3} \\ \frac{3c+1}{2} \end{matrix} \middle| 1 - \frac{(1-x)^3}{(1+2x)^3}\right).$$

The case $c = 1/3$ was found earlier in [BB91].

A pull-back transformation between hypergeometric equations usually gives several identities like (1) between some of the 24 Kummer's solutions of both equations. It is appropriate to look first for suitable pull-back coverings $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$ up to fractional-linear transformations. As we will see, suitable pull-back coverings are determined by appropriate transformations of singular points and local exponent differences.

Once a suitable covering φ is known, it is convenient to use Riemann's P-notation for deriving hypergeometric identities (1) with the argument $\varphi(x)$. Recall that a Fuchsian equation with 3 singular points is determined by the location of those singular points and local exponents there. The linear space of solutions is determined by the same data. It can be defined homologically without reference to hypergeometric equations as a *local system* on the projective line; see [Kat96], [Gra86, Section 1.4]. The notation for it is

$$(9) \qquad P\left\{\begin{matrix} \alpha & \beta & \gamma \\ a_1 & b_1 & c_1 & z \\ a_2 & b_2 & c_2 \end{matrix}\right\},$$

where $\alpha, \beta, \gamma \in \boldsymbol{P}_z^1$ are the singular points, and $a_1, a_2; b_1, b_2; c_1, c_2$ are the local exponents at them, respectively. Recall that second order Fuchsian equations with 3 relevant singularities are defined uniquely by their singularities and local exponents, unlike general Fuchsian equations with more than 3 singular points. Our approach can be entirely formulated in terms of local systems, without reference to hypergeometric equations and their pull-back transformations. By Papperitz' theorem [AAR99, Theorem 2.3.1] we must have

$$a_1 + a_2 + b_1 + b_2 + c_1 + c_2 = 1.$$

We are looking for transformations of local systems of the form

$$(10) \qquad P\left\{\begin{matrix} 0 & 1 & \infty \\ 0 & 0 & \tilde{A} & x \\ 1 - \tilde{C} & \tilde{C} - \tilde{A} - \tilde{B} & \tilde{B} \end{matrix}\right\}$$

$$= \theta(x) P\left\{\begin{matrix} 0 & 1 & \infty \\ 0 & 0 & A & \varphi(x) \\ 1 - C & C - A - B & B \end{matrix}\right\}.$$

The factor $\theta(x)$ should shift local exponents at irrelevant singularities to the values 0 and 1, and it should shift one local exponent at both $x = 0$ and $x = 1$ to the value 0. In intermediate computations, Fuchsian equations with more than 3 singular points naturally occur, but those extra singularities are irrelevant singularities. We extend Riemann's P-notation and write

(11)
$$P\left\{\begin{matrix} \alpha & \beta & \gamma & S_1 & \cdots & S_k & \\ a_1 & b_1 & c_1 & e_1 & \cdots & e_k & z \\ a_2 & b_2 & c_2 & e_1+1 & \cdots & e_k+1 & \end{matrix}\right\}$$

to denote the local system (of solutions of a Fuchsian equation) with irrelevant singularities $S_1, \ldots, S_k$. This notation makes sense if a local system exists (i.e., if the local exponents sum up to the right value); then it can be transformed to a local system like (9). For example, if none of the points $\gamma, S_1, \ldots, S_k$ is the infinity, local system (11) can be identified with

$$\frac{(z-S_1)^{e_1} \ldots (z-S_k)^{e_k}}{(z-\gamma)^{e_1+\cdots+e_k}} P\left\{\begin{matrix} \alpha & \beta & \gamma & \\ a_1 & b_1 & c_1+e_1+\cdots+e_k & z \\ a_2 & b_2 & c_2+e_1+\cdots+e_k & \end{matrix}\right\}.$$

Here is an example of computation with local systems leading to quadratic transformation (3):

$$P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a/2 & t^2 \\ \frac{1}{2}-b & b-a & (a+1)/2 & \end{matrix}\right\} = P\left\{\begin{matrix} 0 & 1 & -1 & \infty & \\ 0 & 0 & 0 & a & t \\ 1-2b & b-a & b-a & a+1 & \end{matrix}\right\}$$

$$= P\left\{\begin{matrix} 0 & 1 & \infty & 2 & \\ 0 & 0 & 0 & a & x=\dfrac{2t}{t+1} \\ 1-2b & b-a & b-a & a+1 & \end{matrix}\right\}$$

$$= (2-x)^a P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & a & x \\ 1-2b & b-a & b & \end{matrix}\right\}.$$

To conclude (3), one has to identify two functions with the local exponent 0 and the value 1 at $t=0$ and $x=0$ (in the first and the last local systems respectively), like in the proof of part 1 of Lemma 2.1.

Once a hypergeometric identity (1) is obtained, it can be composed with Euler's and Pfaff's fractional-linear transformations; we recall them in formulas (16)–(18) below. Geometrically, these transformations permute the singularities $1, \infty$ (on $\boldsymbol{P}^1_z$ or $\boldsymbol{P}^1_x$) and their local exponents. Besides, simultaneous permutation of the local exponents at $x=0$ and $z=0$ usually implies a similar identity to (1), as presented in the following lemma.

Lemma 2.3. *Suppose that a pull-back transformation induces identity* (1) *in an open neighborhood of* $x=0$. *Then* $\varphi(x)^{1-C} \sim Kx^{1-\tilde{C}}$ *as* $x \to 0$ *for some constant K, and the following identity holds* (*if both hypergeometric functions are well-defined*):

(12)
$$
{}_2\mathrm{F}_1\left(\begin{matrix} 1+\tilde{A}-\tilde{C}, 1+\tilde{B}-\tilde{C} \\ 2-\tilde{C} \end{matrix}\,\middle|\, x\right)
$$

$$
= \theta(x)\frac{\varphi(x)^{1-C}}{Kx^{1-\tilde{C}}}\, {}_2\mathrm{F}_1\left(\begin{matrix} 1+A-C, 1+B-C \\ 2-C \end{matrix}\,\middle|\, \varphi(x)\right).
$$

Proof. The asymptotic relation $\varphi(x)^{1-C} \sim Kx^{1-\tilde{C}}$ as $x \to 0$ is clear from the transformation of local exponents. (We are ensured that $\theta(0) = 1$.) Further, we have relation (10) and the relations

$$
P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & A & \varphi(x) \\ 1-C & C-A-B & B & \end{matrix}\right\}
$$

$$
= \varphi(x)^{1-C}P\left\{\begin{matrix} 0 & 1 & \infty & \\ C-1 & 0 & A+1-C & \varphi(x) \\ 0 & C-A-B & B+1-C & \end{matrix}\right\},
$$

$$
P\left\{\begin{matrix} 0 & 1 & \infty & \\ 0 & 0 & \tilde{A} & x \\ 1-\tilde{C} & \tilde{C}-\tilde{A}-\tilde{B} & \tilde{B} & \end{matrix}\right\}
$$

$$
= x^{1-\tilde{C}}P\left\{\begin{matrix} 0 & 1 & \infty & \\ \tilde{C}-1 & 0 & \tilde{A}+1-\tilde{C} & x \\ 0 & \tilde{C}-\tilde{A}-\tilde{B} & \tilde{B}+1-\tilde{C} & \end{matrix}\right\}.
$$

From here we get the right identification of local systems for (12). ∎

A general pull-back transformation converts a hypergeometric equation to a Fuchsian differential equation with several singularities. To find proper candidates for pull-back coverings $\varphi : P^1_x \to P^1_z$, we look first for possible pull-back transformations of hypergeometric equations to Fuchsian equations with (at most) 3 relevant singularities. These Fuchsian equations can be always transformed to hypergeometric equations by suitable fractional-linear pull-back transformations, and vice versa. Relevant singular points and local exponent differences for the transformed equation are determined by the covering φ only. Here are simple rules which determine singularities and local exponent differences for the transformed equation.

Lemma 2.4. *Let* $\varphi : P^1_x \to P^1_z$ *be a finite covering. Let* H_1 *denote a Fuchsian equation on* P^1_z, *and let* H_2 *denote the pull-back transformation of* H_1 *under* (6). *Let* $S \in P^1_x$, $Q \in P^1_z$ *be points such that* $\varphi(S) = Q$.

1. *The point* S *is a logarithmic point for* H_2 *if and only if the point* Q *is a logarithmic point for* H_1.

2.   *If the point Q is non-singular for H_1, then the point S is not a relevant singularity for H_2 if and only if the covering φ does not branch at S.*

3.   *If the point Q is a singular point for H_1, then the point S is not a relevant singularity for H_2 if and only if the following two conditions hold*:

- *The point Q is not logarithmic.*
- *The local exponent difference at Q is equal to $1/k$, where k is the branching index of φ at S.*

Proof.   First we note that if the point S is not a relevant singularity, then it is either a non-singular point or an irrelevant singularity.   Therefore S is not a relevant singularity if and only if it is not a logarithmic point and the local exponent difference is equal to 1.

Let p, q denote the local exponents for H_1 at the point Q.   Let k denote the branching order of φ at S.   Then the local exponent difference at S is equal to $k(p - q)$.   To see this, note that if $m \in \mathbf{C}$ is the order of $\theta(x)$ at S, the local exponents at S are equal to $kp + m$ and $kq + m$.   This fact is clear if Q is not logarithmic, when the local exponents can be read from solutions.   In general one has to use the indicial polynomial to determine local exponents.

The first statement is clear, since local solutions of H_1 at S can be pull-backed to local solutions of H_2 at Q, and local solutions of H_2 at Q can be push-forwarded to local solutions of H_1 at S.

If the point Q is non-singular, the point S is not logarithmic by the first statement, so S is a not a relevant singularity if and only if $k = 1$.

If the point Q is singular, then the local exponent difference at S is equal to 1 if and only if the local exponent difference $|p - q|$ is equal to $1/k$.   ∎

The following Lemma gives an estimate for the number of points S to which part 3 of Lemma 2.4 applies, and it gives a relation between local exponent differences of two hypergeometric equations related by a pull-back transformation and the degree of the pull-back transformation.   In this paper we make the convention that real local exponent differences are non-negative, and complex local exponent differences have the argument in the interval $(-\pi, \pi]$.

Lemma 2.5.   *Let $\varphi : \mathbf{P}_x^1 \to \mathbf{P}_z^1$ be a finite covering, and let d denote the degree of φ.*

1.   *Let Δ denote a set of 3 points on $\mathbf{P}_z^1$.   If all branching points of φ lie above Δ, then there are exactly $d + 2$ distinct points on $\mathbf{P}_x^1$ above Δ.   Otherwise there are more than $d + 2$ distinct points above Δ.*

2.   *Let H_1 denote a hypergeometric equation on $\mathbf{P}_z^1$, and let H_2 denote a pull-back transformation of H_1 with respect to φ.   Suppose that H_2 is hyper-geometric equation as well.   Let e_1, e_2, e_3 denote the local exponent differ-*

ences for H_1, and let e'_1, e'_2, e'_3 denote the local exponent differences for H_2. Then

$$(13) \qquad\qquad d(e_1 + e_2 + e_3 - 1) = e'_1 + e'_2 + e'_3 - 1.$$

Proof. For a point $S \in \mathbf{P}^1_x$ let $\mathrm{ord}_S \, \varphi$ denote the branching order of φ at S. By Hurwitz formula [Har77, Corollary IV.2.4] we have $-2 = -2d + D$, where

$$D = \sum_{S \in \mathbf{P}^1_x} (\mathrm{ord}_S \, \varphi - 1).$$

Therefore $D = 2d - 2$. The number of points above $\varDelta$ is

$$3d - \sum_{\varphi(S) \in \varDelta} (\mathrm{ord}_S \, \varphi - 1) \geq 3d - D = d + 2.$$

We have the equality if and only if all branching points of φ lie above $\varDelta$.

Now we show the second statement. For a point $S \in \mathbf{P}^1_z$ or $S \in \mathbf{P}^1_x$, let $\mathrm{led}(S)$ denote the local exponent difference for H_1 or H_2 (respectively) at S. The following sums make sense:

$$\sum_{S \in \mathbf{P}^1_x} (\mathrm{led}(S) - 1) = \sum_{Q \in \mathbf{P}^1_z} \sum_{\varphi(S)=Q} (\mathrm{led}(S) - 1)$$

$$= \sum_{Q \in \mathbf{P}^1_z} \left(d \, \mathrm{led}(Q) - \sum_{\varphi(S)=Q} 1 \right)$$

$$= d \sum_{Q \in \mathbf{P}^1_z} (\mathrm{led}(Q) - 1) + D.$$

The first sum is equal to $e'_1 + e'_2 + e'_3 - 3$. The last expression is equal to $d(e_1 + e_2 + e_3 - 3) + 2d - 2$. ∎

3. The classification scheme

The core problem is to classify pull-back transformations of hypergeometric equations to Fuchsian equations with at most 3 relevant singular points. By Lemma 2.4, a general pull-back transformation gives a Fuchsian equation with quite many relevant singular points, especially above the set $\{0, 1, \infty\} \subset \mathbf{P}^1_z$. In order to get a Fuchsian equation with so few singular points, we have to restrict parameters (or local exponent differences) of the original hypergeometric equation, and usually we can allow branching only above the set $\{0, 1, \infty\} \subset \mathbf{P}^1_z$.

