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Connection Matrices Associated with the Generalized Hypergeometric Function ${}_3F_2$

By

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Abstract. Connection matrices associated with the generalized hypergeometric function ${}_3F_2$ are determined. Our approach is based on the integral representation, and the theory of twisted homology groups (homology with coefficients in local system). We also consider the related monodromy representation and the monodromy-invariant Hermitian form.

Key Words and Phrases. Generalized hypergeometric function, Twisted homology, Connection matrices, Monodromy representation, Monodromy-invariant Hermitian form.

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The *generalized hypergeometric function* ${}_3F_2$ is the analytic continuation of the generalized hypergeometric series defined by

$$ {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};z\right)=\sum_{k=0}^{\infty}\frac{(\alpha_1)_k(\alpha_2)_k(\alpha_3)_k}{(\beta_1)_k(\beta_2)_k k!}z^k,\qquad |z|<1, \tag{0.1}$$

where $(a)_k=a(a+1)\dots(a+k-1)$. It satisfies the differential equation ${}_3E_2$ with regular singular points $z=0,1,\infty$:

$$\begin{aligned}&z^2(1-z)F'''+\{\beta_1+\beta_2+1-(\alpha_1+\alpha_2+\alpha_3+3)z\}zF''\\&\quad+\{\beta_1\beta_2-(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1+\alpha_1+\alpha_2+\alpha_3+1)z\}F'-\alpha_1\alpha_2\alpha_3F=0.\end{aligned}$$

The characteristic exponents of ${}_3E_2$ at singularities are

$$\begin{aligned}&0,1-\beta_1,1-\beta_2\qquad \text{at } z=0,\\&0,1,\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3\qquad \text{at } z=1,\\&\alpha_1,\alpha_2,\alpha_3\qquad \text{at } z=\infty.\end{aligned}$$

If $\beta_1,\beta_2,\beta_1-\beta_2\notin \boldsymbol{Z}$, a set of linearly independent solutions around the origin 0 is given by

$$
{}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};z\right),
$$
$$
(-z)^{1-\beta_1}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_1+1,\alpha_2-\beta_1+1,\alpha_3-\beta_1+1\\ 2-\beta_1,\beta_2-\beta_1+1\end{matrix};z\right),
$$
$$
(-z)^{1-\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_2+1,\alpha_2-\beta_2+1,\alpha_3-\beta_2+1\\ \beta_1-\beta_2+1,2-\beta_2\end{matrix};z\right),
$$

and, if $\alpha_i-\alpha_j$ $(1\le i<j\le 3)\notin \boldsymbol{Z}$, that around ∞ is given by

$$
(-z)^{-\alpha_1}{}_3F_2\left(\begin{matrix}\alpha_1,\alpha_1-\beta_1+1,\alpha_1-\beta_2+1\\ \alpha_1-\alpha_2+1,\alpha_1-\alpha_3+1\end{matrix};\frac{1}{z}\right),
$$
$$
(-z)^{-\alpha_2}{}_3F_2\left(\begin{matrix}\alpha_2,\alpha_2-\beta_1+1,\alpha_2-\beta_2+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1\end{matrix};\frac{1}{z}\right),
$$
$$
(-z)^{-\alpha_3}{}_3F_2\left(\begin{matrix}\alpha_3,\alpha_3-\beta_1+1,\alpha_3-\beta_2+1\\ \alpha_3-\alpha_1+1,\alpha_3-\alpha_2+1\end{matrix};\frac{1}{z}\right).
$$

The first purpose is to give a connection matrix which expresses the fundamental set of solutions around the infinity ∞ in terms of that of solutions around the origin 0 (Theorem 2.4), which was first shown by Thomae [17] in the nineteenth century (See also the work by Winkler [18]). The second purpose is to give an explicit expression of the non-holomorphic solution around 1 (Theorem 3.2) and determine the connection coefficients by which the non-holomorphic solution around 1 is expressed around the origin 0 (Theorem 3.4). It is noteworthy that the solutions around 1 are not represented by the hypergeometric series having $1-z$ as a variable: the situation is completely different from the case of the Gauss hypergeometric equation. For the solutions around $z=1$, we refer the reader to Nørlund [15] and Bühring [4].

