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Existence Results for Second Order Differential Equations with Nonlocal Conditions in Banach Spaces

By

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Abstract. This work is concerned with implicit second order abstract differential equations with nonlocal conditions. Assuming that the involved operators satisfy some compactness properties, we establish the existence of local mild solutions, the existence of global mild solutions and the existence of asymptotically almost periodic solutions.

Key Words and Phrases. Second-order abstract Cauchy problem, Differential equations in abstract spaces, Nonlocal conditions, Cosine function of operators, Almost periodic functions.

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1. Introduction

This paper is devoted to the study of existence of mild solutions for initial value problems described as an implicit second order abstract differential equations with nonlocal conditions. Specifically, we are concerned with problems that can be modeled as an abstract Cauchy problem on a Banach space X of the form

$$\frac{d}{dt}(x'(t) - g(t, x(t), x'(t))) = Ax(t) + f(t, x(t), x'(t)), \qquad t \in I, \tag{1.1}$$

$$x(0) = x_0 + p(x, x'), \tag{1.2}$$

$$x'(0) = y_0 + q(x, x'), \tag{1.3}$$

where A is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators on X, I can be the interval $[0, a]$ or the unbounded interval $[0, \infty)$, $g(\cdot), f(\cdot) : I \times X^2 \to X$, and $p(\cdot), q(\cdot) : C(I, X)^2 \to X$ are appropriate functions.

System (1.1)–(1.3) is simultaneously a generalization of the classical second order abstract Cauchy problem, and a generalization of some systems studied

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recently by Staněk in [14, 15, 16, 17]. In these works, Staněk studied different problems related to the existence of solutions for a class of second order functional differential equations modeled of the form

$$(x'(t) + g(t, x(t), x'(t)))' = f(t, x(t), x'(t)), \qquad t \in I = [0, T], \tag{1.4}$$

$$r(x(0), x'(0), x(T)) = \varphi(x), \tag{1.5}$$

$$\omega(x(0), x(T), x'(T)) = \psi(x), \tag{1.6}$$

where $r, \omega : \boldsymbol{R}^3 \to \boldsymbol{R}$, $f(\cdot), g(\cdot) : I \times \boldsymbol{R}^2 \to \boldsymbol{R}$, $\varphi, \psi : C^0(I, \boldsymbol{R}) \to \boldsymbol{R}$ are appropriate functions, and $C^0(I, \boldsymbol{R})$ denotes an adequate subspace of the space of continuous functions $C(I, \boldsymbol{R})$. It is important to observe that the problem (1.4)–(1.6) does not include partial evolution equations. This fact is the main motivation of this paper.

Throughout this work, A denotes the infinitesimal generator of a strongly continuous cosine function $C(t)$ of bounded linear operators on X and $S(t)$ is the sine function associated with $C(t)$, which is defined by $S(t)x = \int_0^t C(s)x\,ds$, $x \in X$, $t \in \boldsymbol{R}$. We refer the reader to [2] for the basic concepts about cosine functions. We next only mention a few properties and notations needed to establish our results. We represent by $[D(A)]$ the domain of A endowed with the graph norm $\|x\|_A = \|x\| + \|Ax\|$, $x \in D(A)$. Moreover, the notation E stands for the space consisting of vectors $x \in X$ for which the function $C(\cdot)x$ is of class C^1. It was proved by Kisińsky [7] that E endowed with the norm

$$\|x\|_1 = \|x\| + \sup_{0 \le t \le 1} \|AS(t)x\|, \qquad x \in E,$$

is a Banach space. The operator valued function

$$G(t) = \begin{bmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{bmatrix}$$

is a strongly continuous group of bounded linear operators on the space $E \times X$, generated by the operator

$$\mathscr{A} = \begin{bmatrix} 0 & I \\ A & 0 \end{bmatrix}$$

defined on $D(A) \times E$. It follows from this property that $AS(t) : E \to X$ is a bounded linear operator, and that $AS(t)x \to 0$, $t \to 0$, for each $x \in E$. Furthermore, if $x : [0, \infty) \to X$ is a locally integrable function, then $y(t) = \int_0^t S(t-s)x(s)ds$ defines an E-valued continuous function.

Some important properties of the second order abstract Cauchy problem were studied in Travis and Webb [18, 19]. Specifically, the existence of solutions for the second order abstract Cauchy problem

$$(1.7) \qquad x''(t) = Ax(t) + h(t), \qquad 0 \le t \le a,$$

$$(1.8) \qquad x(0) = x_0, \qquad x'(0) = y_0,$$

where $h : [0, a] \to X$ is an integrable function, was discussed in [18]. Similarly, the existence of solutions for the semilinear second order abstract Cauchy problem was considered in [19]. We only mention here that the function $x(\cdot)$ given by

$$(1.9) \qquad x(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)h(s)ds, \qquad 0 \le t \le a,$$

is called mild solution of (1.7)–(1.8). Moreover, when $x_0 \in E$, the function $x(\cdot)$ is continuously differentiable and

$$x'(t) = AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)h(s)ds.$$

This paper has four sections. In section 2, we discuss the existence of mild solutions for some partial second order differential problems with nonlocal conditions in bounded intervals. In section 3, we are concerned with the existence of global and almost-periodic solutions in the interval $[0, \infty)$. In the last section, we apply our results to study a pair of concrete situations.

The terminology and notations are those generally used in functional analysis. In particular, if $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ are Banach spaces, we denote by $\mathscr{L}(Y, Z)$ the Banach space of the bounded linear operators from Y into Z and, we abbreviate this notation to $\mathscr{L}(Y)$ whenever $Z = Y$. Throughout this paper, $B_r(z, Z)$ denotes the closed ball with center at z and radius $r > 0$ in the space Z. Additionally, for a bounded function $\xi : [0, a] \to [0, \infty)$ and $0 \le t \le a$, we will use the notation $\xi_t = \sup\{\xi(s) : s \in [0, t]\}$.

To complete these remarks, we mention that most of our proofs are based on the following well known result ([3, Theorem 6.5.4]).

Lemma 1.1. *Let D be a closed convex subset of a Banach space X such that $0 \in D$. Let $F : D \to D$ be a completely continuous map. Then the set $\{x \in D : x = \lambda F(x), 0 < \lambda < 1\}$ is unbounded or the map F has a fixed point in D.*

2. Existence of mild solutions

In this section we are concerned with initial value problems defined on a bounded interval $I = [0, a]$. We denote by $N \ge 1$ and $\tilde{N} \ge 0$ certain constants such that $\|C(t)\| \le N$ and $\|S(t)\| \le \tilde{N}$ for every $t \in I$. Furthermore, we represent by $\tilde{N}_1 = \sup_{0 \le t \le a} \|AS(t)\|$, when $AS(t)$ is considered as an operator in the space $\mathscr{L}(E,X)$. As usual, we write $C(I,X)$ for the space of continuous functions from I into X endowed with the norm of uniform convergence.

We begin by studying the initial value problem

$$\frac{d}{dt}[x'(t)-g(t,x(t))] = Ax(t)+f(t,x(t)), \qquad t\in I, \tag{2.1}$$

$$x(0) = x_0 + p(x), \tag{2.2}$$

$$x'(0) = y_0 + q(x), \tag{2.3}$$

where $x_0, y_0 \in X$ and $p(\cdot), q(\cdot) : C(I,X) \to X$ are continuous functions that take closed bounded sets into bounded sets. We set $N_p(r) = \sup\{\|p(x)\| : x \in C(I,X), \|x\| \le r\}$ and $N_q(r) = \sup\{\|q(x)\| : x \in C(I,X), \|x\| \le r\}$.

We assume that f and g fulfill the following general properties.

(H-1) The functions $f,g : I\times X \to X$ satisfy the Carathéodory conditions:
   (i) $f(t,\cdot), g(t,\cdot) : X \to X$ are continuous a.e. $t\in I$.
   (ii) For each $x\in X$ the functions $f(\cdot,x), g(\cdot,x) : I \to X$ are strongly measurable.

(H-2) There exist integrable functions $m_f, m_g : I \to [0,\infty)$ and there exist continuous nondecreasing functions $W_f, W_g : [0,\infty) \to (0,\infty)$ such that

$$\|f(t,x)\| \le m_f(t)W_f(\|x\|), \qquad (t,x)\in I\times X,$$

$$\|g(t,x)\| \le m_g(t)W_g(\|x\|), \qquad (t,x)\in I\times X.$$

We next abbreviate the exposition by writing $W = \max\{W_f, W_g\}$.