We classify pull-back transformations between hypergeometric equations (and algebraic transformations of Gauss hypergeometric functions) in the following five principal steps:

1. Let H_1 denote hypergeometric equation (4), and consider its pull-back transformation (6). Let H_2 denote the pull-backed differential equation, and let T denote the number of singular points of H_2. Let Δ denote the subset $\{0, 1, \infty\}$ of $\boldsymbol{P}_z^1$, and let d denote the degree of the covering $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$ in transformation (6). We consequently assume that exactly $N \in \{0, 1, 2, 3\}$ of the 3 local exponent differences for H_1 at Δ are *restricted* to the values of the form $1/k$, where k is a positive integer. If $k = 1$ then the corresponding point of Δ is assumed to be not logarithmic, as we cannot get rid of singularities above a logarithmic point.

2. In each assumed case, use Lemma 2.4 and determine all possible combinations of the degree d and restricted local exponent differences. Let $k_1, \ldots, k_N$ denote the denominators of the restricted differences. By part 4 of Lemma 2.4,

$$T \geq [\text{the number of singular points above } \Delta]$$

$$\geq d + 2 - [\text{the number of non-singular points above } \Delta]$$

$$\geq d + 2 - \sum_{j=1}^{N} \left\lfloor \frac{d}{k_j} \right\rfloor.$$

Since we wish $T \leq 3$, we get the following restrictive inequality in integers:

$$(14) \qquad\qquad d - \sum_{j=1}^{N} \left\lfloor \frac{d}{k_j} \right\rfloor \leq 1.$$

To skip specializations of cases with smaller N, we may assume that $d \geq \max(k_1, \ldots, k_N)$. A preliminary list of possibilities can be obtained by dropping the rounding down in (14); this gives a weaker but more convenient inequality

$$(15) \qquad\qquad \frac{1}{d} + \sum_{j=1}^{N} \frac{1}{k_j} \geq 1.$$

3. For each combination of d and restricted local exponent differences, determine possible branching patterns for φ such that the transformed equation H_2 would have at most three singular points. In most cases we can allow branching points only above Δ, and we have to take the maximal number $\left\lfloor d/k_j \right\rfloor$ of non-singular points above the point with the local exponent difference $1/k_j$.

4. For each possible branching pattern, determine all rational functions $\varphi(x)$ which define a covering with that branching pattern. For $d \leq 6$ this can be done using a computer by a straightforward method of undetermined coefficients. In [Vid05, Section 3] a more appropriate algorithm is introduced which uses differentiation of $\varphi(x)$. In many cases this problem has precisely one solution up to fractional-linear transformations. But not for any branching pattern a covering exists, and there can be several different coverings with the same branching pattern. For infinite families of branching patterns we are able to give a general, algorithmic or explicit characterization of corresponding coverings. For instance, if hypergeometric solutions can be expressed very explicitly, we can identify the local systems in (10) up to unknown factor $\theta(x)$. Then quotients of corresponding hypergeometric solutions (aka *Schwarz maps*) can be identified precisely, which gives a straightforward way to determine $\varphi(x)$.

5. Once a suitable covering $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$ is computed, there always exist corresponding pull-back transformations. Two-term identities like (1) can be computed using extended Riemann's P-notation of Section 2. We have two-term identities for each singular point S of the transformed equation such that $\varphi(S) \subset \varDelta$, as in the proof of part 1 of Lemma 2.1. Once we fix S, $\varphi(S)$ as $x = 0$, $z = 0$ respectively, permutations of local exponents and other singularities give identities (1) which are related by Euler's and Pfaff's transformations and Lemma 2.3. If the transformed equation has less than 3 actual singularities, one can consider any point above $\varDelta$ in this manner. Some of the obtained identities may be the same up to change of free parameters.

Now we sketch explicit appliance of the above procedure. When $N = 0$, i.e., when no local exponent differences are restricted, then $d = 1$ by formula (15). This gives Euler's and Pfaff's fractional-linear transformations. When $N = 1$, we have the following cases:

• $k_1 = 2$, $d = 2$. This gives the classical quadratic transformations. See Section 4.

• $k_1 = 1$, d any. The z-point with the local exponent difference $1/k_1$ is assumed to be non-logarithmic, so the equation H_1 has only two relevant singularities. As we show in Lemma 5.1 below, the two unrestricted local exponent differences must be equal. As it turns out, the covering φ branches only above the two points with unrestricted local exponent differences. If the triple of local exponent differences for H_1 is $(1, p, p)$, the triple of local exponent differences for H_2 is $(1, dp, dp)$. Formally, this case has a continuous family of fractional-linear pull-back transformations, but that does not give interesting hypergeometric identities.

When $N = 2$, we have the following cases:

Local exponent differences		Degree d	Branching pattern above the regular singular points	Covering composition
$(1/k_1, 1/k_2, p)$	above			
$(1/2, 1/3, p)$	$(1/2, p, 2p)$	3	$2 + 1 = 3 = 2 + 1$	indecomposable
$(1/2, 1/3, p)$	$(1/3, p, 3p)$	4	$2 + 2 = 3 + 1 = 3 + 1$	indecomposable
$(1/2, 1/3, p)$	$(1/3, 2p, 2p)$	4	$2 + 2 = 3 + 1 = 2 + 2$	no covering
$(1/2, 1/3, p)$	$(p, p, 4p)$	6	$2 + 2 + 2 = 3 + 3 = 4 + 1 + 1$	2×3
$(1/2, 1/3, p)$	$(2p, 2p, 2p)$	6	$2 + 2 + 2 = 3 + 3 = 2 + 2 + 2$	2×3 or 3×2
$(1/2, 1/3, p)$	$(p, 2p, 3p)$	6	$2 + 2 + 2 = 3 + 3 = 3 + 2 + 1$	no covering
$(1/2, 1/4, p)$	$(p, p, 2p)$	4	$2 + 2 = 4 = 2 + 1 + 1$	2×2
$(1/3, 1/3, p)$	(p, p, p)	3	$3 = 3 = 1 + 1 + 1$	indecomposable

Table 1:   Transformations of Gauss hypergeometric functions with 1 free parameter, except for dihedral and degenerate functions

- If $\max(k_1, k_2) > 2$, the possibilities are listed in Table 1. Steps 2 and 3 of the classification scheme are straightforward, and a snapshot of possibilities after them is presented by the first four columns of Table 1. The notation for a branching pattern in the fourth column gives $d + 2$ branching orders for the points above Δ; branching orders at points in the same fiber are separated by the $+$ signs, branching orders for different fibers are separated by the $=$ signs. Step 4 of our scheme gives at most one covering (up to fractional-linear transformations) for each branching pattern. Ultimately, Table 1 yields precisely the classical transformations of degree 3, 4, 6 due to Goursat [Gou81]; see Section 4. It is straightforward to figure out possible composits of small degree coverings, and then identify them with the unique coverings for Table 1. Degrees of constituents for decomposable coverings are listed in the last column from right (for the constituent transformation from H_1) to left. Note that one degree 6 covering has two distinct decompositions; a corresponding hypergeometric transformation is given in formula (28) below.
- $k_1 = 2$, $k_2 = 2$, d any. The monodromy group of H_1 is a dihedral group. The hypergeometric functions can be expressed very explicitly, see Section 6. The triple $(1/2, 1/2, p)$ of local exponent differences for H_1 is transformed either to $(1/2, 1/2, dp)$ for any d, or to $(1, dp/2, dp/2)$ for even d.
- $k_1 = 1$; k_2 and d are any positive integers. The z-point with the local exponent difference $1/k_1$ is not logarithmic, so the triple of local exponent differences for H_1 must be $(1, 1/k_2, 1/k_2)$. The monodromy group is a finite cyclic group. Possible transformations are outlined in Section 5.

When $N = 3$, we have the following three very distinct cases:

- $1/k_1 + 1/k_2 + 1/k_3 > 1$. The monodromy groups of H_1 and H_2 are finite, the hypergeometric functions are algebraic. The degree d is unbounded. Klein's theorem [Kle77] implies that any hypergeometric equation

with a finite monodromy group (or equivalently, with algebraic solutions) is a pull-back transformation of a *standard hypergeometric equation* with the same monodromy group. These are the most interesting pull-back transformations for this case. Equations with finite cyclic monodromy groups are mentioned in the previous subcase; their transformations are considered in Section 5. Equations with finite dihedral monodromy groups are considered in Section 6. Equations with the tetrahedral, octahedral or icosahedral projective monodromy groups are characterized in Section 7.

• $1/k_1 + 1/k_2 + 1/k_3 = 1$. Non-trivial hypergeometric solutions of H_1 are incomplete elliptic integrals, see Section 8. The degree d is unbounded, different transformations with the same branching pattern are possible. Most interesting transformations pull-back the equation H_1 into itself, so that $H_2 = H_1$; these transformations come from endomorphisms of the corresponding elliptic curve.

• $1/k_1 + 1/k_2 + 1/k_3 < 1$. Here we have transformations of hyperbolic hypergeometric functions, see Section 9. The list of these transformations is finite, the maximal degree of their coverings is 24. Existence of some of these transformations is shown in [Hod18], [Beu02], [AK03].

The degree of transformations is determined by formula (13), except in the case of incomplete elliptic integrals. If all local exponent differences are real numbers in the interval $(0, 1]$, the covering $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$ is defined over $\boldsymbol{R}$ and it branches only above $\{0, 1, \infty\} \subset \boldsymbol{P}_z^1$, then it induces a tessellation of the Schwarz triangle for H_2 into Schwarz triangles for H_1, as outlined in [Hod18, Beu02] or [Vid05, Section 2]. Recall that a *Schwarz triangle* for a hypergeometric equation is the image of the upper half-plane under a Schwarz map for the equation. The described tessellation is called *Coxeter decomposition*. If it exists, formula (13) can be interpreted nicely in terms of areas of the Schwarz triangles for H_1 and H_2 in the spherical or hyperbolic metric. Out of the classical transformations, only the cubic transformation with the branching pattern $3 = 3 = 1 + 1 + 1$ does not allow a Coxeter decomposition; see formula (23) below.

The following sections form an overview of algebraic transformations for different types of Gauss hypergeometric functions. We also mention some three-term identities with Gauss hypergeometric functions. Non-classical cases are considered more thoroughly in other articles [Vid08a], [Vid08b], [Vid08c], [Vid05].

4. Classical transformations

Formally, Euler's and Pfaff's fractional-linear transformations [AAR99, Theorem 2.2.5]

$$(16) \qquad {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ c \end{matrix}\middle|\, z\right) = (1-z)^{-a}\, {}_2\mathrm{F}_1\left(\begin{matrix} a, c-b \\ c \end{matrix}\middle|\, \frac{z}{z-1}\right)$$

$$(17) \qquad\qquad = (1-z)^{-b}\, {}_2\mathrm{F}_1\left(\begin{matrix} c-a, b \\ c \end{matrix}\middle|\, \frac{z}{z-1}\right)$$

$$(18) \qquad\qquad = (1-z)^{c-a-b}\, {}_2\mathrm{F}_1\left(\begin{matrix} c-a, c-b \\ c \end{matrix}\middle|\, z\right).$$

can be considered as pull-back transformations of degree 1. These are the only transformations without restrictions on the parameters (or local exponent differences) of a hypergeometric function under transformation. In a geometrical sense, they permute the local exponents at $z = 1$ and $z = \infty$. In general, permutation of the singular points $z = 0$, $z = 1$, $z = \infty$ and local exponents at them gives 24 Kummer's hypergeometric series solutions to the same hypergeometric differential equation. Any three hypergeometric solutions are linearly related, of course.

To present other classical and non-classical transformations, we introduce the following notation. Let $(p_1, q_1, r_1) \overset{d}{\leftarrow} (p_2, q_2, r_2)$ schematically denote a pull-back transformation of degree d, which transforms a hypergeometric equation with the local exponent differences p_1, q_1, r_1 to a hypergeometric equation with the local exponent differences p_2, q_2, r_2. The order of local exponents in a triple is irrelevant. Note that the arrow follows the direction of the covering $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$.

The list of classical transformations comes from the data of Table 1. Here is the list of classical transformations with indecomposable φ, up to Euler's and Pfaff's fractional-linear transformations and the conversion of Lemma 2.3.