On the other hand, the theory of integral representations of special functions, initiated by Aomoto [1] [2] [3], have been developed during these three decades after the name of the *twisted de Rham theory*. From this viewpoint, it should be settled to derive the connection matrices by using the integral representation of the form:

$$
(0.2)\quad {}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};z\right)=\frac{\Gamma(\beta_1)\Gamma(\beta_2)}{\Gamma(\alpha_1)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\alpha_2)}
$$
$$
\int_1^{+\infty}\int_{t_1}^{+\infty} t_1^{\alpha_1-\beta_2}t_2^{\alpha_3-\beta_1}(t_1-1)^{\beta_2-\alpha_2-1}(t_2-z)^{-\alpha_3}(t_2-t_1)^{\beta_1-\alpha_1-1}dt_2dt_1,
$$

where $\operatorname{Re}\alpha_1,\operatorname{Re}(\beta_1-\alpha_1),\operatorname{Re}\alpha_2,\operatorname{Re}(\beta_2-\alpha_2)>0$, and the branch of each factor is fixed to be zero on the integration domain (which we shall call a twisted cycle).

The third purpose is to settle the associated monodromy representation in our framework (Theorem 4.1). The method used here is elaborated in [10]. For studies on the monodromy representations, we refer the reader to Okubo-Takano-Yoshida [16], Kato [6] and Kato-Noumi [7]. It is noteworthy that Okubo-Takano-Yoshida also gives the connection relation corresponding to our Theorem 3.4 in Theorem 3 in [16]. Finally, calculating the intersection numbers retated our twisted cycles leads to the monodromy-invariant Hermitian form for ${}_3F_2$'s (Theorem 4.2).

In this paper, we frequently use the symbols

$$e(A) = \exp(\pi\sqrt{-1}A), \qquad s(A) = \sin(\pi A)$$

and

$$\lambda_{ijk\ldots l} = \lambda_i + \lambda_j + \lambda_k + \cdots + \lambda_l, \qquad e_{ijk\ldots l} = e(\lambda_{ijk\ldots l}),$$

for abbreviation.

1. Preliminaries

Let $u(t) = \prod_i f_i(t)^{\alpha_i}$ be a multivalued function on $T \subset \boldsymbol{C}^m$, where $\alpha_i \in \boldsymbol{C}$ and T is the complement of the singular locus $\bigcup_i \{f_i(t) = 0\}$ in $\boldsymbol{C}^m$. Let $\mathscr{L}$ be the local system (locally constant sheaf) defined by u: the sheaf consisting of the local solutions of $dL = L\omega$ for $\omega = du(t)/u(t)$.

Let $H_m(T, \mathscr{L})$ be the m-th homology group with coefficients in $\mathscr{L}$, $H_m^{\mathrm{lf}}(T, \mathscr{L})$ the m-th locally finite homology group with coefficients in $\mathscr{L}$. Elements of these twisted homology groups, called *twisted cycles* or *loaded cycles*, are represented by ∂-closed twisted (finite or locally finite) chains

$$C = \sum_{\Delta} a_\Delta \Delta \otimes v_\Delta, \qquad (a_\Delta \in \boldsymbol{C}),$$

where each Δ is an m-simplex and v_Δ a section of $\mathscr{L}$ on Δ.

If each factor $f_i(t)$ of $u(t)$ is defined over $\boldsymbol{R}$, and D is a domain of the real manifold $T_{\boldsymbol{R}}$ (the real locus of T), then it is convenient to load D with a section

$$u_D(t) = \prod_i (\varepsilon_i f_i(t))^{\alpha_i}$$

of $\mathscr{L}$ on D, and to make a loaded cycle $D \otimes u_D(t)$, where $\varepsilon_i = \pm$ is so determined that $\varepsilon_i f_i(t)$ is positive on D, and the argument of $\varepsilon_i f_i(t)$ is assigned to be zero. This choice of a section is said to be *standard*.

In this paper, we adopt mainly the standard loading. Thus, we frequently omit the assignment of loading and denote just the topological cycles for simplicity. For example, when $u(t) = t^{\alpha}(1-t)^{\beta}$, we denote by $\overrightarrow{(0,1)}$ to express $\overrightarrow{(0,1)} \otimes u(t)$, and $\overrightarrow{(1,\infty)}$ for $\overrightarrow{(1,\infty)} \otimes t^{\alpha}(t-1)^{\beta}$.