When $p(\cdot)$ is bounded on $C(I,X)$, we denote $N_p = \sup\{\|p(x)\| : x \in C(I,X)\}$. Similarly, when $q(\cdot)$ is bounded on $C(I,X)$ we set $N_q = \sup\{\|q(x)\| : x\in C(I,X)\}$. In the case both $p(\cdot)$ as $q(\cdot)$ are bounded, we denote

$$c = N(\|x_0\| + N_p) + \tilde{N}(\|y_0\| + N_q + c_0), \tag{2.4}$$

where $c_0 = \sup\{\|g(0,y)\| : \|y\| \le \|x_0\| + N_p\}$.

By comparing with the expression (1.9), we introduce the following concept of mild solution.

Definition 2.1. A continuous function $x : I \to X$ is said to be a mild solution of the problem (2.1)–(2.3) if the integral equation

$$x(t) = C(t)(x_0+p(x)) + S(t)[y_0 - g(0,x(0)) + q(x)] + \int_0^t C(t-s)g(s,x(s))ds + \int_0^t S(t-s)f(s,x(s))ds, \qquad t\in I,$$

is verified.

We are now in a position to establish our first result of existence.

Theorem 2.1. *Assume that conditions* (H-1) *and* (H-2) *are verified and that*

$$\int_0^a (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W(s)}. \tag{2.5}$$

Suppose, furthermore, that the following conditions hold.

(a) *For every $t > 0$ and $r > 0$, the sets $U(t,r) = \{S(t)f(s,x) : s \in [0,a], \|x\| \le r\}$ and $U_1(t,r) = \{g(s,x) : s \in [0,t], \|x\| \le r\}$ are relatively compact in X.*

(b) *The function $p(\cdot)$ is bounded and completely continuous.*

(c) *The function $q(\cdot)$ is bounded and for every $t \in I$ and every $r > 0$ the set $V(t,r) = \{S(t)q(x) : \|x\| \le r\}$ is relatively compact in X.*

Then there exists a mild solution of problem (2.1)–(2.3).

Proof. We define the map $\Gamma : C(I,X) \to C(I,X)$ by

$$\begin{aligned} \Gamma x(t) = {} & C(t)(x_0 + p(x)) + S(t)[y_0 + q(x) - g(0,x(0))] \\ & + \int_0^t C(t-s)g(s,x(s))ds + \int_0^t S(t-s)f(s,x(s))ds \end{aligned} \tag{2.6}$$

for $t \in I$. Clearly Γ is well defined and a standard application of the Lebesgue dominated convergence theorem allows us to assert that Γ is continuous.

In order to use Lemma 1.1, we obtain an a priori bound for the solutions of the integral equation $x = \lambda\Gamma(x)$, $\lambda \in (0,1)$. Let $x^\lambda \in C(I,X)$ be a solution of $x^\lambda = \lambda\Gamma(x^\lambda)$, $\lambda \in (0,1)$. Using the previous notations, we get

$$\begin{aligned} \|x^\lambda(t)\| \le {} & N(\|x_0\| + N_p) + \tilde{N}(\|y_0\| + N_q + \|g(0,x^\lambda(0))\|) \\ & + \int_0^t (Nm_g(s) + \tilde{N}m_f(s))W(\|x^\lambda(s)\|)ds. \end{aligned}$$

Denoting by $\beta_\lambda(t)$ the right hand side of the last inequality, we get that

$$\beta_\lambda'(t) \le (Nm_g(t) + \tilde{N}m_f(t))W(\beta_\lambda(t)).$$

In view of that $x^\lambda(0) = \lambda\Gamma(x^\lambda)(0) = \lambda(x_0 + p(x^\lambda))$, we have that

$$\|g(0,x^\lambda(0))\| \le \sup\{\|g(0,y)\| : \|y\| \le \|x_0\| + N_p\}$$

and $\beta_\lambda(0) \le c$. This yields that

$$\int_c^{\beta_\lambda(t)} \frac{ds}{W(s)} \le \int_{\beta_\lambda(0)}^{\beta_\lambda(t)} \frac{ds}{W(s)} \le \int_0^t (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W(s)}.$$

If we suppose that the set $\{\beta_\lambda : \lambda \in (0,1)\}$ is not bounded, from the preceding estimate we get

$$\int_c^\infty \frac{ds}{W(s)} \leq \int_0^a (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W(s)}$$

which is a contradiction. Consequently, $\{\beta_\lambda : \lambda \in (0,1)\}$ is a bounded set and, as an immediate consequence, we infer that $\{x^\lambda : \lambda \in (0,1)\}$ is bounded in $C(I,X)$.

In what follows, we prove that Γ is completely continuous. To this end, we introduce the decomposition $\Gamma = \Gamma_1 + \Gamma_2$ where

$$\Gamma_1 u(t) = C(t)(x_0 + p(u)) + S(t)[y_0 + q(u) - g(0,u(0))], \qquad t \in I,$$

$$\Gamma_2 u(t) = \int_0^t C(t-s)g(s,u(s))ds + \int_0^t S(t-s)f(s,u(s))ds, \qquad t \in I.$$

It is easy to see that our hypotheses imply that Γ_1 is completely continuous. We will next prove that Γ_2 takes bounded sets into relatively compact ones. To abbreviate our notations, we set $B_r = B_r(0, C(I,X))$.

Initially we shall show that the set $\Gamma_2 B_r = \{\Gamma_2 x : x \in B_r\}$ is equicontinuous on I. For fixed $t \in I$ and $\varepsilon > 0$, since $C(\cdot)$ is strongly continuous, from the condition (a) we obtain that there is $\delta > 0$ such that

$$\|(C(t+h-s) - C(t-s))g(s,x(s))\| < \varepsilon, \qquad x \in B_r,\ s \in [0,t],$$

when $|h| \leq \delta$. Furthermore, since the sine function verifies the Lipschitz condition $\|S(t_1) - S(t_2)\| \leq N|t_1 - t_2|$, for $x \in B_r$ and $|h| \leq \delta$ with $t + h \in I$, we can estimate

$$\begin{aligned}
&\|\Gamma_2 x(t+h) - \Gamma_2 x(t)\| \\
&\quad \leq \int_0^t \|(C(t+h-s) - C(t-s))g(s,x(s))\|ds + N\int_t^{t+h} \|g(s,x(s))\|ds \\
&\qquad + \int_0^t \|S(t+h-s) - S(t-s)\|\,\|f(s,x(s))\|ds + \tilde{N}\int_t^{t+h} \|f(s,x(s))\|ds \\
&\quad \leq \varepsilon t + W(r)\int_t^{t+h} (Nm_g(s) + \tilde{N}m_f(s))ds + NhW(r)\int_0^t m_f(s)ds,
\end{aligned}$$

which establishes the assertion.

We next prove that $\Gamma_2 B_r(t) = \{\Gamma_2 x(t) : x \in B_r\}$ is a relatively compact set in X for every $t \in I$. For fixed $t \in I$ and $\varepsilon > 0$, since $U_1(t,r)$ is a relatively compact set and $C(\cdot)$ is strongly continuous, we have that $U_2(t,r) = \{C(t-s)g(s,x(s)) : 0 \leq s \leq t, x \in B_r\}$ is relatively compact. On the other hand, using again the already mentioned Lipschitz continuity of $S(\cdot)$, we can choose $\delta > 0$ and a partition of $[0, t]$ into points $0 = s_1 < s_2 < \cdots < s_k = t$ such that $s_{i+1} - s_i \le \delta$ and

$$\left\| \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} [S(s) - S(s_i)] f(t-s, x(t-s)) ds \right\| \le \delta N W_f(r) \int_0^t m_f(s) ds$$
$$\le \varepsilon.$$

Collecting these remarks and applying the mean value theorem for the Bochner integral ([9]), we get

$$\begin{aligned} \Gamma_2 x(t) &= \int_0^t C(t-s) g(s, x(s)) ds + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} S(s_i) f(t-s, x(t-s)) ds \\ &\quad + \sum_{i=1}^{k-1} \int_{s_i}^{s_{i+1}} (S(s) - S(s_i)) f(t-s, x(t-s)) ds \\ &\in \overline{tco(U_2(t,r))} + \sum_{i=1}^{k-1} (s_{i+1} - s_i) \overline{co(U(t, s_i, r))} + B_\varepsilon(0, X), \end{aligned}$$

where co is used to denote the convex hull of a set. Consequently, $\Gamma_2 B_r(t)$ is a totally bounded set, and hence $\Gamma_2 B_r(t)$ is relatively compact in X.

From the Ascoli-Arzelà theorem, we infer that $\Gamma_2 B_r$ is relatively compact in $C(I, X)$, which completes the proof that Γ is completely continuous.

Finally, employing Lemma 1.1 we conclude that Γ has a fixed point in $C(I, X)$ which is a mild solution of (2.1)–(2.3). $\square$

In most of situations of practical interest the sine function is compact. This is the motivation for the next result.