- $(1/2, p, q) \overset{2}{\leftarrow} (2p, q, q)$. These are classical quadratic transformations. All two-term quadratic transformations of hypergeometric functions can be obtained by composing (2) or (3) with Euler's and Pfaff's transformations. An example of a three-term relation under a quadratic transformation is the following (see also Remark 5.2 below, and [Erd53, 2.11(3)]):

$$(19) \qquad {}_2\mathrm{F}_1\left(\begin{matrix} a, b \\ \frac{a+b+1}{2} \end{matrix}\middle|\, x\right) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a+1}{2}\right)\Gamma\left(\frac{b+1}{2}\right)}\, {}_2\mathrm{F}_1\left(\begin{matrix} \frac{a}{2}, \frac{b}{2} \\ \frac{1}{2} \end{matrix}\middle|\, (1-2x)^2\right)$$

$$+ (1-2x)\frac{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{a+b+1}{2}\right)}{\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)}\, {}_2\mathrm{F}_1\left(\begin{matrix} \frac{a+1}{2}, \frac{b+1}{2} \\ \frac{3}{2} \end{matrix}\middle|\, (1-2x)^2\right).$$

- $(1/2, 1/3, p) \overset{3}{\leftarrow} (1/2, p, 2p)$. These are well-known Goursat's cubic transformations. Two-term transformations follow from the following three formulas, along with Euller's and Pfaff's transformations and application of Lemma 2.3 to (22):

$$(20) \quad {}_2F_1\left(\begin{matrix} a, (2a+1)/2 \\ (4a+2)/3 \end{matrix}\,\middle|\, x\right) = \left(1 - \frac{3x}{4}\right)^{-a} {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ (4a+5)/6 \end{matrix}\,\middle|\, \frac{27x^2(1-x)}{(4-3x)^3}\right),$$

$$(21) \quad {}_2F_1\left(\begin{matrix} a, (2a+1)/2 \\ (4a+5)/6 \end{matrix}\,\middle|\, x\right) = (1+3x)^{-a}\, {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ (4a+5)/6 \end{matrix}\,\middle|\, \frac{27x(1-x)^2}{(1+3x)^3}\right),$$

$$(22) \quad {}_2F_1\left(\begin{matrix} a, (2a+1)/6 \\ 1/2 \end{matrix}\,\middle|\, x\right) = \left(1 + \frac{x}{3}\right)^{-a} {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ 1/2 \end{matrix}\,\middle|\, \frac{x(9-x)^2}{(3+x)^3}\right).$$

- $(1/3, 1/3, p) \overset{3}{\leftarrow} (p, p, p)$. These are less-known cubic transformations. Let ω denote a primitive cubic root of unity, so $\omega^2 + \omega + 1 = 0$. Since singular points of the transformed equation are all the same, there is only one two-term formula (up to changing the parameter):

$$(23) \quad {}_2F_1\left(\begin{matrix} a, (a+1)/3 \\ (2a+2)/3 \end{matrix}\,\middle|\, x\right)$$

$$= (1 + \omega^2 x)^{-a}\, {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ (2a+2)/3 \end{matrix}\,\middle|\, \frac{3(2\omega+1)x(x-1)}{(x+\omega)^3}\right).$$

A three-term formula is the following (see also [Erd53, 2.11(38)]):

$$(24) \quad {}_2F_1\left(\begin{matrix} a, (a+1)/3 \\ (2a+2)/3 \end{matrix}\,\middle|\, x\right)$$

$$= 3^{a-1}(1 + \omega^2 x)^{-a}\left[\frac{\Gamma((2a+2)/3)\Gamma(a/3)}{\Gamma(2/3)\Gamma(a)}\, {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ 2/3 \end{matrix}\,\middle|\, \left(\frac{x+\omega^2}{x+\omega}\right)^3\right)\right.$$

$$- \frac{1+\omega x}{1+\omega^2 x}\frac{\Gamma((2a+2)/3)\Gamma((a+2)/3)}{\Gamma(4/3)\Gamma(a)}$$

$$\left. \times\, {}_2F_1\left(\begin{matrix} (a+1)/3, (a+2)/3 \\ 4/3 \end{matrix}\,\middle|\, \left(\frac{x+\omega^2}{x+\omega}\right)^3\right)\right].$$

- $(1/2, 1/3, p) \overset{4}{\leftarrow} (1/3, p, 3p)$. These are indecomposable Goursat's transformations of degree 4. Two-term transformations follow from the following three formulas, if we compose them with Euller's and Pfaff's transformations and apply Lemma 2.3 to (27):

$$(25) \quad {}_2F_1\left(\begin{matrix} 4a/3, (4a+1)/3 \\ (4a+1)/2 \end{matrix}\,\middle|\, x\right) = \left(1 - \frac{8x}{9}\right)^{-a} {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ (4a+5)/6 \end{matrix}\,\middle|\, \frac{64x^3(1-x)}{(9-8x)^3}\right),$$

$$(26) \quad {}_2F_1\left(\begin{matrix} 4a/3, (4a+1)/3 \\ (4a+5)/6 \end{matrix} \middle| x\right) = (1+8x)^{-a}\, {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ (4a+5)/6 \end{matrix} \middle| \frac{64x(1-x)^3}{(1+8x)^3}\right),$$

$$(27) \quad {}_2F_1\left(\begin{matrix} 4a/3, (4a+1)/6 \\ 2/3 \end{matrix} \middle| x\right) = (1-x)^{-a}\, {}_2F_1\left(\begin{matrix} a/3, (3-2a)/6 \\ 2/3 \end{matrix} \middle| \frac{-x(8+x)^3}{64(1-x)^3}\right).$$

As recorded in Table 1, there are four ways to compose quadratic and cubic transformations to higher degree transformations of hypergeometric functions. This gives three different pull-back transformations of degree 4 and 6. The composition transformations can be schematically represented as follows:

$$(1/2, 1/4, p) \overset{2}{\leftarrow} (1/2, p, p) \overset{2}{\leftarrow} (p, p, 2p),$$

$$(1/2, 1/3, p) \overset{3}{\leftarrow} (1/2, p, 2p) \overset{2}{\leftarrow} (p, p, 4p),$$

$$(1/2, 1/3, p) \overset{3}{\leftarrow} (1/2, p, 2p) \overset{2}{\leftarrow} (2p, 2p, 2p),$$

$$(1/2, 1/3, p) \overset{2}{\leftarrow} (1/3, 1/3, 2p) \overset{3}{\leftarrow} (2p, 2p, 2p).$$

The last two compositions should produce the same covering, since computations show that the pull-back $(1/2, 1/3, p) \overset{6}{\leftarrow} (2p, 2p, 2p)$ is unique up to fractional-linear transformations; see [Vid05, Section 3]. Indeed, one may check that the identity

$$(28) \qquad {}_2F_1\left(\begin{matrix} 2a, (2a+1)/3 \\ (4a+2)/3 \end{matrix} \middle| x\right)$$

$$= (1-x+x^2)^{-a}\, {}_2F_1\left(\begin{matrix} a/3, (a+1)/3 \\ (4a+5)/6 \end{matrix} \middle| \frac{27}{4}\frac{x^2(x-1)^2}{(x^2-x+1)^3}\right)$$

is a composition of (3) and (21), and also a composition of (23), (3) and (16). Note that these two compositions use different types of cubic transformations.

5. Hypergeometric equations with two singularities

Here we outline transformations of hypergeometric equations with two relevant singularities; their monodromy group is abelian. The explicit classification scheme of Section 3 refers to this case three times. These equations form a special sample of degenerate hypergeometric equations [Vid07]. For the degenerate cases, not all usual hypergeometric formulas for fractional-linear transformations or other classical algebraic transformations may hold, since the structure of 24 Kummer's solutions degenerates; see [Vid07, Table 1]. Here we consider only the new case of pull-back transformations of the hypergeometric equations with the cyclic monodromy group.

If a Fuchsian equation has the local exponent difference 1 at some point, that point can be a non-singular point, an irrelevant singularity or a logarithmic point. Here is how the logarithmic case is distinguished for hypergeometric equations.

Lemma 5.1. *Consider hypergeometric equation* (4), *and let* $P \in \{0, 1, \infty\}$. *Suppose that the local exponent difference at S is equal to* 1. *Then the point S is logarithmic if and only if (absolute values of) the two local exponent differences at the other two points of the set* $\{0, 1, \infty\}$ *are not equal.*

Proof. Because of fractional-linear transformations, we may assume that S is the point $z = 0$, and the local exponents there are 0 and 1. Therefore $C = 0$. Then the point $z = 0$ is either a non-singular point or a logarithmic point. It is non-singular if and only if $AB = 0$. If $B = 0$, then local exponent differences at $z = 1$ and $z = \infty$ are both equal to A. ∎

This lemma implies that a hypergeometric equation has (at most) two relevant singularities if and only if the local exponent difference at one of the three points $z = 0$, $z = 1$, $z = \infty$ is 1, and the local exponent differences at the other two points are equal. After applying a suitable fractional-linear transformation to this situation we may assume that the point $z = 0$ is non-singular. Like in the proof of Lemma 5.1, we have $C = 0$ and we may take $B = 0$. Then we are either in the case $n = m = 0$ of [Vid07, Section 7 or 8], or in the case $n = m = \ell = 0$ of [Vid07, Section 9]. Most of the 24 Kummer's solutions have to be interpreted either as the constant 1 or the power function $(1 - z)^{-a}$. The only interesting hypergeometric function (up to Euler's and Pfaff's transformations) is the following:

$$(29) \qquad {}_2\mathrm{F}_1\left(\begin{matrix} 1 + a, 1 \\ 2 \end{matrix} \,\middle|\, z \right) = \begin{cases} \dfrac{1 - (1 - z)^{-a}}{az}, & \text{if } a \neq 0, \\[2mm] -\dfrac{1}{z} \log(1 - z), & \text{if } a = 0. \end{cases}$$

For general a, pull-back transformation (6) of the considered hypergeometric equation to a hypergeometric equation branches only above the points $z = 1$ and $z = \infty$. Indeed, if the covering $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$ branches above other point, then these branching points would be singular by part 2 of Lemma 2.4, and there would be at least 3 singular points above $\{1, \infty\} \subset \boldsymbol{P}_z^1$ by part 1 of Lemma 2.5. To keep the number of singular points down to 3, the covering φ should branch only above $\{1, \infty\}$. Up to fractional-linear transformations on $\boldsymbol{P}_x^1$, these coverings have the form $(1 - z) \mapsto (1 - x)^d$, or

$$(30) \qquad z \mapsto x\phi_{d-1}(x), \qquad \text{where } \phi_{d-1}(x) = \frac{1 - (1 - x)^d}{x}.$$

Note that $\phi_{d-1}(x)$ is a polynomial of degree $d-1$. A corresponding hypergeometric identity is

$$(31) \qquad {}_2F_1\left(\begin{array}{c} 1+da, 1 \\ 2 \end{array}\middle| x\right) = \frac{\phi_{d-1}(x)}{d} \, {}_2F_1\left(\begin{array}{c} 1+a, 1 \\ 2 \end{array}\middle| x\phi_{d-1}(x)\right).$$

This transformation is obvious from the explicit expressions in (29).

Formally, we additionally have a continuous family $z \mapsto 1 - \beta + \beta z$ of fractional-linear pull-back transformations which fix the two points $z = 1$ and $z = \infty$. However, they do not give interesting hypergeometric identities since Kummer's series at those two points are trivial.

If $|a| = 1/k$ for an integer $k > 1$, there are more pull-back transformations of hypergeometric equations with the local exponent differences $(1, a, a)$. In this case, the monodromy group is a finite cyclic group, of order k. Pull-backed equations will have a cyclic monodromy group as well, possibly of smaller order. On the other hand, the mentioned Klein's theorem [Kle77] implies that any hypergeometric equation with a cyclic monodromy group of order k is a pull-back of a hypergeometric equation with the local exponent differences $(1, 1/k, 1/k)$. These pull-back transformations can be easily computed from explicit terminating solutions of the target differential equation. According to [Vid07, Section 7], a general hypergeometric equation with a completely reducible (but non-trivial) monodromy representation has the local exponents $(m+n+1, a, a+n-m)$, where $a \notin \mathbf{Z}$ and $n, m \in \mathbf{Z}$ are non-negative. A basis of terminating solutions is

$$(32) \qquad {}_2F_1\left(\begin{array}{c} -n, a-m \\ -m-n \end{array}\middle| z\right), \qquad (1-z)^{-a} \, {}_2F_1\left(\begin{array}{c} -m, -a-n \\ -m-n \end{array}\middle| z\right).$$

The monodromy group is finite cyclic if $a = \ell/k$ with co-prime positive $k, \ell \in \mathbf{Z}$. The terminating solutions can be written as terminating hypergeometric series at $z = 1$ as well:

$$ {}_2F_1\left(\begin{array}{c} -n, a-m \\ -m-n \end{array}\middle| z\right) = \frac{(1+a)_n m!}{(m+n)!} \, {}_2F_1\left(\begin{array}{c} -n, a-m \\ 1+a \end{array}\middle| 1-z\right), \qquad \text{etc.} $$

The quotient of two solutions in (32) defines a *Schwarz map* for the hypergeometric equation. In the simplest case $n = m = 0$, $a = 1/k$, the Schwartz map is just $(1-z)^{1/k}$. Klein's pull-back transformation for $(1, 1/k, 1/k) \xleftarrow{d} (m+n+1, \ell/k, \ell/k+n-m)$ is obtained from identification of the two Schwarz maps. The pull-back covering is defined by

$$(33) \quad (1-z) \mapsto (1-x)^\ell \, {}_2F_1\left(\begin{array}{c} -n, \ell/k-m \\ -m-n \end{array}\middle| x\right)^k \Bigg/ {}_2F_1\left(\begin{array}{c} -m, -\ell/k-n \\ -m-n \end{array}\middle| x\right)^k.$$