Under some genericity condition on the exponents α_i, we have the isomorphism, called the *regularization*,

$$\mathrm{reg} : H_m^{\mathrm{lf}}(T, \mathscr{L}) \to H_m(T, \mathscr{L}),$$

which is the inverse of the natural map $\imath : H_m(T, \mathscr{L}) \to H_m^{\mathrm{lf}}(T, \mathscr{L})$.

For example, in case $T = \boldsymbol{C} \backslash \{0, 1\}$ and $u(t) = t^{\alpha}(t-1)^{\beta}$, where $\alpha, \beta, \alpha + \beta \in \boldsymbol{R} \backslash \boldsymbol{Z}$, a regularization (regularized cycle) $\mathrm{reg}\, C \in H_1(T, \mathscr{L})$ of $C = (0,1) \in H_1^{\mathrm{lf}}(T, \mathscr{L})$ can be given by

$$\mathrm{reg}\, C = \left\{ \frac{1}{d_{\alpha}} S(\varepsilon; 0) + \overrightarrow{[\varepsilon, 1-\varepsilon]} - \frac{1}{d_{\beta}} S(1-\varepsilon; 1) \right\} \otimes u(t),$$

where $d_a = e(2a) - 1$ and the symbol $S(a; z)$ stands for the positively oriented circle centered at the point z with starting and ending point a, ε is a small positive number and the argument of each factor of $u(t)$ on the oriented circle $S(\varepsilon; 0)$ or $S(1-\varepsilon; 1)$ is defined so that $\arg t$ takes values from 0 to 2π on $S(\varepsilon; 0)$, and $\arg(1-t)$ from 0 to 2π.

We refer the reader to [8] for the construction of regularized cycles in higher dimensional cases.

2. The solutions around ∞ in terms of the solutions around 0

In this section, let $\mathscr{L}_z$ be the local system determined by a function

$$u(t) = t_1^{\lambda_1}(t_1 - 1)^{\lambda_2}(t_2 - z)^{\lambda_3} t_2^{\lambda_4}(t_2 - t_1)^{\lambda_5} \tag{2.1}$$

on

$$T_z = \boldsymbol{C}^2 \backslash \{t_1 - t_2 = 0\} \cup \{t_1 = 0\} \cup \{t_2 = 0\} \cup \{t_1 - 1 = 0\} \cup \{t_2 - z = 0\}.$$

Under the *genericity condition* for the exponents λ_i $(1 \le i \le 5)$:

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_{145},\ \lambda_{125},\ \lambda_{345},\ \lambda_{12345} \notin \boldsymbol{Z}, \tag{2.2}$$

it follows that $\mathrm{rank}\, H_j(T_z, \mathscr{L}_z)$ and $\mathrm{rank}\, H_j^{\mathrm{lf}}(T_z, \mathscr{L}_z)$ vanish for $j \neq 2$ and $\dim H_2(T_z, \mathscr{L}_z) = \dim H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z) = 3$, and that the natural map $\imath : H_2(T_z, \mathscr{L}_z) \to H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z)$ is an isomorphism (See [5], [9], [11]). We assume this genericity condition. Thus the inverse map

$$\mathrm{reg} : H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z) \to H_2(T_z, \mathscr{L}_z)$$

is freely used.

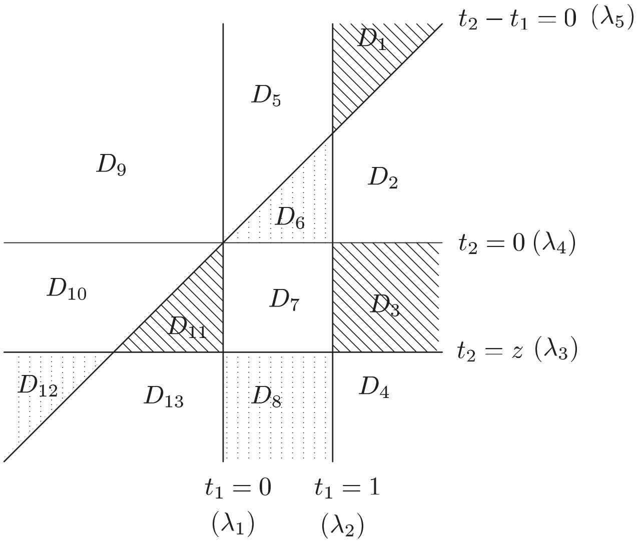


Fig. 1

For convenience to our purpose, we fix a complex variable z to be real such that $z < 0$ and assign the name D_j for $1 \le j \le 13$ to each domain of the real manifold $T_{\boldsymbol{R}}$ (the real locus of $T = T_z$) as in Fig. 1, and use the same name D_j to express the loaded cycle with coefficient $u_{D_j}(t)$ for simplicity. Here the orientation of D_j is fixed to be natural one induced from $T_{\boldsymbol{R}}$.