Corollary 2.1. *Assume that* (H-1) *and* (H-2) *are satisfied, the operator* $S(t)$ *is compact for all* $t \in \boldsymbol{R}$, *and that the following conditions are fulfilled.*

(a) *The function* f *takes bounded closed sets into bounded sets and for all* $0 \le t \le a$ *and* $r \ge 0$ *the set* $U_1(t, r) = \{g(s, x) : s \in [0, t], \|x\| \le r\}$ *is relatively compact in* X.

(b) *The function* $p(\cdot)$, $q(\cdot)$ *are bounded and* $p(\cdot)$ *is completely continuous. If inequality* (2.5) *holds, then there exists a mild solution of problem* (2.1)–(2.3).

In the next result we remove the conditions that $p(\cdot)$ and $q(\cdot)$ are bounded maps and that $p(\cdot)$ is completely continuous.

Theorem 2.2. *Assume that* (H-1) *and* (H-2) *hold and that the following conditions are fulfilled.*

(a) *For every* $0 < t$ *and* $r > 0$ *the set* $U(t,r) = \{S(t)f(s,x) : s \in [0,a], \|x\| \leq r\}$ *is relatively compact.*

(b) *There exist positive constants* L_g *and* L_p *such that*

$$\|g(t,x_1) - g(t,x_2)\| \leq L_g\|x_1 - x_2\|, \qquad t \in I,\ x_1, x_2 \in X,$$

$$\|p(x) - p(y)\| \leq L_p\|x - y\|, \qquad x, y \in C(I,X).$$

(c) *For every* $t \in I$ *and every* $r > 0$ *the set* $V(t,r) = \{S(t)q(x) : \|x\| \leq r\}$ *is relatively compact in* X.

If, in further,

$$N(L_g a + L_p) + \tilde{N}\left(L_g + \liminf_{r\to+\infty} \frac{N_q(r)}{r}\right) + \tilde{N} \liminf_{r\to+\infty} \frac{W_f(r)}{r}\int_0^a m_f(s)ds < 1,$$

then there exists a mild solution of problem (2.1)–(2.3).

Proof. Let Γ be the map defined by (2.6). Arguing as in the proof of Theorem 2.1, it is easy to see that the map Γ is well defined and continuous. We affirm that there exists $r > 0$ such that $\Gamma(B_r) \subseteq B_r$. In fact, if we assume that the assertion is false, for each $r > 0$ there exists $x^r \in B_r$ such that $\|\Gamma x^r\|_a > r$, which implies that

$$\begin{aligned} r &< \|\Gamma x^r\|_a \\ &\leq N(\|x_0\| + L_p r + \|p(0)\|) + \tilde{N}(\|y_0\| + N_q(r) + L_g r + \|g(0,0)\|) \\ &\quad + N\left[L_g a r + \int_0^a \|g(s,0)\|ds\right] + \tilde{N} W_f(r)\int_0^a m_f(s)ds. \end{aligned}$$

Hence, this yields

$$1 \leq N(L_p + L_g a) + \tilde{N}\left(L_g + \liminf_{r\to+\infty} \frac{N_q(r)}{r}\right) + \tilde{N} \liminf_{r\to+\infty} \frac{W_f(r)}{r}\int_0^a m_f(s)ds,$$

which is an absurd. We now consider the decomposition $\Gamma = \Gamma_1 + \Gamma_2$, where

$$\Gamma_1 x(t) = C(t)(x_0 + p(x)) - S(t)g(0,x(0)) + \int_0^t C(t-s)g(s,x(s))ds,$$

$$\Gamma_2 x(t) = S(t)(y_0 + q(x)) + \int_0^t S(t-s)f(s,x(s))ds,$$

for $t \in I$.

Let $r_0 > 0$ be a constant such that $\Gamma(B_{r_0}) \subseteq B_{r_0}$. Arguing as in the proof of Theorem 2.1, we can establish that Γ_2 is a completely continuous map. In addition, the estimate

$$\|\Gamma_1 u - \Gamma_1 v\|_a \leq (N(L_p + L_g a) + \tilde{N}L_g)\|u - v\|_a$$

yields that Γ_1 is a contraction. Thus, Γ is a condensing map on B_{r_0} and by the Sadovskii fixed point Theorem ([13]), we derive the existence of a mild solution of problem (2.1)–(2.3). $\square$

We next discuss the existence of solutions for the abstract Cauchy problem (1.1)–(1.3). Since the results are similar to those established in the first part of this section, we will only mention the main ideas of the proofs. To study this problem, we introduce the following technical assumptions.

(H-3) The function $f : I \times X \times X \to X$ satisfies the Carathéodory conditions:
- (i) The function $f(t, \cdot) : X \times X \to X$ is continuous a.e. $t \in I$.
- (ii) For each $(x, y) \in X \times X$, the function $f(\cdot, x, y) : I \to X$ is strongly measurable.
- (iii) There exists an integrable function $m_f : I \to [0, \infty)$ and a continuous nondecreasing function $W_f : [0, \infty) \to (0, \infty)$ such that

$$\|f(t, x, y)\| \le m_f(t) W_f(\|x\| + \|y\|), \qquad (t, x, y) \in I \times X \times X.$$

(H-4) The function $g : I \times X \times X \to X$ is continuous, E-valued and verifies the Carathéodory conditions:
- (i) The function $g(t, \cdot) : X \times X \to E$ is continuous a.e. $t \in I$.
- (ii) For each $x, y \in X$ the function $g(\cdot, x, y) : I \to E$ is strongly measurable.
- (iii) There exists an integrable function $\tilde{m}_g : I \to [0, \infty)$ and a continuous nondecreasing function $\tilde{W}_g : [0, \infty) \to (0, \infty)$ such that

$$\|g(t, x, y)\|_1 \le \tilde{m}_g(t) \tilde{W}_g(\|x\| + \|y\|), \qquad (t, x, y) \in I \times X \times X.$$

(H-5) The functions $x_0 + p(\cdot) : C(I, X)^2 \to E$ and $q(\cdot) : C(I, X)^2 \to X$ are continuous and take bounded closed sets into bounded sets.

Proceeding as before, we set

$$N_p(r) = \sup\{\|x_0 + p(x, y)\|_1 : x, y \in C(I, X), \|x\| + \|y\| \le r\}$$

and $N_q(r) = \sup\{\|q(x, y)\| : x, y \in C(I, X), \|x\| + \|y\| \le r\}$. In the case that $p(\cdot)$, respectively $q(\cdot)$, is a bounded map we denote by N_p, respectively N_q, an upper bound of $\|x_0 + p(\cdot)\|_1$ and $\|q(\cdot)\|$, respectively.

In the statements that follow, we use the notations $B_r = \{(x, y) \in C(I, X)^2 : \|x\| + \|y\| \le r\}$ and $W = \max\{W_f, \tilde{W}_g\}$.

We include for completeness the following result which will be frequently used afterwards. For a proof see [5, Lemma 1.1].

Lemma 2.1. *Let $h : [0, a] \to E$ be an integrable and continuous function for the norm of X. Then the function v given by $v(t) = \int_0^t C(t-s)h(s)ds$ is continuously differentiable, $s \mapsto AS(t-s)h(s)$ is integrable on $[0, t]$ and*

$$v'(t) = h(t) + A\int_0^t S(t-s)h(s)ds = h(t) + \int_0^t AS(t-s)h(s)ds.$$

We consider the following concept of mild solution.

Definition 2.2. A continuously differentiable function $x : I \to X$ is said to be a mild solution of problem (1.1)–(1.3) if the integral equation

$$\begin{aligned}(2.7)\qquad x(t) = {} & C(t)(x_0 + p(x, x')) + S(t)[y_0 - g(0, x(0), x'(0)) + q(x, x')] \\ & + \int_0^t C(t-s)g(s, x(s), x'(s))ds \\ & + \int_0^t S(t-s)f(s, x(s), x'(s))ds, \qquad t \in I,\end{aligned}$$

is verified.

Related to this definition, it is worthwhile to point out that if $u(\cdot)$ is a mild solution of problem (1.1)–(1.3) and conditions (H-4), (H-5) hold, then by the properties of second-order abstract Cauchy problem mentioned in the Introduction, and Lemma 2.1 we know that

$$\begin{aligned}x'(t) = {} & AS(t)(x_0 + p(x, x')) + C(t)[y_0 - g(0, x(0), x'(0)) + q(x, x')] \\ & + g(t, x(t), x'(t)) + \int_0^t AS(t-s)g(s, x(s), x'(s))ds \\ & + \int_0^t C(t-s)f(s, x(s), x'(s))ds, \qquad t \in I.\end{aligned}$$

Proceeding as before, we can establish the following results of existence. We omit the proof for the sake of brevity.