The Schwarz maps (or pairs of hypergeometric solutions) are identified here by
the corresponding local exponents at $x = 1$ (placed above $z = 1$) and the same
value at $x = 0$ (placed above $x = 0$). The degree of the transformation is equal
to $\max(nk + \ell, mk)$, by formula (13) as well. Besides, $z \mapsto O(x^{n+m+1})$ at $x = 0$
by the required branching pattern. In particular,

$$(34) \quad {}_2F_1\left(\begin{matrix} -m, -\ell/k - n \\ -m - n \end{matrix} \middle| x\right) \Big/ {}_2F_1\left(\begin{matrix} -n, \ell/k - m \\ -m - n \end{matrix} \middle| x\right) = (1 - x)^{\ell/k} + O(x^{n+m+1})$$

at $x = 0$, hence the quotient of two hypergeometric polynomials is the *Padé
approximation* of $(1 - x)^{\ell/k}$ of precise degree (m, n). For example, the Pade
approximation of $\sqrt{1 - x}$ of degree $(1, 1)$ is $(4 - x)/(4 - 3x)$. Hence the
following pullback must give a transformation $(1, 1/2, 1/2) \overset{3}{\leftarrow} (3, 1/2, 1/2)$:

$$1 - z \mapsto \frac{(1 - x)(x - 4)^2}{(3x - 4)^2}.$$

A corresponding hypergeometric identity is

$$(35) \qquad {}_2F_1\left(\begin{matrix} 3/2, 2 \\ 4 \end{matrix} \middle| x\right) = \frac{4}{4 - 3x}\, {}_2F_1\left(\begin{matrix} 1/2, 1 \\ 2 \end{matrix} \middle| \frac{x^3}{(3x - 4)^2}\right).$$

Transformation (33) is Klein's pull-back transformation if $\gcd(k, \ell) = 1$.
Otherwise the transformed hypergeometric equation has a smaller monodromy
group. These transformations must factor via (30) with $d = \gcd(k, \ell)$, and
Klein's transformation between equations with the smaller monodromy group.
Even $\ell/k \in \mathbf{Z}$ can be allowed if the transformed equation has no logarithmic
points. The condition for that is $\ell/k > m$; see [Vid07, Corollary 2.3 part (2)].
Under this condition, one may even allow $k = 1$ and consider transformations
$(1, 1, 1) \overset{\ell+n}{\leftarrow} (m + n + 1, \ell, \ell + n - m)$. All hypergeometric equations with the
trivial monodromy group can be obtained in this way, by Klein's theorem.
Solutions of these hypergeometric equations are analyzed in [Vid07, Section
8]. A hypergeometric equation with the local exponent differences $(1, 1, 1)$ can
be transformed to $y'' = 0$ by fractional-linear transformations. We underscore
that transformation (33) specializes nicely even for $k = 1$ if only logarithmic
solutions are not involved; the corresponding two-term hypergeometric identities
are trivial.

Remark 5.2. Algebraic transformations of Gauss hypergeometric functions
often hold only in some part of the complex plane, even after standard analytic
continuation. For example, formula (2) is obviously false at $x = 1$. Formula
(2) holds when $\operatorname{Re}(x) < 1/2$, as the standard z-cut $(1, \infty)$ is mapped into the line
$\operatorname{Re}(x) = 1/2$ under the transformation $z = 4x(1 - x)$.

An extreme example of this kind is the following transformation of a hypergeometric function to a rational function:

$$(36) \qquad {}_2F_1\left(\begin{array}{c} 1/2, 1 \\ 2 \end{array}\middle| -\frac{4x^3(x-1)^2(x+2)}{(3x-2)^2}\right) = \frac{2-3x}{(1-x)^2(x+2)}.$$

This identity holds in a neighborhood of $x = 0$, but it certainly does not hold around $x = 1$ or $x = -2$. Apparently, standard cuts for analytic continuation for the hypergeometric function isolate the three points $x = 0$, $x = 1$, $x = -2$. Note that

$$ {}_2F_1\left(\begin{array}{c} 1/2, 1 \\ 2 \end{array}\middle| z\right) = \frac{2-2\sqrt{1-z}}{z} $$

is a two-valued algebraic function on $\boldsymbol{P}_z^1$. Its composition in (36) with the degree 6 rational function apparently consists of two disjoint branches. The second branch is the rational function $(3x-2)/x^3$, which is the correct evaluation of the left-hand side of (36) around the points $x = 1$, $x = -2$ (check the power series.)

Many identities like (36) can be produced for hypergeometric functions of this section with $1/a \in \boldsymbol{Z}$. The pull-backed hypergeometric equations should be Fuchsian equations with the trivial monodromy group. More generally, any algebraic hypergeometric function can be pull-backed to a rational function. Other algebraic hypergeometric functions are considered in the following two sections.

Three-term hypergeometric identities may also have limited region of validity. But it may happen that branch cuts of two hypergeometric terms cancel each other in a three-term identity. For example, standard branch cuts for the hypergeometric functions on the right-hand side of (19) are the intervals $[1, \infty)$ and $(-\infty, 0]$ on the real line. But identity (19) is valid on $\boldsymbol{C}\backslash[1, \infty)$, if we agree to evaluate the right-hand consistently on the interval $(-\infty, 0]$: either using analytic continuation of both terms from the upper half-plane, or from the lower half-plane.

6. Dihedral functions

Hypergeometric equations with (infinite or finite) dihedral monodromy group are characterized by the property that two local exponent differences are rational numbers with the denominator 2. By a quadratic pull-back transformation, these equations can be transformed to Fuchsian equations with at most 4 singularities and with a cyclic monodromy group. Explicit expressions and transformations for these functions are considered thoroughly in [Vid08a]. Here we look at transformations of hypergeometric equations which have two

local exponent differences equal to $1/2$. The explicit classification scheme of Section 3 refers to this case twice.

The starting hypergeometric equation for new transformations has the local exponent differences $(1/2, 1/2, a)$. Hypergeometric solutions of such an equation can be written explicitly. In particular, quadratic transformation (2) with $b = a + 1$ implies

$$(37) \qquad {}_2F_1\left(\begin{matrix} a/2, (a+1)/2 \\ a+1 \end{matrix} \middle| z\right) = \left(\frac{1+\sqrt{1-z}}{2}\right)^{-a}.$$

Other explicit formulas are

$$(38) \qquad {}_2F_1\left(\begin{matrix} a/2, (a+1)/2 \\ 1/2 \end{matrix} \middle| z\right) = \frac{(1-\sqrt{z})^{-a} + (1+\sqrt{z})^{-a}}{2},$$

$$(39) \quad {}_2F_1\left(\begin{matrix} (a+1)/2, (a+2)/2 \\ 3/2 \end{matrix} \middle| z\right) = \begin{cases} \dfrac{(1-\sqrt{z})^{-a} - (1+\sqrt{z})^{-a}}{2a\sqrt{z}}, & \text{if } a \neq 0, \\[2ex] \dfrac{1}{2\sqrt{z}} \log \dfrac{1+\sqrt{z}}{1-\sqrt{z}}, & \text{if } a = 0. \end{cases}$$

General dihedral Gauss hypergeometric functions are contiguous to these ${}_2F_1$ functions. As shown in [Vid08a], explicit expressions for them can be given in terms of terminating Appell's F_2 or F_3 series. For example, generalizations of (37)–(38) are

$$(40) \quad {}_2F_1\left(\begin{matrix} a/2, (a+1)/2 + \ell \\ a + k + \ell + 1 \end{matrix} \middle| 1 - z\right)$$

$$= z^{k/2}\left(\frac{1+\sqrt{z}}{2}\right)^{-a-k-\ell} F_3\left(\begin{matrix} k+1, \ell+1; -k, -\ell \\ a+k+\ell+1 \end{matrix} \middle| \frac{\sqrt{z}-1}{2\sqrt{z}}, \frac{1-\sqrt{z}}{2}\right),$$

$$(41) \quad \frac{((a+1)/2)_n}{(1/2)_n} \, {}_2F_1\left(\begin{matrix} a/2, n + (a+1)/2 \\ -m + 1/2 \end{matrix} \middle| z\right)$$

$$= \frac{(1+\sqrt{z})^{-a}}{2} F_2\left(\begin{matrix} a; -m, -n \\ -2m, -2n \end{matrix} \middle| \frac{2\sqrt{z}}{1+\sqrt{z}}, \frac{2}{1+\sqrt{z}}\right)$$

$$+ \frac{(1-\sqrt{z})^{-a}}{2} F_2\left(\begin{matrix} a; -m, -n \\ -2m, -2n \end{matrix} \middle| \frac{2\sqrt{z}}{\sqrt{z}-1}, \frac{2}{1-\sqrt{z}}\right).$$

Here m, n are assumed to be non-negative integers.

For general a, there are two types of transformations:

• $(1/2, 1/2, a) \xleftarrow{d} (1/2, 1/2, da)$. These are the only transformations to a dihedral monodromy group as well, as there is a singularity above the point with the local exponent difference a. Identification of explicit Schwarz

maps gives the following recipe for computing the pull-back coverings $\varphi : \boldsymbol{P}_x^1 \to \boldsymbol{P}_z^1$. Expand $(1 + \sqrt{x})^d$ in the form $\theta_1(x) + \theta_2(x)\sqrt{x}$ with $\theta_1(x), \theta_2(x) \in \boldsymbol{C}[x]$. Then $\varphi(x) = x\theta_2^2(x)/\theta_1^2(x)$ gives a pull-back transformation of dihedral hypergeometric equations. Explicitly,

$$\theta_1(x) = \sum_{k=0}^{\lfloor d/2 \rfloor} \binom{d}{2k} x^k = {}_2F_1\left(\begin{matrix} -d/2, -(d-1)/2 \\ 1/2 \end{matrix} \middle| x \right),$$

$$\theta_2(x) = \sum_{k=0}^{\lfloor (d-1)/2 \rfloor} \binom{d}{2k+1} x^k = d \, {}_2F_1\left(\begin{matrix} -(d-1)/2, -(d-2)/2 \\ 3/2 \end{matrix} \middle| x \right).$$

A particular transformation of hypergeometric functions is the following:

$$(42) \quad {}_2F_1\left(\begin{matrix} da/2, (da+1)/2 \\ 1/2 \end{matrix} \middle| x \right) = \theta_1(x)^{-a} \, {}_2F_1\left(\begin{matrix} a/2, (a+1)/2 \\ 1/2 \end{matrix} \middle| \frac{x\theta_2(x)^2}{\theta_1(x)^2} \right).$$

It is instructive to check this transformation using (38). Other transformations from the same pull-back covering are given in [Vid08a, Section 6]. Particularly interesting are the following formulas; they hold for odd or even d, respectively:

$$_2F_1\left(\begin{matrix} da/2, -da/2 \\ 1/2 \end{matrix} \middle| x \right) = {}_2F_1\left(\begin{matrix} a/2, -a/2 \\ 1/2 \end{matrix} \middle| d^2x \, {}_2F_1\left(\begin{matrix} (1-d)/2, (1+d)/2 \\ 3/2 \end{matrix} \middle| x \right)^2 \right),$$

$$_2F_1\left(\begin{matrix} da/2, -da/2 \\ 1/2 \end{matrix} \middle| x \right) = {}_2F_1\left(\begin{matrix} a/2, -a/2 \\ 1/2 \end{matrix} \middle| d^2x(1-x) \, {}_2F_1\left(\begin{matrix} 1-d/2, 1+d/2 \\ 3/2 \end{matrix} \middle| x \right)^2 \right).$$

The branching pattern of $\varphi(x)$ is

$$1 + 2 + 2 + \cdots + 2 = d = 1 + 2 + 2 + \cdots + 2, \qquad \text{if } d \text{ is odd,}$$

$$1 + 1 + 2 + 2 + \cdots + 2 = d = 2 + 2 + \cdots + 2, \qquad \text{if } d \text{ is even.}$$

- $(1/2, 1/2, a) \xleftarrow{2\ell} (1, \ell a, \ell a)$, and $d = 2\ell$ is even. These are transformations to hypergeometric equations of Section 5. They are compositions of the mentioned quadratic transformation and the transformations $(1/2, 1/2, a) \xleftarrow{d} (1/2, 1/2, da)$ or $(1, a, a) \xleftarrow{d} (1, da, da)$ described above.

If $a = 1/k$ with k a positive integer, the monodromy group is the finite dihedral group with $2k$ elements, and hypergeometric solutions are algebraic. Klein's theorem [Kle77] implies that any hypergeometric equation with a finite dihedral monodromy group is a pull-back from a hypergeometric equation with the local exponent differences $(1/2, 1/2, 1/k)$ and the same monodromy group. The pull-back transformation can be computed by the similar method: identification of explicit Schwarz maps, using the mentioned explicit evaluations with

terminating Appell's F_2 or F_3 series. That leads to expressing a polynomial in $\sqrt{x}$ in the form $\theta_1(x) + \sqrt{x}\theta_2(x)$ as above.