Then the loaded cycles $\operatorname{reg} D_1$, $\operatorname{reg} D_3$, $\operatorname{reg} D_{11}$ or D_1, D_3, D_{11} give a basis of $H_2(T_z, \mathscr{L}_z)$ or $H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z)$, and hence a fundamental set of solutions around the origin.

Proposition 2.1. (1) *If* $\operatorname{Re}(-\lambda_{345} - 1), \operatorname{Re}(\lambda_5 + 1), \operatorname{Re}(-\lambda_{12345} - 2), \operatorname{Re}(\lambda_2 + 1) > 0,$ we have

$$\begin{aligned}
\iint_{\operatorname{reg} D_1} u_{D_1}\, dt_1 dt_2 &= \iint_{D_1} u_{D_1}\, dt_1 dt_2 \\
&= \int_1^{+\infty} \int_{t_1}^{+\infty} t_1^{\lambda_1}(t_1 - 1)^{\lambda_2}(t_2 - z)^{\lambda_3} t_2^{\lambda_4}(t_2 - t_1)^{\lambda_5} dt_2 dt_1 \\
&= B(-\lambda_{345} - 1, \lambda_5 + 1) B(-\lambda_{12345} - 2, \lambda_2 + 1) \\
&\quad \times {}_3F_2\left(\begin{matrix} -\lambda_{345} - 1, -\lambda_{12345} - 2, -\lambda_3 \\ -\lambda_{34}, -\lambda_{1345} - 1 \end{matrix} ; z \right).
\end{aligned}$$

(2) *If* $\operatorname{Re}(\lambda_3 + 1), \operatorname{Re}(\lambda_4 + 1), \operatorname{Re}(-\lambda_{125} - 1), \operatorname{Re}(\lambda_2 + 1) > 0,$ *we have*

$$\begin{aligned}\iint_{\mathrm{reg}\,D_3} u_{D_3}\,dt_1dt_2 &= \iint_{D_3} u_{D_3}\,dt_1dt_2\\ &= \int_1^{+\infty}\int_z^0 t_1^{\lambda_1}(t_1-1)^{\lambda_2}(t_2-z)^{\lambda_3}(-t_2)^{\lambda_4}(t_1-t_2)^{\lambda_5}dt_2dt_1\\ &= B(\lambda_3+1,\lambda_4+1)B(-\lambda_{125}-1,\lambda_2+1)\\ &\quad\times(-z)^{\lambda_{34}+1}{}_3F_2\left({-\lambda_5,-\lambda_{125}-1,\lambda_4+1 \atop \lambda_{34}+2,-\lambda_{15}};z\right).\end{aligned}$$

(3) *If* $\mathrm{Re}(\lambda_1+1),\mathrm{Re}(\lambda_5+1),\mathrm{Re}(\lambda_{145}+2),\mathrm{Re}(\lambda_3+1)>0$, *we have*

$$\begin{aligned}\iint_{\mathrm{reg}\,D_{11}} u_{D_{11}}\,dt_1dt_2 &= \iint_{D_{11}} u_{D_{11}}\,dt_1dt_2\\ &= \int_z^0\int_{t_2}^0 (-t_1)^{\lambda_1}(1-t_1)^{\lambda_2}(t_2-z)^{\lambda_3}(-t_2)^{\lambda_4}(t_1-t_2)^{\lambda_5}dt_1dt_2\\ &= B(\lambda_1+1,\lambda_5+1)B(\lambda_{145}+2,\lambda_3+1)\\ &\quad\times(-z)^{\lambda_{1345}+2}{}_3F_2\left({-\lambda_2,\lambda_{145}+2,\lambda_1+1 \atop \lambda_{15}+2,\lambda_{1345}+3};z\right).\end{aligned}$$