Theorem 2.3. *Assume that properties* (H-3), (H-4), (H-5) *are satisfied and that the following conditions hold*:

(a) *For each $r > 0$, $U(r) = f(I \times B_r)$ is a relatively compact set in X and $U_1(r) = g(I \times B_r)$ is a relatively compact set in E.*

(b) *The functions $x_0 + p(\cdot)$ and $q(\cdot)$ are completely continuous with values in E and X, respectively.*

(c) *There exists a constant $L_g \geq 0$ such that*

$$\|g(t, x_1, y_1) - g(t, x_2, y_2)\| \leq L_g(\|x_1 - x_2\| + \|y_1 - y_2\|).$$

If, in further,

$$(N+\tilde{N}_1)\liminf_{r\to+\infty}\frac{N_p(r)}{r}+(N+\tilde{N})\liminf_{r\to+\infty}\frac{N_q(r)}{r}+(\tilde{N}+aN+N+1)L_g$$
$$+\liminf_{r\to+\infty}\frac{W(r)}{r}\int_0^a[\tilde{N}_1\tilde{m}_g(s)+(N+\tilde{N})m_f(s)]ds<1,$$

then there exists a mild solution of problem (1.1)–(1.3).

We can establish a similar result using Lipschitz conditions instead of compactness in the space E.

Theorem 2.4. *Assume that properties* (H-3), (H-4), (H-5) *are satisfied and that the following conditions hold.*

(a) *For each* $r>0$, $U(r)=f(I\times B_r)$ *is a relatively compact set in* X.

(b) *The function* $q(\cdot)$ *is completely continuous with values in* X.

(c) *There is a constant* $L_g\geq 0$ *such that*

$$\|g(t,x_1,y_1)-g(t,x_2,y_2)\|_1\leq L_g(\|x_1-x_2\|+\|y_1-y_2\|),$$

for all $x_1,x_2,y_1,y_2\in X$.

(d) *There exists a constant* $L_p\geq 0$ *such that*

$$\|p(u_1,v_1)-p(u_2,v_2)\|_1\leq L_p(\|u_1-u_2\|+\|v_1-v_2\|),$$

for all $u_1,u_2,v_1,v_2\in C(I,X)$.

If, in further,

$$(\tilde{N}_1+N)L_p+(N+\tilde{N})\liminf_{r\to+\infty}\frac{N_q(r)}{r}+(\tilde{N}+a\tilde{N}_1+(a+1)N+1)L_g$$
$$+(N+\tilde{N})\liminf_{r\to+\infty}\frac{W_f(r)}{r}\int_0^a m_f(s)ds<1,$$

then there exists a mild solution of problem (1.1)–(1.3).

3. Global solutions

In this section, we discuss the existence of global and asymptotically almost periodic mild solutions for the nonlocal problem (2.1)–(2.3). For this reason, we modify our previous notations. In what follows, I represent the interval $[0,\infty)$, and we assume that $C(t)$ and $S(t)$ are uniformly bounded on I. We denote by N and $\tilde{N}$ positive constants such that $\|C(t)\|\leq N$ and $\|S(t)\|\leq\tilde{N}$, for all $t\geq 0$. The conditions (H-1) and (H-2) are referred to this interval I

and, we assume that the functions m_f, m_g are only locally integrable. Furthermore, we abbreviate the exposition by writing $W = \max\{W_f, W_g\}$ and $m = \max\{m_f, m_g\}$. Similarly, a function $x : I \to X$ is said to be a mild solution of problem (2.1)–(2.3) if it verifies the Definition 2.1. We begin by studying the existence of global solutions.

3.1. Existence of solutions on $[0, \infty)$

We next study the existence of mild solutions in the space of continuous functions with weight. Let $h : [0, \infty) \to (0, \infty)$ be a continuous nondecreasing function such that $h(0) = 1$ and $h(t) \to \infty$, as $t \to \infty$. In what follows $C_h^0(X)$ denotes the space of all continuous functions $x : [0, \infty) \to X$ such that $h(t)^{-1}\|x(t)\| \to 0$, $t \to \infty$, endowed with the norm $\|x\|_h = \sup_{t \geq 0} h(t)^{-1}\|x(t)\|$, and $C_0(X)$ denotes the space consisting of continuous functions $x : [0, \infty) \to X$ that vanish at infinity endowed with the norm of the uniform convergence. In this subsection we assume that the functions $p(\cdot), q(\cdot) : C_h^0(X) \to X$ are continuous and bounded. We use the notations $N_p = \sup\{\|p(x)\| : x \in C_h^0(X)\}$, $N_q = \sup\{\|q(x)\| : x \in C_h^0(X)\}$, and the constant c has the expression introduced in (2.4).

The following property is well known, we include it here for future reference.

Lemma 3.1. *A set $W \subseteq C_0(X)$ is relatively compact if and only if W is equicontinuous, the functions $x(t) \to 0$, $t \to \infty$, uniformly for $x \in W$, and the orbits $W(t)$ are relatively compact in X for all $t \geq 0$.*

In the next statement, we denote by $\gamma : [0, \infty) \to [0, \infty)$ the function defined by $\gamma(s) = Nm_g(s) + \tilde{N}m_f(s)$.

Theorem 3.1. *Assume that conditions* (H-1) *and* (H-2) *hold, and that*

$$\liminf_{r \to \infty} \frac{1}{r} \sup_{t \geq 0} \frac{1}{h(t)} \int_0^t \gamma(s) W(rh(s)) ds < 1.$$

Suppose further that the following conditions are fulfilled.

(a) *For each $t \in I$, $t' \leq t$ and $r \geq 0$, $\{S(t')f(s, x) : 0 \leq s \leq t, \|x\| \leq r\}$ and $\{g(s, x) : 0 \leq s \leq t, \|x\| \leq r\}$ are relatively compact sets in X.*

(b) *The functions $p(\cdot)$, $q(\cdot)$ are bounded and $p(\cdot), S(t)q(\cdot) : C_h^0(X) \to X$ are completely continuous, for each $t \geq 0$.*

(c) *For every $L \geq 0$, $h(t)^{-1} \int_0^t m(s) W(Lh(s)) ds \to 0$, as $t \to \infty$.*

Then there exists a mild solution $x(\cdot) \in C_h^0(X)$ of problem (2.1)–(2.3).

Proof. For each $x \in C_h^0(X)$, we define $\Gamma x(t)$ by means of (2.6). Clearly $\Gamma(x)$ is a continuous function. Moreover,

$$\|\Gamma x(t)\| \leq N(\|x_0\| + \|p(x)\|) + \tilde{N}(\|y_0\| + \|q(x)\| + \|g(0, x(0))\|) + \int_0^t [N m_g(s) + \tilde{N} m_f(s)] W(\|x(s)\|) ds. \tag{3.1}$$

Since $\|x(s)\| \leq \|x\|_h h(s)$ the above expression yields that

$$\frac{\|\Gamma x(t)\|}{h(t)} \leq \frac{N}{h(t)}(\|x_0\| + \|p(x)\|) + \frac{\tilde{N}}{h(t)}(\|y_0\| + \|q(x)\| + \|g(0, x(0))\|) + \frac{1}{h(t)}\int_0^t [N m_g(s) + \tilde{N} m_f(s)] W(\|x\|_h h(s)) ds, \tag{3.2}$$

and applying condition (c), it follows that $h(t)^{-1}\|\Gamma x(t)\|$ converges to zero as $t \to \infty$. This shows that Γ is a well defined map from $C_h^0(X)$ into $C_h^0(X)$. The inequality (3.2) also shows that $h(t)^{-1}\|\Gamma x(t)\| \to 0$ as $t \to \infty$, uniformly for x in a bounded subset of $C_h^0(X)$. Using this property we can easily show that Γ is continuous.

On the other hand, if $x_\lambda \in C_h^0(X)$ is a solution of the equation $\lambda \Gamma(x_\lambda) = x_\lambda$, for $0 < \lambda < 1$, we find that $\|x_\lambda(0)\| \leq \|\Gamma(x_\lambda)(0)\| \leq N(\|x_0\| + N_p)$. Therefore, it follows from (3.1) that

$$\|x_\lambda(t)\| \leq c + \int_0^t \gamma(s) W(\|x_\lambda(s)\|) ds,$$

which in turn implies that

$$\frac{\|x_\lambda(t)\|}{h(t)} \leq \frac{c}{h(t)} + \frac{1}{h(t)}\int_0^t \gamma(s) W(\|x_\lambda\|_h h(s)) ds$$

and

$$\|x_\lambda\|_h \leq c + \sup_{t \geq 0} \frac{1}{h(t)} \int_0^t \gamma(s) W(\|x_\lambda\|_h h(s)) ds.$$

If we assume that the set $\{\|x_\lambda\|_h : 0 < \lambda < 1\}$ is unbounded, taking $r = \|x_\lambda\|_h$, we obtain that

$$1 \leq \liminf_{r \to \infty} \frac{1}{r} \sup_{t \geq 0} \frac{1}{h(t)} \int_0^t \gamma(s) W(rh(s)) ds,$$

which is an absurd. Consequently, we conclude that $\{\|x_\lambda\|_h : 0 < \lambda < 1\}$ is a bounded set.