Theorem 6.1. *Let k, ℓ, m, n be positive integers, and suppose that $k \geq 2$, $\gcd(k,\ell) = 1$. Let us denote*

$$G(x) = x^{m/2} F_3 \left(\begin{matrix} m+1, n+1; -m, -n \\ 1 + \ell/k \end{matrix} \,\middle|\, \frac{\sqrt{x}+1}{2\sqrt{x}}, \frac{1+\sqrt{x}}{2} \right).$$

This is a polynomial in $\sqrt{x}$. We can write

$$\left(1 + \sqrt{x}\right)^\ell G(x)^k = \Theta_1(x) + x^{m+1/2}\Theta_2(x),$$

so that $\Theta_1(x)$ and $\Theta_2(x)$ are polynomials in x. Then the rational function $\Phi(x) = x^{2m+1}\Theta_2(x)^2/\Theta_1(x)^2$ defines Klein's pull-back covering $(1/2, 1/2, 1/k) \xleftarrow{d} (m+1/2, n+1/2, \ell/k)$. The degree d of this rational function is equal to $(m+n)k + \ell$.

Proof. This is Theorem 7.1 in [Vid08a]. ■

The condition $\gcd(k,\ell) = 1$ can be replaced by the weaker condition $\ell/k \notin \mathbf{Z}$, but then the transformed hypergeometric equation has a smaller dihedral monodromy group, and it factors via the transformation in (42) with $d = \gcd(k,\ell)$. Even more, $\ell/k \in \mathbf{Z}$ can be allowed, if the transformed equation has no logarithmic solutions. Sufficient and necessary conditions for that are given in [Vid08a, Theorem 2.1]. The branching pattern for all these coverings has the following pattern:

• Above the two points with the local exponent difference $1/2$, there are two points with the branching orders $2m+1$, $2n+1$, and the remaining points are simple branching points.

• Above the point with the local exponent difference $1/k$, there is one point with the ramification order ℓ, and $m+n$ points with the ramification order k.

Any covering $(1/2, 1/2, 1/k) \xleftarrow{d} (m+1/2, n+1/2, \ell/k)$ is unique up to fractional-linear transformations, as Schwarz maps are identified uniquely. Transformations from the local exponent differences $(1/2, 1/2, 1/k)$ to hypergeometric equations with finite cyclic monodromy groups are either the mentioned degeneration $\ell/k \in \mathbf{Z}$, or compositions with the quadratic transformation $(1/2, 1/2, 1/k) \xleftarrow{2} (1, 1/k, 1/k)$. Other transformations involving dihedral Gauss hypergeometric functions are special cases of classical transformations.

For the purposes of Theorem 6.1, the function $G(x)$ can be alternatively defined as follows:

$$(43) \qquad \left(1 + \sqrt{x}\right)^{(m+n)+\ell} F_2 \left(\begin{matrix} -\ell/k - m - n; -m, -n \\ -2m, -2n \end{matrix} \,\middle|\, \frac{2\sqrt{x}}{1+\sqrt{x}}, \frac{2}{1+\sqrt{x}} \right)^k.$$

The two definitions differ by a constant multiple. The F_2 and F_3 sums are related by reversing the order of summation in both directions in the rectangular sums, as noted in [Vid08a].

For an example, consider the case $n = 1$, $m = 0$, $\ell = 1$ of Theorem 6.1. To compute the transformation $(1/2, 1/2, 1/k) \overset{k+1}{\longleftarrow} (1/2, 3/2, 1/k)$ we need to expand

$$(1 + \sqrt{x})\left(1 - \frac{\sqrt{x}}{k}\right)^k = \theta_3(x) + x^{3/2}\theta_4(x).$$

Straightforward computation shows that

$$(44) \qquad \theta_3(x) = {}_2F_1\left(\begin{matrix} -k/2, -(k+1)/2 \\ -1/2 \end{matrix} \,\middle|\, \frac{x}{k^2}\right),$$

$$\theta_4(x) = \frac{k^2 - 1}{3k^2} \, {}_2F_1\left(\begin{matrix} -(k-2)/2, -(k-3)/2 \\ 5/2 \end{matrix} \,\middle|\, \frac{x}{k^2}\right).$$

A transformation of hypergeometric functions is

$$(45) \qquad {}_2F_1\left(\begin{matrix} -1/(2k), -1/2 - 1/(2k) \\ -1/2 \end{matrix} \,\middle|\, x\right)$$

$$= \theta_3(x)^{1/k} \, {}_2F_1\left(\begin{matrix} -1/(2k), 1/2 - 1/(2k) \\ 1/2 \end{matrix} \,\middle|\, \frac{x^3\theta_4(x)^2}{\theta_3(x)^2}\right).$$

On the other hand,

$${}_2F_1\left(\begin{matrix} -1/(2k), -1/2 - 1/(2k) \\ 1 - 1/k \end{matrix} \,\middle|\, 1 - z\right) = \frac{k - \sqrt{z}}{k - 1}\left(\frac{1 + \sqrt{z}}{2}\right)^{1/k}$$

by formula (40). Note that the construction in (44) breaks down if $k = 1$; a hypergeometric equation with the local exponent differences $(1/2, 3/2, 1)$ has logarithmic solutions.

As computed in [Vid08a, Section 7], the polynomials $\Theta_1(x)$, $\Theta_2(x)$ of Theorem 6.1 in the case $n = 1$, $m = 0$, $\ell = 2$ can be expressed as terminating ${}_3F_2$ series.

7. Algebraic Gauss hypergeometric functions

Algebraic Gauss hypergeometric functions form a classical subject of mathematics. These functions were classified by Schwarz [Sch72]. Recall that a Fuchsian equation has a basis of algebraic solutions if and only if its monodromy group is finite. Finite *projective monodromy groups* for second order equations are either cyclic, or dihedral, or the tetrahedral group iso-

morphic to A_4, or the octahedral group isomorphic to S_4, or the icosahedral group isomorphic to A_5. An important characterization of second order Fuchsian equations with finite monodromy group was given by Klein [Kle77, Kle78]: all these equations are pull-backs of a few *standard hypergeometric equations* with algebraic solutions. In particular, this holds for hypergeometric equations with finite monodromy groups. The corresponding standard equation depends on the projective monodromy group:

- Second order equations with a cyclic monodromy group are pull-backs of a hypergeometric equation with the local exponent differences $(1, 1/k, 1/k)$, where k is a positive integer. Klein's transformations to general hypergeometric equations with a cyclic monodromy group are considered in Section 5 above.

- Second order equations with a finite dihedral monodromy group are pull-backs of a hypergeometric equation with the local exponent differences $(1/2, 1/2, 1/k)$, where $k \geq 2$. Klein's transformations to general hypergeometric equations with a dihedral monodromy group are considered in Section 6 above.

- Second order equations with the tetrahedral projective monodromy group are pull-backs of a hypergeometric equation with the local exponent differences $(1/2, 1/3, 1/3)$. Hypergeometric equations with this monodromy group are contiguous to hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/3)$ or $(1/3, 1/3, 2/3)$.

- Second order equations with the octahedral projective monodromy group are pull-backs of a hypergeometric equation with the local exponent differences $(1/2, 1/3, 1/4)$. Hypergeometric equations with this monodromy group are contiguous to hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/4)$ or $(2/3, 1/4, 1/4)$.

- Second order equations with the icosahedral projective monodromy group are pull-backs of a hypergeometric equation with the local exponent differences $(1/2, 1/3, 1/5)$. Hypergeometric equations with this monodromy group are contiguous to hypergeometric equations with the local exponent differences $(1/2, 1/3, 1/5)$, $(1/2, 1/3, 2/5)$, $(1/2, 1/5, 2/5)$, $(1/3, 1/3, 2/5)$, $(1/3, 2/3, 1/5)$, $(2/3, 1/5, 1/5)$, $(1/3, 2/5, 3/5)$, $(1/3, 1/5, 3/5)$, $(1/5, 1/5, 4/5)$ or $(2/5, 2/5, 2/5)$.

A general algorithm for computation of Klein's coverings is given in [vHW05]. The algorithm is based on finding semi-invariants of the monodromy group by solving appropriate symmetric powers of the given second order differential equation. A more effective algorithm specifically for hypergeometric equations with finite monodromy groups is given in [Vid08b]. This algorithm is based on identification of explicit Schwarz maps for the given and the corresponding standard hypergeometric equation.

Klein's pull-back transformations are most interesting among the transformations of algebraic $_2F_1$ functions. As we will show soon, all other transformations between hypergeometric equations with a finite monodromy group are special cases of classical transformations, expect the series of transformations of Sections 5, 6, and one degree 5 transformation between standard icosahedral and octahedral hypergeometric equations.

First we sketch the algorithm in [Vid08b] for computing Klein's pull-back transformations of hypergeometric equations with the tetrahedral, octahedral or icosahedral projective monodromy groups. The mentioned contiguous orbits of hypergeometric functions determine *Schwarz types* of algebraic $_2F_1$ functions. There is one dihedral, 2 tetrahedral, 2 octahedral and 10 icosahedral Schwarz types.

The algorithm in [Vid08b] uses explicit evaluation of algebraic Gauss hypergeometric functions, called *Darboux evaluations*. The geometric idea behind them is to pull-back a hypergeometric equation with a finite monodromy group to a Fuchsian differential equation with a cyclic monodromy group. Pull-backed hypergeometric solutions can be expressed in terms of radical functions, like in formulas (37)–(41) for dihedral functions. The minimal degree for these *Darboux pull-backs* to a cyclic monodromy group is 3, 4 or 5 for, respectively, tetrahedral, octahedral and icosahedral differential equations. The quadratic transformation $(1/2, 1/2, a) \xleftarrow{2} (1, a, a)$ in Section 6 is actually a Darboux pull-back in the dihedral case. Here are a few examples of Darboux evaluations for larger finite monodromy groups:

$$
_2F_1\left(\begin{matrix} 1/4, -1/12 \\ 2/3 \end{matrix} \,\middle|\, \frac{x(x+4)^3}{4(2x-1)^3}\right) = \frac{1}{(1-2x)^{1/4}},
$$

$$
_2F_1\left(\begin{matrix} 1/2, -1/6 \\ 2/3 \end{matrix} \,\middle|\, \frac{x(x+2)^3}{(2x+1)^3}\right) = \frac{1}{\sqrt{1+2x}},
$$

$$
_2F_1\left(\begin{matrix} 7/24, -1/24 \\ 3/4 \end{matrix} \,\middle|\, \frac{108x(x-1)^4}{(x^2+14x+1)^3}\right) = \frac{1}{(1+14x+x^2)^{1/8}},
$$

$$
_2F_1\left(\begin{matrix} 1/6, -1/6 \\ 1/4 \end{matrix} \,\middle|\, \frac{27x(x+1)^4}{2(x^2+4x+1)^3}\right) = \frac{(1+2x)^{1/4}}{\sqrt{1+4x+x^2}},
$$

$$
_2F_1\left(\begin{matrix} 13/60, -7/60 \\ 3/5 \end{matrix} \,\middle|\, \frac{1728x(x^2-11x-1)^5}{(x^4+228x^3+494x^2-228x+1)^3}\right)
$$

$$
= \frac{1-7x}{(1-228x+494x^2+228x^3+x^4)^{7/20}},
$$

$$_2F_1\left(\begin{matrix} 7/20, -1/20 \\ 4/5 \end{matrix}\ \middle|\ \frac{64x(x^2 - x - 1)^5}{(x^2 - 1)(x^2 + 4x - 1)^5}\right)$$

$$= \frac{(1 + x)^{7/20}}{(1 - x)^{1/20}(1 - 4x - x^2)^{1/4}}.$$

Some of minimal Darboux pull-back coverings for icosahedral functions are defined not on $\boldsymbol{P}_x^1$, but on a genus 1 curve. For example,

$$_2F_1\left(\begin{matrix} 8/15, -1/15 \\ 4/5 \end{matrix}\ \middle|\ \frac{54(\xi_1 + 5x)^3(1 - 2\xi_1 + 6x)^5}{(16x^2 - 1)(\xi_1 - 5x)^2(1 - 2\xi_1 - 14x)^5}\right)$$

$$= \frac{(1 + 4x)^{8/15}(\xi_1 + 5x)^{1/6}x^{1/15}}{(1 - 2\xi_1 - 14x)^{1/3}(\xi_1 - 3x)^{3/10}},$$

$$_2F_1\left(\begin{matrix} 7/10, -1/10 \\ 4/5 \end{matrix}\ \middle|\ \frac{16\xi_2(1 + x - x^2)^2(1 - \xi_2)^2}{(1 + \xi_2 + 2x)(1 + \xi_2 - 2x)^5}\right)$$

$$= \frac{(1 - \xi_2 + 2x)^{1/15}(1 - \xi_2)^{3/5}}{(1 + \xi_2 + 2x)^{7/30}\sqrt{1 + \xi_2 - 2x}},$$

where

$$\xi_1 = \sqrt{x(1 + x)(1 + 16x)} \qquad \text{and} \qquad \xi_2 = \sqrt{x(1 + x - x^2)}.$$

These formulas can be checked with a computer algebra package by expanding both sides in power series in x or $\sqrt{x}$. In [Vid08b], a few of these evaluations are computed for each Schwarz type of algebraic Gauss hypergeometric functions. Using contiguous relations, one can find a Darboux evaluation for any algebraic $_2F_1$ function. For comparison, in Section 6 we used general formulas (with terminating Appell's F_2 or F_3 sums) for dihedral $_2F_1$ functions, instead of applying contiguous relations.