Proof. (1) When $t_1=u_1^{-1}$, $t_2=u_1^{-1}u_2^{-1}$, the Jacobian is calculated as

$$\frac{\partial(t_1,t_2)}{\partial(u_1,u_2)}=\frac{1}{u_1^3u_2^2}.$$

Hence, the change of the integration variables such as $t_1=u_1^{-1}$, $t_2=u_1^{-1}u_2^{-1}$ implies that

$$\begin{aligned}&\int_1^{+\infty}\int_{t_1}^{+\infty} t_1^{\lambda_1}(t_1-1)^{\lambda_2}(t_2-z)^{\lambda_3}t_2^{\lambda_4}(t_2-t_1)^{\lambda_5}dt_2dt_1\\ &\quad=\int_0^1\int_0^1 u_1^{-\lambda_{12345}-3}(1-u_1)^{\lambda_2}u_2^{-\lambda_{345}-2}(1-u_2)^{\lambda_5}(1-zu_1u_2)^{\lambda_3}du_2du_1.\end{aligned}$$

On the other hand, the binomial theorem

$$(1-zu_1u_2)^{\lambda_3}=\sum_{n\geq 0}\frac{(-\lambda_3)_n}{n!}(zu_1u_2)^n$$

leads to

$$
\begin{aligned}
&\int_0^1 u_2^{-\lambda_{345}-2}(1-u_2)^{\lambda_5}(1-zu_1u_2)^{\lambda_3}du_2\\
&\quad=\sum_{n\geq 0}\frac{(-\lambda_3)_n}{n!}(zu_1)^n\int_0^1 u_2^{-\lambda_{345}-2+n}(1-u_2)^{\lambda_5}du_2\\
&\quad=\frac{\Gamma(-\lambda_{345}-1)\Gamma(\lambda_5+1)}{\Gamma(-\lambda_{34})}\sum_{n\geq 0}\frac{(-\lambda_3)_n(-\lambda_{345}-1)_n}{n!(-\lambda_{34})_n}(zu_1)^n
\end{aligned}
$$

for $|u_1|, |z| < 1$. Therefore, we obtain the equality

$$
\begin{aligned}
&\int_0^1\int_0^1 u_1^{-\lambda_{12345}-3}(1-u_1)^{\lambda_2}u_2^{-\lambda_{345}-2}(1-u_2)^{\lambda_5}(1-zu_1u_2)^{\lambda_3}du_2du_1\\
&\quad=B(-\lambda_{345}-1,\lambda_5+1)\\
&\qquad\times\sum_{n\geq 0}\frac{(-\lambda_3)_n(-\lambda_{345}-1)_n}{n!(-\lambda_{34})_n}z^n\int_0^1 u_1^{-\lambda_{12345}-3+n}(1-u_1)^{\lambda_2}du_1\\
&\quad=B(-\lambda_{345}-1,\lambda_5+1)B(-\lambda_{12345}-2,\lambda_2+1)\\
&\qquad\times\sum_{n\geq 0}\frac{(-\lambda_3)_n(-\lambda_{345}-1)_n(-\lambda_{12345}-2)_n}{n!(-\lambda_{34})_n(-\lambda_{1345}-1)_n}z^n.
\end{aligned}
$$

This proves the equality we need.

The equality (2) follows from the same argument by the change of the variable as $t_2 \to zt_2$. The equality (3) follows from the same argument by the change of the variables as $t_1 \to t_2t_1$ and $t_2 \to zt_2$. This completes the proof of Proposition. □

The loaded cycles $D_6, D_8, D_{12} \in H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z)$ or $\operatorname{reg} D_6, \operatorname{reg} D_8, \operatorname{reg} D_{12} \in H_2(T_z, \mathscr{L}_z)$ give a fundamental set of solutions around the infinity ∞.