On the other hand, if $B_r = B_r(0, C_h^0(X))$, for $r > 0$, it is not difficult to see that the set $\{h^{-1}\Gamma x : x \in B_r\}$ fulfills the conditions of Lemma 3.1, which implies that this set is relatively compact in $C_0(X)$ and, consequently $\Gamma(B_r)$ is relatively compact in $C_h^0(X)$. Finally, applying Lemma 1.1 we get the existence of a mild solution of problem (2.1)–(2.3). $\square$

3.2. Existence of asymptotically almost periodic solutions

We now study the existence of asymptotically almost periodic (a.a.p) solutions of (2.1)–(2.3). For the basic concepts about almost periodic (a.p.) and asymptotically almost periodic functions we refer to [21]. Furthermore, the reader can consult [6, 11, 12] for recent developments about the existence of almost periodic and asymptotically almost periodic solutions, and the extension to the admissibility of functions spaces, of the abstract Cauchy problem. In what follows, we denote by $C_b(X)$ the space consisting of bounded continuous functions from $[0, \infty)$ into X endowed with the norm of uniform convergence, and the notation $AP(X)$ (resp. $AAP(X)$) stands for the subspace of $C_b(X)$ formed by the functions $x : [0, \infty) \to X$ which are a.p. (resp. a.a.p.) with the norm inherits from $C_b(X)$. Moreover, a strongly continuous function $F : [0, \infty) \to \mathscr{L}(X, Y)$ is said (strongly) a.p. if for each $x \in X$ the function $F(\cdot)x : [0, \infty) \to Y$ is a.p. We refer to [1] for the characterization of almost periodic cosine functions and to [4] for similar results for almost periodic sine functions. Our results will be based on some well known criteria of compactness in $AP(X)$ and $AAP(X)$ ([21]). In particular, it follows from [21, Theorem 6.3] that if $F : [0, \infty) \to \mathscr{L}(X, Y)$ is a.p. and U is a relatively compact subset of X, then $V = \{F(\cdot)x : x \in U\}$ is relatively compact in $AP(Y)$. We can strengthen this property for the sine function.

Proposition 3.1. *Assume that $S(\cdot)$ is almost periodic and that $U \subseteq X$. If the set $\{S(t)x : x \in U, t \geq 0\}$ is relatively compact in X, then $V = \{S(\cdot)x : x \in U\}$ is relatively compact in $AP(X)$.*

Proof. Let us fix $\delta > 0$. Since $S(\delta)U$ is relatively compact in X, by the previous remark we can affirm that the set $V_\delta = \{S(\cdot)S(\delta)x : x \in U\}$ is relatively compact in $AP(X)$. On the other hand, for each $\varepsilon > 0$ there is $\delta > 0$ such that $\|(I - C(s))S(t)x\| \leq \varepsilon$, for all $0 \leq s \leq \delta$, $x \in U$ and all $t \geq 0$. It follows from this estimate that

$$\left\| S(t)x - \frac{1}{\delta}S(t)S(\delta)x \right\| = \left\| \frac{1}{\delta}\int_0^\delta (I - C(s))S(t)x\,ds \right\| \leq \varepsilon.$$

Combining this property with the decomposition

$$S(t)x = \frac{1}{\delta} S(t)S(\delta)x + S(t)x - \frac{1}{\delta} S(t)S(\delta)x,$$

we get that $V \subseteq \delta^{-1} V_\delta + B_\varepsilon(0, C_b(X))$, which readily implies that V is a relatively compact set. $\square$

The condition about the compactness of $\{S(t)x : x \in U, t \geq 0\}$ considered in the previous proposition is verified in several situations. We begin with a definition. The function $S(\cdot)$ is said uniformly almost periodic if $S : [0, \infty) \to \mathscr{L}(X)$ is almost periodic for the norm of operators.

Remark 3.1. Assume that $S(\cdot)$ is almost periodic and that U is a bounded subset of X. If one of the following conditions hold:
(i) U is relatively compact.
(ii) $S(\cdot)$ is uniformly almost periodic and $S(t)$ is compact for all $t \in \boldsymbol{R}$,
then $\{S(t)x : x \in U, t \geq 0\}$ is relatively compact.

By the results of Lutz [8] the case (ii) includes the periodic sine functions.

On the other hand, for each function $f \in AAP(X)$ there are uniquely determined functions $f_1 \in AP(X)$ and $f_2 \in C_0(X)$ such that $f = f_1 + f_2$. Let $V \subseteq AAP(X)$ and denote $V_i = \{f_i : f \in V\}$, $i = 1, 2$. It is easy to see that V is relatively compact in $AAP(X)$ if and only if V_1 is relatively compact in $AP(X)$ and V_2 is relatively compact in $C_0(X)$. Combining this property with Lemma 3.1 and [21, Theorem 6.3] the following criterion is obtained.

Lemma 3.2. *Let $V \subseteq AAP(X)$ be a set with the following properties*:
(a) *V is uniformly equicontinuous on $[0, \infty)$.*
(b) *V is equi-asymptotically almost periodic. That is, for every $\varepsilon > 0$ there are $T_\varepsilon \geq 0$ and a relatively dense set $P_\varepsilon \subseteq [0, \infty)$ such that*

$$\|x(t + \tau) - x(t)\| \leq \varepsilon, \qquad x \in V,\ t \geq T_\varepsilon,\ \tau \in P_\varepsilon.$$

(c) *For each $t \geq 0$, $V(t)$ is relatively compact in X.*
Then V is a relatively compact set in $AAP(X)$.

We will use repeatedly the following property, which is a direct consequence of Lemma 3.1 and the mean value theorem for the Bochner integral ([9]).

Proposition 3.2. *Let Z_i, $i = 1, 2$, be Banach spaces and let $V \subseteq \mathscr{L}^1([0, \infty), Z_1)$. If $F_1 : [0, \infty) \to \mathscr{L}(Z_1, Z_2)$ and $F_2 : [0, \infty) \to \mathscr{L}(Z_2, Z_2)$ are strongly continuous functions that satisfy the following conditions*:
(a) *$\int_L^\infty F_1(s)x(s)ds \to 0$ in Z_2 when $L \to \infty$, uniformly for $x \in V$.*
(b) *For each $t \geq 0$, $\{x(s) : x \in V, 0 \leq s \leq t\}$ is a relatively compact set in Z_1,*

then $W(t) = \{\int_0^t F_1(s)x(s)ds : x \in V\}$, $t \geq 0$, *and* $W = \bigcup_{0 \leq t \leq \infty} W(t)$ *are relatively compact sets in* Z_2. *Furthermore, if* F_2 *is uniformly bounded on* $[0, \infty)$ *and* $\int_t^{t+h} F_1(s)x(s)ds \to 0$, *when* $h \to 0$, *uniformly for* $x \in V$, *then* $U = \{z_x : x \in V\}$, *where* $z_x(t) = F_2(t) \int_t^{\infty} F_1(s)x(s)ds$, *is a relatively compact set in* $C_0(Z_2)$.

Let $x : [0, \infty) \to X$ be a locally integrable function. In the next results, we denote by $z_x, y_x : [0, \infty) \to X$ the functions defined by $z_x(t) = \int_0^t C(t-s)x(s)ds$ and $y_x(t) = \int_0^t S(t-s)x(s)ds$, $t \geq 0$.

Corollary 3.1. *Assume that the sine function* $S(\cdot)$ *is almost periodic and* $V \subseteq \mathscr{L}^1([0, \infty), X)$ *is a set with the following properties*:

(a) $\int_L^{\infty} \|x(s)\| ds \to 0$ *when* $L \to \infty$, *uniformly for* $x \in V$.

(b) $\int_t^{t+s} \|x(\xi)\| d\xi \to 0$, *when* $s \to 0$, *uniformly for* $x \in V$ *and* $t \geq 0$.

(c) *For each* $t \geq 0$, $\{x(s) : 0 \leq s \leq t, x \in V\}$ *is a relatively compact set.*

Then $\{y_x : x \in V\}$ *and* $\{z_x : x \in V\}$ *are relatively compact sets in* $AAP(X)$.