A suitable ramification pattern for Klein's pull-back covering, say for a transformation $(1/2, 1/3, 1/4) \overset{d}{\leftarrow} (n + 1/2, m + 1/3, \ell + 1/4)$ of local exponent differences, is easy to set up. In the setting of Section 2, it is convenient to assume that $x = 0$ lies above $z = 0$ and assign local exponent differences with the largest denominator (say, 4) to these points. Hypergeometric solutions of the given and its standard hypergeometric equations at these points can be *a priori* identified (up to a constant multiple, at worst) by their local exponents. This gives identification of Schwarz maps (for both hypergeometric equations) up to a constant multiple. The constant multiple can be determined by a separate routine for each Schwarz type. Elimination of the variables in Darboux evaluations gives an algebraic relation between the arguments of

the given and its standard hypergeometric equations, which must give Klein's pull-back covering. The degree of Klein's pull-back covering for a transformation $(1/2, 1/3, 1/k) \overset{d}{\leftarrow} (e_0, e_1, e_\infty)$ can be computed from (13):

$$(46) \qquad d = \frac{6k}{6-k}(e_0 + e_1 + e_\infty - 1).$$

Here is a list of Klein's pull-back coverings computed in [Vid08b]. The triples of local exponent differences for the transformed equation are given on the left. The standard hypergeometric equations have the local exponent differences $(1/2, 1/3, 1/3)$ or $(1/2, 1/3, 1/5)$.

$$(1/2, 2/3, 2/3): \qquad z = -\frac{x^2(4x-5)^3}{(5x-4)^3},$$

$$(3/2, 1/3, 1/3): \qquad z = -\frac{x(x^2 - 42x - 7)^3}{(7x^2 + 42x - 1)^3},$$

$$(1/2, 1/3, 4/3): \qquad z = -\frac{x(256x^2 - 448x + 189)^3}{27(28x - 27)^3},$$

$$(1/2, 2/3, 4/3): \qquad z = \frac{19683x^2(4x-1)^3}{(256x^3 - 192x^2 + 21x - 4)^3},$$

$$(1/2, 1/3, 5/3): \qquad z = -\frac{19683x(128x - 125)^3}{(16384x^3 - 30720x^2 + 14880x - 625)^3},$$

$$(3/2, 1/3, 2/3): \qquad z = -\frac{729x(5x^2 + 14x + 125)^3}{(4x^3 + 15x^2 - 690x - 625)^3},$$

$$(1/3, 2/3, 5/3): \qquad z = \frac{4x(256x^3 - 640x^2 + 520x - 135)^3}{27(x-1)^2(32x - 27)^3},$$

$$(2/3, 2/3, 4/3): \qquad z = -\frac{x^2(x-1)^2(16x^2 - 16x + 5)^3}{4(5x^2 - 5x + 1)^3},$$

$$(2/3, 4/3, 4/3): \qquad z = -\frac{108x^4(x-1)^4(27x^2 - 27x + 7)^3}{(189x^4 - 378x^3 + 301x^2 - 112x + 16)^3},$$

$$(1/2, 2/3, 1/5): \qquad z = \frac{x(102400x^2 - 11264x - 11)^5}{(180224000x^3 + 4325376x^2 - 21252x + 1)^3},$$

$$(1/5, 1/5, 6/5):$$

$$z = \frac{108x(1-x)(512x^2 - 512x + 3)^5}{(1048576x^6 - 3145728x^5 + 3244032x^4 - 1245184x^3 + 94848x^2 + 3456x + 1)^3}.$$

Once a pull-back covering is known, hypergeometric identities are easy to derive. For example,

$$(47) \quad {}_2F_1\left(\begin{matrix} 1/4, -5/12 \\ 1/3 \end{matrix} \middle| x\right) = \left(1 - \frac{5x}{4}\right)^{1/4} {}_2F_1\left(\begin{matrix} 1/4, -1/12 \\ 2/3 \end{matrix} \middle| -\frac{x^2(4x-5)^3}{(5x-4)^3}\right),$$

$$(48) \quad {}_2F_1\left(\begin{matrix} -1/4, -7/12 \\ 2/3 \end{matrix} \middle| x\right)$$

$$= (1 - 42x - 7x^2)^{1/4} \, {}_2F_1\left(\begin{matrix} 1/4, -1/12 \\ 2/3 \end{matrix} \middle| -\frac{x(x^2 - 42x - 7)^3}{(7x^2 + 42x - 1)^3}\right).$$

Or similarly, let $z = \varphi_{14}(x)$ be the degree 14 covering for the $(2/3, 4/3, 4/3)$ tetrahedral case. A hypergeometric identity is:

$$(49) \quad {}_2F_1\left(\begin{matrix} -1/2, -7/6 \\ -1/3 \end{matrix} \middle| x\right) = \frac{(189x^4 - 378x^3 + 301x^2 - 112x + 16)^{1/4}}{2}$$

$$\times {}_2F_1\left(\begin{matrix} 1/4, -1/12 \\ 2/3 \end{matrix} \middle| \varphi_{14}(x)\right).$$

A list of other transformations between algebraic Gauss hypergeometric equations is not long. If the starting equation is not a standard (tetrahedral, octahedral or icosahedral) equation, at least one of the local exponent differences is effectively non-restricted, so we can only have special cases of classical transformations. If one of the possible monodromy groups (i.e., icosahedral, octahedral, tetrahedral, dihedral or cyclic, including trivial) can be a subgroup of another, there is a transformation between two standard hypergeometric equations with those monodromy groups. These transformations factor following the possible subgroup relations between the monodromy groups. The transformations that reduce the monodromy group to a largest proper subgroup are special cases of transformations considered in Sections 4 through 6, except the transformation between standard icosahedral and tetrahedral hypergeometric equations. The pull-back covering has degree 5:

$$\varphi_5(x) = \frac{50(5 + 3\sqrt{-15})x(1024x - 781 - 171\sqrt{-15})^3}{(128x + 7 + 33\sqrt{-15})^5}.$$

Here is a corresponding hypergeometric identity:

$$(50) \quad {}_2F_1\left(\begin{matrix} 1/4, -1/12 \\ 2/3 \end{matrix} \middle| x\right)$$

$$= \left(1 + \frac{7 - 33\sqrt{-15}}{128}x\right)^{1/12} {}_2F_1\left(\begin{matrix} 11/60, -1/60 \\ 2/3 \end{matrix} \middle| \varphi_5(x)\right).$$

This transformation is derived in [AK03, Section 5.1] as well. In general, if a standard hypergeometric equation is transformed to a (not necessary standard) hypergeometric equation with smaller monodromy group, that transformation must factor via the corresponding transformation between standard equations and Klein's transformation preserving the smaller monodromy group.

Remark 7.1. Hypergeometric equations with a finite monodromy group can be pull-backed to differential equations with the trivial monodromy group. Then algebraic Gauss hypergeometric functions are transformed to rational functions (perhaps on a higher genus curve). The minimal transformation degree for these transformations is the order of the monodromy group, which is 12, 24, 60 for tetrahedral, octahedral, icosahedral equations, respectively. If the transformed equation is hypergeometric, its degree is given by formula (13).

A priori, it seems possible that a pull-back to hypergeometric equation with the trivial monodromy group can have all its 3 singularities outside the fibers above $\{0, 1, \infty\} \subset \boldsymbol{P}_z^1$. This situation would be an exception to part 1 of Lemma 2.1: we would have a transformation between hypergeometric equations without two-term identities between their hypergeometric solutions. A simple candidate for such a pull-back transformation could transform the local exponent differences as $(1/2, 1/2, 1/2) \overset{10}{\leftarrow} (2, 2, 2)$; it would have 5 simple ramification points above each of the 3 points with the local exponent difference $1/2$. It two hypergeometric equations with the trivial monodromy group can be transformed to each other, they are related by a chain of transformations considered in Sections 4 through 7. In Klein's standard hypergeometric equations are involved, there is a unique (because of identification of Schwarz maps) transformation, which is a composition of considered transformations of standard equations reducing the projective monodromy group with Klein's transformation keeping the smallest monodromy group. Otherwise we have a classical transformation. As we observed, all these transformations allow two-term hypergeometric identities. (Particularly see the statement just before Remark 5.2). In the indicated composition of pull-back transformations, we can still have a two-term identity if we keep a fractional local exponent difference at $z = 0$ up till the last transformation (possibly acting on a hypergeometric equation with the trivial monodromy group). In particular, computations confirm that no pull-back covering for the transformation $(1/2, 1/2, 1/2) \overset{10}{\leftarrow} (2, 2, 2)$ exists.

8. Elliptic integrals

Here we consider algebraic transformations for solutions of hypergeometric equations with the local exponent differences $(1/2, 1/4, 1/4)$, or $(1/2, 1/3, 1/6)$,

or $(1/3, 1/3, 1/3)$. For each of these equations some of hypergeometric solutions are trivial (i.e., constants or power functions), while other solutions are incomplete elliptic integrals (up to a possible power factor). Here are representative interesting solutions of hypergeometric equations with the mentioned triples of local exponent differences:

$$(51) \qquad {}_2F_1\left(\begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix}\,\middle|\, z\right) = \frac{z^{-1/4}}{4}\int_0^z t^{-3/4}(1-t)^{-1/2}dt$$

$$= \frac{z^{-1/4}}{2}\int_{1/\sqrt{z}}^{\infty} \frac{dx}{\sqrt{x^3 - x}},$$

$$(52) \qquad {}_2F_1\left(\begin{matrix} 1/2, 1/6 \\ 7/6 \end{matrix}\,\middle|\, z\right) = \frac{z^{-1/6}}{6}\int_0^z t^{-5/6}(1-t)^{-1/2}dt$$

$$= \frac{z^{-1/6}}{2}\int_{1/\sqrt[3]{z}}^{\infty} \frac{dx}{\sqrt{x^3 - 1}},$$

$$(53) \qquad {}_2F_1\left(\begin{matrix} 1/3, 2/3 \\ 4/3 \end{matrix}\,\middle|\, z\right) = \frac{z^{-1/3}}{3}\int_0^z t^{-2/3}(1-t)^{-2/3}dt$$

$$= z^{-1/3}\int_{1/\sqrt[3]{z}}^{\infty} \frac{dx}{(x^3 - 1)^{2/3}}.$$

Here we substituted $t = x^{-2}$ or $t = x^{-3}$ into the immediate integral expressions. As we see, the three hypergeometric functions can be transformed to integrals of holomorphic forms on the genus 1 curves

$$(54) \qquad y^2 = x^3 - x, \qquad y^2 = x^3 - 1, \qquad x^3 + y^3 = 1,$$

respectively. In fact, the integrand functions define algebraic curves isomorphic respectively to these three cubic curves; see [Vid08c] for details. Let E_1, E_2, E_3 be three curves defined in (54), respectively. We consider them as elliptic curves (with the classical group structure) by fixing the point at infinity for E_1 and E_2, or the infinite point $(1 : -1 : 0)$ for E_3, as the neutral element of the group structure. The elliptic curves E_2 and E_3 are isomorphic. Non-trivial solutions of hypergeometric equations with the local exponent differences $(2/3, 1/6, 1/6)$ are genus 2 hyperelliptic integrals. For example,

$$ {}_2F_1\left(\begin{matrix} 1/3, 1/6 \\ 7/6 \end{matrix}\,\middle|\, z\right) = \frac{z^{-1/6}}{6}\int_0^z t^{-5/6}(1-t)^{-1/3}dt$$

$$= \frac{z^{-1/6}}{2^{1/3}}\int_{\theta(z)}^{\infty} \frac{X\, dX}{\sqrt{X^6 + 1}}, \qquad \text{where } \theta(z) = \frac{(1-z)^{1/3}}{2^{1/3}z^{1/6}}.$$

Here the substitution is $t \to (\sqrt{X^6 + 1} - X^3)^2$.