Proposition 2.2. (1) *If* $\mathrm{Re}(\lambda_4+1), \mathrm{Re}(\lambda_5+1), \mathrm{Re}(\lambda_{145}+2), \mathrm{Re}(\lambda_2+1) > 0$, *we have*

$$
\begin{aligned}
\iint_{\operatorname{reg} D_6} u_{D_6}\,dt_1dt_2 &= \iint_{D_6} u_{D_6}\,dt_1dt_2\\
&=\int_0^1\int_0^{t_1} t_1^{\lambda_1}(1-t_1)^{\lambda_2}(t_2-z)^{\lambda_3}t_2^{\lambda_4}(t_1-t_2)^{\lambda_5}dt_2dt_1\\
&=B(\lambda_4+1,\lambda_5+1)B(\lambda_{145}+2,\lambda_2+1)\\
&\quad\times(-z)^{\lambda_3}{}_3F_2\left(\begin{matrix}-\lambda_3,\lambda_{145}+2,\lambda_4+1\\ \lambda_{15}+2,\lambda_{1245}+3\end{matrix};\frac{1}{z}\right).
\end{aligned}
$$

(2) *If* $\mathrm{Re}(-\lambda_{345}-1), \mathrm{Re}(\lambda_3+1), \mathrm{Re}(\lambda_1+1), \mathrm{Re}(\lambda_2+1) > 0$, *we have*

$$\begin{aligned}\iint_{\mathrm{reg}\, D_8} u_{D_8}\, dt_1 dt_2 &= \iint_{D_8} u_{D_8}\, dt_1 dt_2 \\ &= \int_0^1 \int_{-\infty}^{z} t_1^{\lambda_1}(1-t_1)^{\lambda_2}(z-t_2)^{\lambda_3}(-t_2)^{\lambda_4}(t_1-t_2)^{\lambda_5} dt_1 dt_2 \\ &= B(-\lambda_{345}-1, \lambda_3+1) B(\lambda_1+1, \lambda_2+1) \\ &\quad \times (-z)^{\lambda_{345}+1} {}_3F_2\left(\begin{matrix} -\lambda_5, -\lambda_{345}-1, \lambda_1+1 \\ -\lambda_{45}, \lambda_{12}+2 \end{matrix}; \frac{1}{z}\right).\end{aligned}$$

(3) *If* $\mathrm{Re}(-\lambda_{125}-1), \mathrm{Re}(\lambda_5+1), \mathrm{Re}(-\lambda_{12345}-2), \mathrm{Re}(\lambda_3+1) > 0$, *we have*

$$\begin{aligned}\iint_{\mathrm{reg}\, D_{12}} u_{D_{12}}\, dt_1 dt_2 &= \iint_{D_{12}} u_{D_{12}}\, dt_1 dt_2 \\ &= \int_{-\infty}^{z} \int_{-\infty}^{t_2} (-t_1)^{\lambda_1}(1-t_1)^{\lambda_2}(z-t_2)^{\lambda_3}(-t_2)^{\lambda_4}(t_2-t_1)^{\lambda_5} dt_1 dt_2 \\ &= B(-\lambda_{125}-1, \lambda_5+1) B(\lambda_3+1, -\lambda_{12345}-2) \\ &\quad \times (-z)^{\lambda_{12345}+2} {}_3F_2\left(\begin{matrix} -\lambda_2, -\lambda_{125}-1, -\lambda_{12345}-2 \\ -\lambda_{1245}-1, -\lambda_{12} \end{matrix}; \frac{1}{z}\right).\end{aligned}$$

Proof. The same argument as in Proposition 2.1 implies the equality (1) by the the change of the variable as $t_2 \to t_1 t_2$, (2) by $t_2 \to z/t_2$, and (3) by $t_1 \to t_2/t_1$ and $t_2 \to z/t_2$. This completes the proof. □

The intersection of the complex line $L_{t_1} : t_1 = h$ for $1 < h < \infty$ and the domains D_4, D_3, D_2, D_1 consists of the segments $(-\infty, z)$, $(z, 0)$, $(0, h)$, $(h, +\infty)$ in the t_2-plane. Thus, it is seen that a trivial loop with counterclockwise direction in the upper half plane of the t_2-plane is homologous to

$$e_{345}\, \mathrm{reg}\overrightarrow{(-\infty, z)} + e_{45}\, \mathrm{reg}\overrightarrow{(z, 0)} + e_5\, \mathrm{reg}\overrightarrow{(0, h)} + \mathrm{reg}\overrightarrow{(h, +\infty)}$$

and a trivial loop with clockwise direction in the lower half plane of the t_2-plane is homologous to

$$e_{345}^{-1}\, \mathrm{reg}\overrightarrow{(-\infty, z)} + e_{45}^{-1}\, \mathrm{reg}\overrightarrow{(z, 0)} + e_5^{-1}\, \mathrm{reg}\overrightarrow{(0, h)} + \mathrm{reg}\overrightarrow{(h, +\infty)}.$$