Proof. It follows from [4, Corollary 3.1] that each function y_x is asymptotically almost periodic. Moreover, by Proposition 3.2 we have that the sets $\{\int_0^{\infty} C(s)x(s)ds : x \in V\}$ and $\{\int_0^{\infty} S(s)x(s)ds : x \in V\}$ are included in a compact subset of X. This implies that the set consisting of functions $S(\cdot)\int_0^{\infty} C(s)x(s)ds - C(\cdot)\int_0^{\infty} S(s)x(s)ds$ for $x \in V$ is relatively compact in $AP(X)$. Similarly, applying again Proposition 3.2 we get that the set formed by the functions

$$C(t)\int_t^{\infty} S(s)x(s)ds - S(t)\int_t^{\infty} C(s)x(s)ds,$$

for $x \in V$, is relatively compact in $C_0(X)$. Combining these remarks with the decomposition

$$\begin{aligned} y_x(t) = S(t)\int_0^{\infty} C(s)x(s)ds \; - \; S(t)\int_t^{\infty} C(s)x(s)ds \\ - C(t)\int_0^{\infty} S(s)x(s)ds + C(t)\int_t^{\infty} S(s)x(s)ds, \end{aligned}$$

we obtain that $\{y_x : x \in V\}$ is a relatively compact set in $AAP(X)$.

On the other hand, we claim that the function z_x is uniformly continuous for each $x \in V$. To establish this assertion, we take $L > 0$. Since $C(\cdot)$ is almost periodic, from condition (c) follows that

$$\|C(t+s)x(\xi) - C(t)x(\xi)\| \to 0, \qquad s \to 0,$$

and this convergence is uniform for $t \geq 0$, $0 \leq \xi \leq L$ and $x \in V$. Therefore,

$$\begin{aligned}
\|z_x(t+s) - z_x(t)\| &\leq \int_0^t \|C(t+s-\xi)x(\xi) - C(t-\xi)x(\xi)\| d\xi \\
&\quad + \left\| \int_t^{t+s} C(t+s-\xi)x(\xi) d\xi \right\| \\
&\leq \int_0^L \sup_{t \geq 0, x \in V} \|C(t+s-\xi)x(\xi) - C(t-\xi)x(\xi)\| d\xi \\
&\quad + 2N \int_L^\infty \|x(\xi)\| d\xi + N \int_t^{t+s} \|x(\xi)\| d\xi.
\end{aligned}$$

Using conditions (a) and (b), we can choose L appropriately so that the right hand side of the above inequality converges to 0, as $s \to 0$, uniformly for $t \geq 0$. In view of that z_x is the derivative of y_x, it follows from [21, Theorem 5.2] that $z_x \in AAP(X)$. Moreover, the above estimate also shows that the functions z_x, $x \in V$, are uniformly equicontinuous.

Finally, we establish that $\{z_x : x \in V\}$ is equi-asymptotically almost periodic. It follows from Proposition 3.2 that $\{z_x(t) : x \in V\}$ is a relatively compact set for all $t \geq 0$. For a fixed $\varepsilon > 0$, applying condition (a), we infer the existence of $T_\varepsilon > 0$ such that $\int_{T_\varepsilon}^\infty \|x(s)\| ds \leq \varepsilon/6N$, for all $x \in V$. In addition, since the set $\{C(\cdot)x(s) : 0 \leq s \leq T_\varepsilon\}$ is uniformly almost periodic, there is a relatively dense set $P_\varepsilon \subseteq [0, \infty)$ such that

$$\|C(\xi + \tau)x(s) - C(\xi)x(s)\| \leq \frac{\varepsilon}{3T_\varepsilon},$$

for all $\xi \geq 0$, $0 \leq s \leq T_\varepsilon$ and all $\tau \in P_\varepsilon$. Hence, for $t \geq T_\varepsilon$, we get

$$\begin{aligned}
\|z_x(t+\tau) - z_x(t)\| &\leq \int_0^{T_\varepsilon} \|C(t+\tau-s)x(s) - C(t-s)x(s)\| ds \\
&\quad + 3N \int_{T_\varepsilon}^\infty \|x(s)\| ds \\
&\leq \varepsilon,
\end{aligned}$$

which shows the assertion. We complete the proof applying Lemma 3.2 to the set $\{z_x : x \in V\}$. □

Using this result and proceeding as in the proof of Proposition 3.1, we can obtain the compactness of $\{y_x : x \in V\}$ under some weaker conditions.

Corollary 3.2. *Assume that $S(\cdot)$ is almost periodic, $V \subseteq \mathscr{L}^1([0,\infty), X)$ and conditions* (a) *and* (b) *of Corollary* 3.1 *are satisfied. If for each $t, \delta \geq 0$, the set $\{S(\delta)x(s) : 0 \leq s \leq t, x \in V\}$ is relatively compact in X, then $\{y_x : x \in V\}$ is relatively compact in $AAP(X)$.*

We are now in a position to establish the main result of this work. In this result, we consider that $p(\cdot), q(\cdot) : AAP(X) \to X$ are bounded continuous functions. As before N_p and N_q denote the upper bound of $\|p(\cdot)\|$ and $\|q(\cdot)\|$, respectively, and c is given by (2.4).

Theorem 3.2. *Assume that $S(\cdot)$ is almost periodic and that the conditions* (H-1) *and* (H-2) *hold with functions $m_f(\cdot)$ and $m_g(\cdot)$ integrable on $[0,\infty)$ and that*

$$\int_0^\infty (Nm_g(s) + \tilde{N}m_f(s))ds < \int_c^\infty \frac{ds}{W(s)}. \tag{3.3}$$

Assume further that the following conditions are fulfilled:

(a) *For each $t, t' > 0$ and each constant $r \geq 0$, $\{S(t')f(s,x) : 0 \leq s \leq t, \|x\| \leq r\}$ and $\{g(s,x) : 0 \leq s \leq t, \|x\| \leq r\}$ are relatively compact sets in X.*

(b) *The function $p(\cdot) : AAP(X) \to X$ is bounded and completely continuous.*

(c) *The function $q(\cdot)$ is bounded and $\{S(t)q(x) : t \geq 0, \|x\| \leq r\}$ is a relatively compact set in X for each $r > 0$.*

Then there exists a mild solution $u(\cdot) \in AAP(X)$ of problem (2.1)–(2.3).

Proof. For each $x \in AAP(X)$, we define $\Gamma x(t)$ by means of (2.6). By the integrability of functions $m_f(\cdot)$ and $m_g(\cdot)$, and proceeding as in the proof of Corollary 3.1 for the functions $f(s, x(s))$ and $g(s, x(s))$, we infer that $\Gamma(x) \in AAP(X)$. Furthermore, if we take a sequence $(x_n)_n$ that converges to x in the space $AAP(X)$, then $S(t-s)f(s, x_n(s)) \to S(t-s)f(s, x(s))$ and $C(t-s)g(s, x_n(s)) \to C(t-s)g(s, x(s))$, as $n \to \infty$, a.e. on $[0,t]$. Let $L = \sup\{\|x\|_\infty, \|x_n\|_\infty : n \in \boldsymbol{N}\}$. From the inequalities

$$\|C(t-s)g(s, x_n(s)) - C(t-s)g(s, x(s))\| \leq 2Nm_g(s)W_g(L),$$

$$\|S(t-s)f(s, x_n(s)) - S(t-s)f(s, x(s))\| \leq 2\tilde{N}m_f(s)W_f(L),$$

and using again the integrability of $m_f(\cdot)$ and $m_g(\cdot)$, we conclude that $\Gamma x_n(t) \to \Gamma x(t)$, when $n \to \infty$, and that this convergence is uniform on $[0,\infty)$. This yields that Γ is a continuous map.

On the other hand, proceeding as in the proof of Theorem 2.1, we can conclude that the set of functions $\{x_\lambda \in AAP(X) : \lambda\Gamma(x_\lambda) = x_\lambda, 0 < \lambda < 1\}$ is uniformly bounded on $[0,\infty)$.

Finally, we shall show that Γ is completely continuous. In order to establish this assertion we take a bounded set $V \subseteq AAP(X)$. Since $p(\cdot)$ is completely continuous, the set of functions $\{C(\cdot)(x_0 + p(x)) : x \in V\}$ is relatively compact in $AP(X)$. Similarly, it follows from conditions (a) and (b), and applying Proposition 3.1 that $\{S(\cdot)(y_0 + q(x) + g(0, x(0))) : x \in V\}$ is relatively compact in $AP(X)$. In addition, since the sets of functions $\Lambda_1 = \{g(\cdot, x(\cdot)) : x \in V\}$ and $\Lambda_2 = \{f(\cdot, x(\cdot)) : x \in V\}$ satisfy the hypotheses of Corollary 3.1 and Corollary 3.2, respectively, we infer that $\Gamma(V)$ is relatively compact in $AAP(X)$.