| Local exponent differences | | Degree | Branching above |
below	above	d	the regular singular points
$(1/2, 1/4, 1/4)$	$(1/2, 1/4, 1/4)$	$4n$	$2n * 2 = n * 4 = (n-1) * 4 + 2 + 1 + 1$
$(1/2, 1/4, 1/4)$	$(1/2, 1/4, 1/4)$	$4n + 1$	$2n * 2 + 1 = n * 4 + 1 = n * 4 + 1$
$(1/2, 1/4, 1/4)$	$(1/2, 1/4, 1/4)$	$4n + 2$	$(2n + 1) * 2 = n * 4 + 2 = n * 4 + 1 + 1$
$(1/2, 1/3, 1/6)$	$(1/2, 1/3, 1/6)$	$6n$	$3n * 2 = 2n * 3 = (n-1) * 6 + 3 + 2 + 1$
$(1/2, 1/3, 1/6)$	$(1/2, 1/3, 1/6)$	$6n + 1$	$3n * 2 + 1 = 2n * 3 + 1 = n * 6 + 1$
$(1/2, 1/3, 1/6)$	$(1/2, 1/3, 1/6)$	$6n + 3$	$(3n + 1) * 2 + 1 = (2n + 1) * 3 = n * 6 + 2 + 1$
$(1/2, 1/3, 1/6)$	$(1/2, 1/3, 1/6)$	$6n + 4$	$(3n + 2) * 2 = (2n + 1) * 3 + 1 = n * 6 + 3 + 1$
$(1/2, 1/3, 1/6)$	$(1/3, 1/3, 1/3)$	$6n$	$3n * 2 = 2n * 3 = (n-1) * 6 + 2 + 2 + 2$
$(1/2, 1/3, 1/6)$	$(1/3, 1/3, 1/3)$	$6n$	$3n * 2 = (2n - 1) * 3 + 1 + 1 + 1 = n * 6$
$(1/2, 1/3, 1/6)$	$(1/3, 1/3, 1/3)$	$6n + 2$	$(3n + 1) * 2 = 2n * 3 + 1 + 1 = n * 6 + 2$
$(1/2, 1/3, 1/6)$	$(1/3, 1/3, 1/3)$	$6n + 4$	$(3n + 2) * 2 = (2n + 1) * 3 + 1 = n * 6 + 2 + 2$
$(1/2, 1/3, 1/6)$	$(2/3, 1/6, 1/6)$	$6n$	$3n * 2 = 2n * 3 = (n-1) * 6 + 4 + 1 + 1$
$(1/2, 1/3, 1/6)$	$(2/3, 1/6, 1/6)$	$6n + 2$	$(3n + 1) * 2 = 2n * 3 + 2 = n * 6 + 1 + 1$
$(1/3, 1/3, 1/3)$	$(1/3, 1/3, 1/3)$	$3n$	$n * 3 = n * 3 = (n-1) * 3 + 1 + 1 + 1$
$(1/3, 1/3, 1/3)$	$(1/3, 1/3, 1/3)$	$3n + 1$	$n * 3 + 1 = n * 3 + 1 = n * 3 + 1$

Table 2: Transformations of hypergeometric elliptic integrals

Formula (13) gives no restriction on the degree d of pull-back transformations of the hypergeometric equations under consideration. But (13) requires that the transformed hypergeometric equation must have local exponent differences e_0, e_1, e_∞ such that $e_0 + e_1 + e_\infty = 1$. In particular, all 3 singularities of the transformed equation are relevant singularities, and the pull-back covering branches only above 3 points. Possible branching patterns are presented in Table 2. Multiplicative terms in the last column give the branching order (as the second multiplicant) and the number of points with that branching order in the same fiber (as the first multiplicant).

Coverings with the branching patterns of Table 2 give rise to morphisms between the corresponding (hyper)elliptic curves. For example, a pull-back transformation $(1/2, 1/3, 1/6) \leftarrow (1/3, 1/3, 1/3)$ of degree $6n + 2$ implies the polynomial identity

$$(55) \qquad R_{3n+1}(z)^2 = (z - 1) Q_{2n}(z)^3 - z^2 P_n(z)^6$$

for some polynomials $P_n(z)$, $Q_{2n}(z)$, $R_{3n+1}(z)$ of degree n, $2n$, $3n + 1$, respectively. This gives the following morphism from E_3 to E_2:

$$(56) \qquad (x, y) \mapsto \left(\frac{xy Q_{2n}(x^{-3})}{P_n(x^{-3})^2}, \frac{x^3 R_{3n+1}(x^{-3})}{P_n(x^{-3})^3} \right).$$

Conversely, a morphism (or endomorphism) between the elliptic curves relates the holomorphic differentials up to a constant multiple, and gives rise to a

transformation of their integrals. If the morphism fixes the upper integration bound ∞ in (51), (52), (53), we get a transformation of the hypergeometric functions as well. This correspondence is investigated thoroughly in [Vid08c]. Here we demonstrate it on several examples.

Most interesting are pull-back transformations of the three mentioned hypergeometric equations to themselves. These transformations correspond to isogeny endomorphisms of the elliptic curves E_1, E_2 or E_3. The ring of isogeny endomorphisms for E_1 is isomorphic to the ring $\mathbf{Z}[i]$ of Gaussian integers [Sil86]. The ring of isogeny endomorphisms for E_2 or E_3 is isomorphic to the ring $\mathbf{Z}[\omega]$, where ω is a primitive cubic root of unity as in (23). Composition of the isogenies corresponds to multiplication in the mentioned rings of algebraic integers. Roots of unity in both $\mathbf{Z}[i]$ and $\mathbf{Z}[\omega]$ correspond to trivial transformations of hypergeometric equations. The degree of a pull-back transformation induced by an endomorphism is equal to the $\mathbf{C}$-norm of the corresponding algebraic integer. In particular, there may be none or several transformations for a fixed degree and branching pattern from Table 2, depending on how many algebraic integers exist with that norm.

The transformations from the local exponent differences $(1/2, 1/3, 1/6)$ to the local exponent differences $(1/3, 1/3, 1/3)$ or $(2/3, 1/6, 1/6)$ are compositions of a classical quadratic transformation and the mentioned endomorphisms of elliptic curves; see [Vid08c, Section 5]. In particular, there are actually no transformations $(1/2, 1/3, 1/6) \leftarrow (1/3, 1/3, 1/3)$ of degree $6n + 4$, even if indicated in Table 2, because there are no transformations $(1/2, 1/3, 1/6) \leftarrow (1/2, 1/3, 1/6)$ of degree $3n + 2$.

Now we consider explicitly pull-back transformations coming from the endomorphisms of E_1. If $(x, y) \mapsto (\psi_x, \psi_y)$ is an isogeny endomorphism of E_1, then the substitution $x \mapsto \psi_x(x, \sqrt{x^3 - x})$ into (51) gives an integral of a holomorphic differential form again. Since the linear space of holomorphic differentials on E_1 is one-dimensional, the transformed differential form must be proportional to $dx/\sqrt{x^3 - x}$. The upper integration bound does not change. Transformation of the lower integration bound gives the transformation $z \mapsto \psi_x(1/\sqrt{z})^{-2}$ of the hypergeometric function into itself, up to a radical factor. Using induction and the addition law on E_1, one can prove [Vid08c, Theorem 2.1] that $\psi_x(1/\sqrt{z})^{-2}$ is a rational function for any isogeny endomorphism, and that its degree is equal to the degree of the isogeny. Conversely [Vid08c, Theorem 2.1], analysis of the first three branching patterns in Table 2 shows that any pull-back transformation $(1/2, 1/4, 1/4) \overset{d}{\leftarrow} (1/2, 1/4, 1/4)$ is induced by an endomorphism of E_1.

As mentioned, the ring of isogeny endomorphisms of E_1 is isomorphic to the ring $\mathbf{Z}[i]$ of Gaussian integers. We identify $i \in \mathbf{Z}[i]$ with the isogeny $(x, y) \mapsto (-x, iy)$. Addition of isogenies is equivalent to the chord-and-tangent

addition law on E_1. Here are a few examples of isogenies on E_1:

$$(x, y) \mapsto \left(\frac{x^2 - 1}{2ix}, \frac{y(x^2 + 1)}{2(i-1)x^2} \right), \quad (x, y) \mapsto \left(\frac{(x^2 + 1)^2}{4x(x^2 - 1)}, \frac{(x^2 + 1)(x^4 - 6x^2 + 1)}{8xy(x^2 - 1)} \right),$$

$$(x, y) \mapsto \left(\frac{x(x^2 - 1 - 2i)^2}{((1 + 2i)x^2 - 1)^2}, \frac{y(x^4 + (2 + 8i)x^2 + 1)(x^2 - 1 - 2i)}{((1 + 2i)x^2 - 1)^3} \right).$$

They correspond to the Gaussian integers $1 + i$, 2, $1 + 2i$, respectively. Here below are the induced algebraic transformations of Gauss hypergeometric functions:

$$(57) \quad {}_2F_1 \left(\begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \middle| z \right) = \frac{1}{\sqrt{1 - z}} \, {}_2F_1 \left(\begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \middle| -\frac{4z}{(z - 1)^2} \right).$$

$$(58) \quad {}_2F_1 \left(\begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \middle| z \right) = \frac{\sqrt{1 - z}}{1 + z} \, {}_2F_1 \left(\begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \middle| \frac{16z(z - 1)^2}{(z + 1)^4} \right).$$

$$(59) \quad {}_2F_1 \left(\begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \middle| z \right) = \frac{1 - z/(1 + 2i)}{1 - (1 + 2i)z} \, {}_2F_1 \left(\begin{matrix} 1/2, 1/4 \\ 5/4 \end{matrix} \middle| \frac{z(z - 1 - 2i)^4}{((1 + 2i)z - 1)^4} \right).$$

The first two identities are special cases of classical transformations. The radical factors on a right-hand side are equal to $[\psi_x(1/\sqrt{z})^{-2}]^{1/4} z^{-1/4}$ times a constant (deducible from the value of hypergeometric series at $z = 0$). The transformations $(1/2, 1/4, 1/4) \overset{d}{\leftarrow} (1/2, 1/4, 1/4)$ form a semi-group under composition, isomorphic to the multiplicative semi-group $\mathbf{Z}[i]^\star/(\pm 1, \pm i)$. The degree of these transformations is equal to the norm $p^2 + q^2$ of a corresponding Gaussian integer $p + qi$. In particular, there are no transformations of degree 21 although Table 2 allows it, because there are no Gaussian integers with this norm. On the other hand, there are several different transformations of degree 25, corresponding to $3 \pm 4i$ or 5. One of them is the composition of (59) with itself, the other is the composition of (59) with the complex conjugate of itself. In fact, algebraic transformations related by the complex conjugation are not related by fractional-linear transformations in general. The addition law on E_1 can be translated into "addition" of the polynomial triples determining the branching points (of order 2 or 4) of explicit pull-back coverings for the first three branching patterns in Table 2; see [Vid08c, Section 2].

Likewise, an isogeny endomorphism on E_2 transforms the holomorphic differential form in (52) into a scalar multiple of itself, and the upper integration bound does not change. The lower integration bound changes as $z \mapsto \psi_x(z^{-1/3})^{-3}$. By induction and the addition law on E_2, this is a rational function determining a desired pull-back covering, and its degree is equal to the degree of the isogeny [Vid08c, Section 3]. Conversely, analysis of respective

cases of Table 2 shows that any pull-back transformation $(1/2, 1/3, 1/6) \overset{d}{\leftarrow}$ $(1/2, 1/3, 1/6)$ is induced by an endomorphism of E_2. These transformations form a semi-group under composition, isomorphic to $\mathbf{Z}[\omega]^*/(\pm 1, \pm \omega, \pm \omega^{-1})$. We identify the cubic root ω with the isogeny $(x, y) \mapsto (\omega x, y)$. Here are examples of explicit transformations corresponding to the algebraic integers $1 - \omega$, 3, $3 + \omega$ of $\mathbf{Z}[\omega]$:

$$(60) \quad {}_2F_1\left(\begin{matrix} 1/2, 1/6 \\ 7/6 \end{matrix} \middle| z \right) = \frac{1}{\sqrt{1 - 4z}} \, {}_2F_1\left(\begin{matrix} 1/2, 1/6 \\ 7/6 \end{matrix} \middle| \frac{27z}{(4z - 1)^3} \right),$$

$$(61) \quad {}_2F_1\left(\begin{matrix} 1/2, 1/6 \\ 7/6 \end{matrix} \middle| z \right) = \frac{1 - 4z}{\sqrt{1 + 96z + 48z^2 - 64z^3}}$$
$$\times \, {}_2F_1\left(\begin{matrix} 1/2, 1/6 \\ 7/6 \end{matrix} \middle| \frac{-729z(4z - 1)^6}{(64z^3 - 48z^2 - 96z - 1)^3} \right),$$

$$(62) \quad {}_2F_1\left(\begin{matrix} 1/2, 1/6 \\ 7/6 \end{matrix} \middle| z \right) = \frac{1 - 4z/(3\omega + 1)}{\sqrt{1 - (44 + 48\omega)z + (48\omega + 16)z^2}}$$
$$\times \, {}_2F_1\left(\begin{matrix} 1/2, 1/6 \\ 7/6 \end{matrix} \middle| \frac{z(4z - 3\omega - 1)^6}{((48\omega + 16)z^2 - (44 + 48\omega)z + 1)^3} \right).$$

Similarly, transformations $(1/3, 1/3, 1/3) \overset{d}{\leftarrow} (1/3, 1/3, 1/3)$ correspond to the isogeny endomorphisms on E_3. Recall that this elliptic curve is isomorphic to E_2. With the chosen addition law on E_3 and identification of a hypergeometric solution as the integral in (53), the isogeny of multiplication by $-1 \in \mathbf{Z}[\omega]$ corresponds to Euler's transformation (16). Transformations of the hypergeometric function for (53) into itself form a semi-group (under composition) isomorphic to $\mathbf{Z}[\omega]^*/(1, \omega, \omega^{-1})$. We identify the cubic root ω with the isogeny $(x, y) \mapsto (\omega^{-1}x, \omega^{-1}y)$. Here are explicit transformations corresponding to $2, 3, 3 + \omega \in \mathbf{Z}[\omega]$:

$$(63) \quad {}_2F_1\left(\begin{matrix} 1/3, 2/3 \\ 4/3 \end{matrix} \middle| z \right) = \frac{1 - z/2}{1 - 2z} \, {}_2F_1\left(\begin{matrix} 1/3, 2/3 \\ 4/3 \end{matrix} \middle| \frac{z(z - 2)^3}{(1 - 2z)^3} \right),$$

$$(64) \quad {}_2F_1\left(\begin{matrix} 1/3, 2/3 \\ 4/3 \end{matrix} \middle| z \right) = \frac{(1 - z + z^2)(1 - z)^{1/3}}{1 + 3z - 6z^2 + z^3}$$
$$\times \, {}_2F_1\left(\begin{matrix} 1/3, 2/3 \\ 4/3 \end{matrix} \middle| \frac{27z(z - 1)(z^2 - z + 1)^3}{(z^3 - 6z^2 + 3z + 1)^3} \right),$$

$$(65) \quad {}_2F_1\left(\begin{array}{c} 1/3, 2/3 \\ 4/3 \end{array}\middle| z\right) = \frac{1 - z - z^2/(3\omega + 2)}{1 + (3\omega + 2)z - (3\omega + 2)z^2}$$

$$\times {}_2F_1\left(\begin{array}{c} 1/3, 2/3 \\ 4/3 \end{array}\middle| \frac{z(z^2 + (3\omega + 2)z - 3\omega - 2)^3}{(1 + (3\omega + 2)z - (3\omega + 2)z^2)^3}\right).$$

An explicit transformation corresponding to $1 - \omega$ can be obtained from (23) with $a = 1$.

As mentioned, transformations from the local exponent differences $(1/2, 1/3, 1/6)$ to the local exponent differences $(1/3, 1/3, 1/3)$ or $(2/3, 1/6, 1/6)$ are compositions of a classical quadratic transformation and the mentioned endomorphisms of elliptic curves. Correspondingly, the morphisms from E_3 or the hyperelliptic curve $Y^2 = X^6 + 1$ to E_2, that leave the infinite points at infinity, factor via isogeny endomorphisms of E_2 and the straightforward morphisms $(x, y) \mapsto (2^{2/3}xy, i - 2ix^3)$ or $(X, Y) \mapsto (-X^2, iY)$, respectively.

9. Hyperbolic hypergeometric functions

Transformations of hyperbolic hypergeometric functions are extensively studied in [Vid05]. There are 9 non-classical transformations in this case, of degree 6, 8, 9, 10, 12, 18 or 24.

Without loss of generality, we may assume $k_1 \leq k_2 \leq k_3 \leq d$. Inequality (15) together with $1/k_1 + 1/k_2 + 1/k_3 < 1$ already implies finitely many possibilities for the tuple (k_1, k_2, k_3, d). Indeed, inequality (15) gives a bound for d when k_1, k_2, k_3 are fixed; then $k_3 \leq d$ gives a bound for k_3 when k_1, k_2 are fixed, etc. But stronger inequalities and conditions follow from [Vid05, Lemma 2.2]. First of all, the transformed equation must have precisely 3 singular points, and the covering $\varphi : P_z^1 \to P_x^1$ branches only above the set $\{0, 1, \infty\} \subset P_z^1$. Then we consequently derive:

$$d - \left\lfloor \frac{d}{k_1} \right\rfloor - \left\lfloor \frac{d}{k_2} \right\rfloor - \left\lfloor \frac{d}{k_3} \right\rfloor = 1, \qquad d\left(1 - \frac{1}{k_1} - \frac{1}{k_2} - \frac{1}{k_3}\right) \leq 1 - \frac{3}{k_3},$$

$$\left(1 - \frac{1}{k_1} - \frac{1}{k_2}\right)k_3^2 - 2k_3 + 3 \leq 0, \qquad \frac{2}{3} \leq \frac{1}{k_1} + \frac{1}{k_2} < 1.$$

With these stronger formulas we get a moderate list of possibilities after Step 2 of our classification scheme. The list of possible branching patterns after Step 3 is presented by the first three columns of Table 3. The branching patterns are determined by the two triples of local exponent differences and the principle that each fiber of $\{0, 1, \infty\} \subset P_z^1$ contains maximal possible number of non-singular points. For each branching pattern there is at most one covering.

Local exponent differences		Degree d	Covering composition	Coxeter decomposition
$(1/k, 1/\ell, 1/m)$	above			
$(1/2, 1/3, 1/7)$	$(1/3, 1/3, 1/7)$	8	indecomposable	no
$(1/2, 1/3, 1/7)$	$(1/2, 1/7, 1/7)$	9	indecomposable	no
$(1/2, 1/3, 1/7)$	$(1/3, 1/7, 2/7)$	10	indecomposable	yes
$(1/2, 1/3, 1/7)$	$(1/7, 1/7, 3/7)$	12	no covering	
$(1/2, 1/3, 1/7)$	$(1/7, 2/7, 2/7)$	12	no covering	
$(1/2, 1/3, 1/7)$	$(1/3, 1/7, 1/7)$	16	no covering	
$(1/2, 1/3, 1/7)$	$(1/7, 1/7, 2/7)$	18	2×9	no
$(1/2, 1/3, 1/7)$	$(1/7, 1/7, 1/7)$	24	3×8	yes
$(1/2, 1/3, 1/8)$	$(1/3, 1/8, 1/8)$	10	indecomposable	no
$(1/2, 1/3, 1/8)$	$(1/4, 1/8, 1/8)$	12	$2 \times 2 \times 3$	yes
$(1/2, 1/3, 1/9)$	$(1/9, 1/9, 1/9)$	12	3×4	no
$(1/2, 1/4, 1/5)$	$(1/4, 1/4, 1/5)$	6	indecomposable	no
$(1/2, 1/4, 1/5)$	$(1/5, 1/5, 1/5)$	8	no covering	

Table 3: Transformations of hyperbolic hypergeometric functions

The coverings were computed by the algorithm in [Vid05, Section 3]; they are characterized in the fourth column of Table 3. The last column indicates existence of Coxeter decompositions described at the end of Section 3. The three cases which admit a Coxeter decomposition are implied in [Hod20] and [Beu02].

Here we give rational functions defining the indecomposable pull-back transformations, and examples of corresponding algebraic transformations of Gauss hypergeometric functions.

- $(1/2, 1/3, 1/7) \xleftarrow{8} (1/3, 1/3, 1/7)$. This transformation was independently computed numerically in [AK03], and later fully presented in [Kit03]. Let ω denote a primitive cubic root of unity as in (23). The covering and an algebraic transformation are:

$$(66) \qquad \varphi_8(x) = \frac{x(x-1)(27x^2 - (723 + 1392\omega)x - 496 + 696\omega)^3}{64((6\omega + 3)x - 8 - 3\omega)^7},$$

$$(67) \qquad {}_2F_1\left(\begin{matrix} 2/21, 5/21 \\ 2/3 \end{matrix} \middle| x\right) = \left(1 - \frac{33 + 39\omega}{49}x\right)^{-1/12} {}_2F_1\left(\begin{matrix} 1/84, 13/84 \\ 2/3 \end{matrix} \middle| \varphi_8(x)\right).$$

Note that the conjugation $\omega = -1 - \omega$ acts in the same way as a composition with fractional-linear transformation interchanging the points $x = 0$ and $x = 1$. This confirms uniqueness of the covering.

- $(1/2, 1/3, 1/7) \xleftarrow{9} (1/2, 1/7, 1/7)$. Let ξ denote an algebraic number satisfying $\xi^2 + \xi + 2 = 0$. The covering and an algebraic transformation are:

$$(68) \quad \varphi_9(x) = \frac{27x(x-1)(49x - 31 - 13\xi)^7}{49(7203x^3 + (9947\xi - 5831)x^2 - (9947\xi + 2009)x + 275 - 87\xi)^3},$$

$$(69) \quad {}_2F_1\left(\begin{array}{c} 3/28, 17/28 \\ 6/7 \end{array}\bigg| x\right)$$

$$= \left(1 + \frac{7(10 - 29\xi)}{8}x - \frac{343(50 - 29\xi)}{512}x^2 + \frac{1029(362 + 87\xi)}{16384}x^3\right)^{-1/28}$$

$$\times {}_2F_1\left(\begin{array}{c} 1/84, 29/84 \\ 6/7 \end{array}\bigg| \varphi_9(x)\right).$$

- $(1/2, 1/3, 1/7) \overset{10}{\leftarrow} (1/3, 1/7, 2/7)$. This transformation was independently computed in [Kit03] as well. The covering and an algebraic transformation are:

$$(70) \quad \varphi_{10}(x) = -\frac{x^2(x-1)(49x - 81)^7}{4(16807x^3 - 9261x^2 - 13851x + 6561)^3},$$

$$(71) \quad {}_2F_1\left(\begin{array}{c} 5/42, 19/42 \\ 5/7 \end{array}\bigg| x\right) = \left(1 - \frac{19}{9}x - \frac{343}{243}x^2 + \frac{16807}{6561}x^3\right)^{-1/28}$$

$$\times {}_2F_1\left(\begin{array}{c} 1/84, 29/84 \\ 6/7 \end{array}\bigg| \varphi_{10}(x)\right).$$

- $(1/2, 1/3, 1/8) \overset{10}{\leftarrow} (1/3, 1/8, 1/8)$. Let β denote an algebraic number satisfying $\beta^2 + 2 = 0$. The covering and an algebraic transformation are:

$$(72) \quad \tilde{\varphi}_{10}(x) = \frac{4x(x-1)(8\beta x + 7 - 4\beta)^8}{(2048\beta x^3 - 3072\beta x^2 - 3264x^2 + 912\beta x + 3264x + 56\beta - 17)^3},$$

$$(73) \quad {}_2F_1\left(\begin{array}{c} 5/24, 13/24 \\ 7/8 \end{array}\bigg| x\right)$$

$$= \left(1 + \frac{16(4 - 17\beta)}{9}x - \frac{64(167 - 136\beta)}{243}x^2 + \frac{2048(112 - 17\beta)}{6561}x^3\right)^{-1/16}$$

$$\times {}_2F_1\left(\begin{array}{c} 1/48, 17/48 \\ 7/8 \end{array}\bigg| \tilde{\varphi}_{10}(x)\right).$$

- $(1/2, 1/4, 1/5) \overset{6}{\leftarrow} (1/4, 1/4, 1/5)$. The covering and an algebraic transformation are:

$$(74) \quad \varphi_6(x) = \frac{4ix(x-1)(4x - 2 - 11i)^4}{(8x - 4 + 3i)^5},$$

$$(75) \qquad {}_2F_1\left(\begin{matrix} 3/20, 7/20 \\ 3/4 \end{matrix}\,\middle|\, x\right) = \left(1 - \frac{8(4+3i)}{25}x\right)^{-1/8} {}_2F_1\left(\begin{matrix} 1/40, 9/40 \\ 3/4 \end{matrix}\,\middle|\, \varphi_6(x)\right).$$

The composite transformations can be schematically represented similarly as in Section 4:

$$(1/2, 1/3, 1/7) \overset{9}{\leftarrow} (1/2, 1/7, 1/7) \overset{2}{\leftarrow} (1/7, 1/7, 2/7),$$

$$(1/2, 1/3, 1/7) \overset{8}{\leftarrow} (1/3, 1/3, 1/7) \overset{3}{\leftarrow} (1/7, 1/7, 1/7),$$

$$(1/2, 1/3, 1/8) \overset{3}{\leftarrow} (1/2, 1/4, 1/8) \overset{2}{\leftarrow} (1/2, 1/8, 1/8) \overset{2}{\leftarrow} (1/4, 1/8, 1/8),$$

$$(1/2, 1/3, 1/9) \overset{4}{\leftarrow} (1/3, 1/3, 1/9) \overset{3}{\leftarrow} (1/9, 1/9, 1/9).$$

Note that the transformation of degree 24 admits a Coxeter decomposition, although it is a composition of two transformations without a Coxeter decomposition. Here is an explicit algebraic transformation of degree 24:

$$(76) \qquad {}_2F_1\left(\begin{matrix} 2/7, 3/7 \\ 6/7 \end{matrix}\,\middle|\, x\right)$$

$$= (1 - 235x + 1430x^2 - 1695x^3 + 270x^4 + 229x^5 + x^6)^{-1/28}$$

$$\times (1 - x + x^2)^{-1/28}\, {}_2F_1\left(\begin{matrix} 1/84, 29/84 \\ 6/7 \end{matrix}\,\middle|\, \varphi_{24}(x)\right),$$

where

$$\varphi_{24}(x) = \frac{1728\,x(x-1)(x^3 - 8x^2 + 5x + 1)^7}{(x^2 - x + 1)^3(x^6 + 229x^5 + 270x^4 - 1695x^3 + 1430x^2 - 235x + 1)^3}.$$

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