The existence of a mild solution of problem (2.1)–(2.3) is now consequence of Lemma 1.1. □

We next establish a result of regularity for the asymptotically almost periodic mild solutions of problem (2.1)–(2.3). This result will be based on the following properties of the almost periodic sine functions.

Lemma 3.3. *Assume that $S(\cdot)$ is almost periodic. Then the following assertions are fulfilled*:

(a) *For each $x \in X$ the function $S(\cdot)x$ is almost periodic for the norm in E.*

(b) *If $AS(\cdot)$ is uniformly bounded on $\boldsymbol{R}$ as an $\mathscr{L}(E, X)$-valued map, then the group G is almost periodic.*

(c) *If $AS(\cdot)$ is uniformly bounded on $\boldsymbol{R}$ as an $\mathscr{L}(E, X)$-valued map and $u : [0, \infty) \to X$ is a function for which there is $y \in D(A)$ such that $u(t) - Ay \in \mathscr{L}^1([0, \infty), X)$, then the function $w_1(\cdot)$ given by $w_1(t) = \int_0^t C(t - s)u(s)ds$ is asymptotically almost periodic.*

(d) *If $AS(\cdot)$ is uniformly bounded on $\boldsymbol{R}$ as an $\mathscr{L}(E, X)$-valued map and $v : [0, \infty) \to E$ is a function for which there is $z \in E$ such that $v(t) - z \in \mathscr{L}^1([0, \infty), E)$, then the function $w_2(\cdot)$ given by $w_2(t) = \int_0^t AS(t - s)v(s)ds$ is asymptotically almost periodic.*

Proof. Let $x \in X$. Since the sine function $S(\cdot)$ is almost periodic, it follows from [4, Theorem 3.2] that $C(\cdot)$ is almost periodic. Consequently, the function $(C(\cdot)x, S(\cdot)x)$ is almost periodic with values in $X \times X$. Thus, given $\varepsilon > 0$ there is a relatively dense set P_ε such that

$$\|C(t+\tau)x - C(t)x\| + \|S(t+\tau)x - S(t)x\| \le \varepsilon, \qquad t \ge 0,\ \tau \in P_\varepsilon.$$

Using that $C(\cdot)$ is uniformly bounded on $\boldsymbol{R}$ and

$$C(t+s)x = C(t)C(s)x + AS(t)S(s)x, \tag{3.4}$$

we obtain

$$\begin{aligned}\|S(t+\tau)x - S(t)x\|_1 &= \|S(t+\tau)x - S(t)x\| + \sup_{0\le h\le 1}\|AS(h)[S(t+\tau)x - S(t)x]\| \\ &\le \varepsilon + \sup_{0\le h\le 1}\|C(h)\|\,\|C(t+\tau)x - C(t)x\| \\ &\quad + \sup_{0\le h\le 1}\|C(t+h+\tau)x - C(t+h)x\| \\ &\le (N+2)\varepsilon,\end{aligned}$$

for all $t \ge 0$ and $\tau \in P_\varepsilon$, which proves the assertion (a).

On the other hand, for all $y \in E$ and $t \ge 0$ we have that

$$\|AS(t+h)y - AS(t)y\| \le \|AS(t)\|_{\mathscr{L}(E,X)}\|(C(h) - I)y\|_1 + \|C(t)\|\,\|AS(h)y\|.$$

If we assume that $\|AS(t)\|_{\mathscr{L}(E,X)}$ is bounded on $\boldsymbol{R}$, it follows from the above inequality that function $AS(\cdot)y$ is uniformly continuous on $[0,\infty)$ and, since $AS(\cdot)y$ is the derivative of the almost periodic function $C(t)y$, applying the Bochner Theorem (see [21, p. 24]), we infer that $AS(\cdot)y$ is almost periodic. Using now the definition of G given in the Introduction, we conclude that the group G is almost periodic, which establishes (b).

Finally, since the functions

$$\begin{bmatrix} v - z \\ 0 \end{bmatrix} = \begin{bmatrix} v \\ 0 \end{bmatrix} - \mathscr{A}\begin{bmatrix} 0 \\ z \end{bmatrix} \in \mathscr{L}^1([0,\infty), E \times X)$$

and

$$\begin{bmatrix} 0 \\ u - Ay \end{bmatrix} = \begin{bmatrix} 0 \\ u \end{bmatrix} - \mathscr{A}\begin{bmatrix} y \\ 0 \end{bmatrix} \in \mathscr{L}^1([0,\infty), E \times X),$$

assertions (c) and (d) are a consequence of (b) and [21, Example 5.1]. □

We next consider $AS(\cdot)$ as a strongly continuous $\mathscr{L}(E,X)$-valued function.

Proposition 3.3. *Assume that $S(\cdot)$ is almost periodic and that the operator-valued function $AS(\cdot)$ is uniformly bounded on $\boldsymbol{R}$. Let $u(\cdot)$ be an asymptotically almost periodic mild solution of problem* (2.1)–(2.3). *If, in further, the following conditions are satisfied*:

(a) *There is $y \in D(A)$ such that the function $[0,\infty) \to X$, $t \mapsto f(t,u(t)) - Ay$, is integrable.*

(b) *The function $[0,\infty) \to X$, $t \mapsto g(t,u(t))$, is asymptotically almost periodic, the values $g(t,u(t)) \in E$ and there is $z \in E$ such that $g(\cdot,u(\cdot)) - z \in \mathscr{L}^1([0,\infty),E)$.*

(c) *$x_0 + p(u) \in E$,*

then $u(\cdot)$ is continuously differentiable and $u'(\cdot)$ is asymptotically almost periodic.

Proof. From our hypotheses and Lemma 2.1, we obtain that u is continuously differentiable and that

$$u'(t) = AS(t)(x_0 + p(u)) + C(t)(y_0 + q(u) + g(0, u(0)) + g(t, u(t)) + \int_0^t AS(t-s)g(s, u(s))ds + \int_0^t C(t-s)f(s, u(s))ds$$

for $t \geq 0$. Combining this equality with the integrability of $f(s, u(s))$ and $g(s, u(s))$, and the assertions of Lemma 3.3, we get that u' is asymptotically almost periodic. □

4. Applications

The one dimensional wave equation modeled as an abstract Cauchy problem has been studied extensively. See for example [20]. In this section, we apply the results established in the previous sections to study some variants of the wave equation.

On the space $X = L^2([0, \pi])$, we consider the operator $Af(\xi) = f''(\xi)$ with domain $D(A) = \{f(\cdot) \in H^2(0, \pi) : f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $C(t)$ on X. Furthermore, A has discrete spectrum, with eigenvalues $-n^2$, $n \in \boldsymbol{N}$, and corresponding normalized eigenvectors $z_n(\xi) = (2/\pi)^{1/2} \sin(n\xi)$. Moreover, $C(t)\varphi = \sum_{n=1}^{\infty} \cos(nt)\langle\varphi, z_n\rangle z_n$ and $S(t)\varphi = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n}\langle\varphi, z_n\rangle z_n$ for $t \in \boldsymbol{R}$. This implies that $\|C(t)\| = 1$, the operator $S(t)$ is compact and $\|S(t)\| = 1$, for every $t \in \boldsymbol{R}$ (see [2] for details).

We consider the following initial value problem with nonlocal conditions defined for $t \in I = [0, a]$ and $\xi \in [0, \pi]$,

$$\frac{\partial}{\partial t}\left[\frac{\partial u(t,\xi)}{\partial t} - F_1(t, \xi, u(t, \xi))\right] = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + F(t, \xi, u(t, \xi)), \tag{4.1}$$

$$u(t, 0) = u(t, \pi) = 0, \tag{4.2}$$

$$u(0, \xi) = x_0(\xi) + \sum_{i=1}^{n} \alpha_i u(t_i, \xi), \tag{4.3}$$

$$\frac{\partial u(0,\xi)}{\partial t} = y_0(\xi) + \sum_{i=1}^{k} \beta_i(u(s_i, \xi)). \tag{4.4}$$

In this statement we assume that $F_1, F : I \times [0, \pi] \times \boldsymbol{R} \to \boldsymbol{R}$ satisfy the conditions:

(C-1) The functions $F(t,\xi,\cdot), F_1(t,\xi,\cdot) : \boldsymbol{R} \to \boldsymbol{R}$ are continuous a.e. $(t,\xi) \in I \times [0,\pi]$.

(C-2) For each $w \in \boldsymbol{R}$, the function $F(\cdot,w) : I \times [0,\pi] \to \boldsymbol{R}$ is measurable and there exist positive measurable functions $\eta_1^F, \eta_2^F : I \times [0,\pi] \to \boldsymbol{R}$ with $\eta_1^F \in \mathscr{L}^2(I \times [0,\pi])$ and $\sup_{0\le\xi\le\pi} \eta_2^F(\cdot,\xi) \in \mathscr{L}^1(I)$ such that

$$|F(t,\xi,w)| \le \eta_1^F(t,\xi) + \eta_2^F(t,\xi)|w|, \qquad t \in I,\ \xi \in [0,\pi],\ w \in \boldsymbol{R}.$$

(C-3) The function $F_1(\cdot,0) \in L^2(I \times [0,\pi])$, and there exists a positive constant L_1 such that

$$|F_1(t,\xi,w_1) - F_1(t,\xi,w_2)| \le L_1|w_1 - w_2|, \qquad t \in I,\ \xi \in [0,\pi],\ w_1, w_2 \in \boldsymbol{R}.$$

Furthermore, $x_0, y_0 \in X$, $0 < t_i$, $s_j < a$, $\alpha_i \in \boldsymbol{R}$ and $\beta_j : \boldsymbol{R} \to \boldsymbol{R}$ are bounded continuous functions for $i = 1,\dots,n$ and $j = 1,\dots,k$.

We define the substitution operators $g(t,x)(\xi) = F_1(t,\xi,x(\xi))$ and $f(t,x)(\xi) = F(t,\xi,x(\xi))$ for $x \in X$. It follows from [10, Proposition V.2.5] that $f(\cdot)$ and $g(\cdot)$ satisfy the Carathéodory conditions (H-1), (H-2), and that $g(\cdot)$ is Lipschitz continuous. Moreover, it is not difficult to see that

$$\|f(t,x)\|_2 \le \left(\int_0^\pi \eta_1^F(t,\xi)^2 d\xi\right)^{1/2} + \sup_{0\le\xi\le\pi} \eta_2^F(t,\xi)\|x\|_2,$$

$$\|g(t,x) - g(t,y)\|_2 \le L_1\|x - y\|_2$$

for all $x, y \in X$ and all $t \in I$, which implies that we can choose

$$m_f(t) = \left(\int_0^\pi \eta_1^F(t,\xi)^2 d\xi\right)^{1/2} + \sup_{0\le\xi\le\pi} \eta_2^F(t,\xi),$$

and $W_f(r) = 1$ for $0 \le r \le 1$ and $W_f(r) = r$, for $r \ge 1$. We complete this representation by defining the maps $p(u)(\xi) = \sum_{i=1}^n \alpha_i u(t_i,\xi)$ and $q(u)(\xi) = \sum_{i=1}^k \beta_i(u(s_i,\xi))$ for $u \in C(I,X)$, where we have abbreviated $u(t)(\xi) = u(t,\xi)$. With these definitions the problem (4.1)–(4.4) can be modeled as the abstract nonlocal Cauchy problem (2.1)–(2.3).

Proposition 4.1. *Assume that the previous conditions are fulfilled. If*

$$\int_0^a \sup_{0\le\xi\le\pi} \eta_2^F(s,\xi)ds + (a+1)L_1 + \sum_{i=1}^n |\alpha_i| < 1,$$

then there exists a mild solution of problem (4.1)–(4.4).

Proof. Since $S(t)$ is compact for all $t \in \boldsymbol{R}$ the existence of a mild solution of problem (4.1)–(4.4) follows directly from Theorem 2.2. $\square$

Finally, we will study the existence of global mild solutions for the problem

$$\frac{\partial}{\partial t}\left[\frac{\partial u(t,\xi)}{\partial t} - \int_0^\pi b(t,\eta,\xi)u(t,\eta)d\eta\right] = \frac{\partial^2 u(t,\xi)}{\partial \xi^2} + F(t,\xi,u(t,\xi)), \tag{4.5}$$

$$u(t,0) = u(t,\pi) = 0, \tag{4.6}$$

$$u(0,\xi) = x_0(\xi) + \int_0^\infty P(u(s,\cdot))(\xi)d\mu(s), \tag{4.7}$$

$$\frac{\partial}{\partial t}u(0,\xi) = y_0(\xi) + \int_0^\infty Q(u(s,\cdot))(\xi)dv(s), \tag{4.8}$$

for $t \in [0,\infty)$ $\xi \in [0,\pi]$. We turn to model this problem on the space $X = L^2([0,\pi])$ and, we assume that the following technical conditions hold, where $I = [0,\infty)$:

(i) The function $F : I \times [0,\pi] \times \boldsymbol{R} \to \boldsymbol{R}$ satisfies condition (C-1), for each $w \in \boldsymbol{R}$ the function $F(\cdot,w) : I \times [0,\pi] \to \boldsymbol{R}$ is measurable, and there exists a positive measurable function $\eta^F(\cdot) : I \times [0,\pi] \to \boldsymbol{R}$ with $\sup_{0\le\xi\le\pi} \eta^F(\cdot,\xi) \in \mathscr{L}^1(I)$ such that

$$|F(t,\xi,w)| \le \eta^F(t,\xi)|w|, \qquad t \ge 0,\ \xi \in [0,\pi],\ w \in \boldsymbol{R}.$$

(ii) The functions $b(s,\eta,\xi)$, $\partial b(s,\eta,\xi)/\partial\xi$, $\partial^2 b(s,\eta,\xi)/\partial\xi^2$ are continuous on $[0,\infty) \times [0,\pi]^2$ and $b(s,\eta,\pi) = b(s,\eta,0) = 0$ for all $(s,\eta) \in I \times [0,\pi]$.

(iii) The maps $P, Q : X \to X$ are continuous and bounded, and P is completely continuous (we refer to [10] for examples of operators that satisfy these properties). Furthermore, μ, v are real functions of bounded variation on $[0,\infty)$.

(iv) $x_0, y_0 \in X$.

Defining the maps $f, g : I \times X \to X$ and $p(\cdot), q(\cdot) : C(I,X) \to X$ by

$$f(t,x)(\xi) = F(t,\xi,x(\xi)); \qquad g(t,x)(\xi) = \int_0^\pi b(t,\eta,\xi)x(\eta)d\eta;$$

$$p(u)(\xi) = \int_0^\infty P(u(s))(\xi)d\mu(s); \qquad q(u)(\xi) = \int_0^\infty Q(u(s))(\xi)dv(s)$$

for $\xi \in [0,\pi]$, the problem (4.5)–(4.8) can be modeled as the abstract Cauchy problem (2.1)–(2.3). It follows from (i) that we can choose the functions $m_f(t) = \eta^F(t,\cdot)_\pi$, $m_g(t) = (\int_0^\pi \int_0^\pi b(t,\eta,\xi)^2 d\eta d\xi)^{1/2}$ and $W(s) = s$. Let $\gamma(t) = m_f(t) + m_g(t)$ and assume that $h(\cdot)$ is a function verifying the general assumptions considered in subsection 3.1. It is easy to see that we can consider $p(\cdot)$ and $q(\cdot)$ as bounded continuous functions defined on $C_h^0(X)$ and $AAP(X)$.

Proposition 4.2. *Assume that the preceding conditions are fulfilled* $h(t)^{-1}\int_0^t \gamma(s)h(s)ds \to 0$ *as* $t \to \infty$, *and* $\sup_{t\geq 0} h(t)^{-1}\int_0^t \gamma(s)h(s)ds < 1$. *Then there exists a mild solution of problem* (4.5)–(4.8) *in* $C_h^0(X)$.

Proof. We infer from condition (ii) that $g(t,\cdot)$ is a $[D(A)]$-valued linear continuous operator. Moreover, it is easy to see that $f(\cdot)$, $g(\cdot)$, $p(\cdot)$ and $q(\cdot)$ satisfy the compactness conditions considered in Theorem 3.1. Since $\sup_{t\geq 0} h(t)^{-1}\int_0^t \gamma(s)h(s)ds < 1$, the assertion is a consequence of Theorem 3.1. □

We mention here that, under very general assumptions on γ, there exists a function h which satisfies the conditions considered in this proposition. For instance, if γ is equi-integrable, that is, for each $\varepsilon > 0$ there is $\delta > 0$ such that for every measurable set $B \subseteq [0,\infty)$, with $\lambda(B) \leq \delta$, then $\int_B \gamma(s)ds \leq \varepsilon$, we have that $h(t) = \exp(\alpha t^2)$, for some $\alpha > 0$ large enough, fulfills the requirements.

Using similar arguments we can prove the next result.

Proposition 4.3. *Assume that the previous conditions are verified and that* $\gamma(\cdot)$ *is integrable on* $[0,\infty)$. *Then there exists an asymptotically almost periodic mild solution of problem* (4.5)–(4.8).

Proof. Since $\int_c^\infty W(s)^{-1}ds = \infty$ the assertion is a consequence of Theorem 3.2 and the fact that $S(\cdot)$ is periodic. □

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