This implies

$$e_{345}\, \mathrm{reg}\, D_4 + e_{45}\, \mathrm{reg}\, D_3 + e_5\, \mathrm{reg}\, D_2 + \mathrm{reg}\, D_1 = 0$$

and

$$e_{345}^{-1} \operatorname{reg} D_4 + e_{45}^{-1} \operatorname{reg} D_3 + e_5^{-1} \operatorname{reg} D_2 + \operatorname{reg} D_1 = 0$$

in the sense of twisted homology. Similarly, we have the following.

$$\begin{aligned}
&\operatorname{reg} D_1 + e_5 \operatorname{reg} D_2 + e_{45} \operatorname{reg} D_3 + e_{345} \operatorname{reg} D_4 = 0,\\
&\operatorname{reg} D_5 + e_5 \operatorname{reg} D_6 + e_{45} \operatorname{reg} D_7 + e_{345} \operatorname{reg} D_8 = 0,\\
&\operatorname{reg} D_9 + e_4 \operatorname{reg} D_{10} + e_{45} \operatorname{reg} D_{11} + e_{34} \operatorname{reg} D_{12} + e_{345} \operatorname{reg} D_{13} = 0,\\
&\operatorname{reg} D_9 + e_1 \operatorname{reg} D_5 + e_{15} \operatorname{reg} D_6 + e_{12} \operatorname{reg} D_1 + e_{125} \operatorname{reg} D_2 = 0,\\
&\operatorname{reg} D_{10} + e_5 \operatorname{reg} D_{11} + e_{15} \operatorname{reg} D_7 + e_{125} \operatorname{reg} D_3 = 0,\\
&\operatorname{reg} D_{12} + e_5 \operatorname{reg} D_{13} + e_{15} \operatorname{reg} D_8 + e_{125} \operatorname{reg} D_4 = 0,
\end{aligned}$$

and

$$\begin{aligned}
&\operatorname{reg} D_1 + e_5^{-1} \operatorname{reg} D_2 + e_{45}^{-1} \operatorname{reg} D_3 + e_{345}^{-1} \operatorname{reg} D_4 = 0,\\
&\operatorname{reg} D_5 + e_5^{-1} \operatorname{reg} D_6 + e_{45}^{-1} \operatorname{reg} D_7 + e_{345}^{-1} \operatorname{reg} D_8 = 0,\\
&\operatorname{reg} D_9 + e_4^{-1} \operatorname{reg} D_{10} + e_{45}^{-1} \operatorname{reg} D_{11} + e_{34}^{-1} \operatorname{reg} D_{12} + e_{345}^{-1} \operatorname{reg} D_{13} = 0,\\
&\operatorname{reg} D_9 + e_1^{-1} \operatorname{reg} D_5 + e_{15}^{-1} \operatorname{reg} D_6 + e_{12}^{-1} \operatorname{reg} D_1 + e_{125}^{-1} \operatorname{reg} D_2 = 0,\\
&\operatorname{reg} D_{10} + e_5^{-1} \operatorname{reg} D_{11} + e_{15}^{-1} \operatorname{reg} D_7 + e_{125}^{-1} \operatorname{reg} D_3 = 0,\\
&\operatorname{reg} D_{12} + e_5^{-1} \operatorname{reg} D_{13} + e_{15}^{-1} \operatorname{reg} D_8 + e_{125}^{-1} \operatorname{reg} D_4 = 0.
\end{aligned}$$

Here the former ones are derived from the trivial loops in the upper half plane of the t_1-planes with fixed t_2 or the t_2-planes with fixed t_1, and the latter ones are derived from the trivial loops in the corresponding lower half planes.

These equations give the relations to express each twisted cycle $\operatorname{reg} D_j$ in terms of the cycles

$$\operatorname{reg} D_1, \operatorname{reg} D_3, \operatorname{reg} D_{11} \in H_2(T_z, \mathscr{L}_z),$$

which correspond to the fundamental set of solutions around 0.

Proposition 2.3. *If* λ_i $(1 \leq i \leq 5)$, $\lambda_{125}, \lambda_{145}, \lambda_{345}, \lambda_{12345}, \lambda_{34}, \lambda_{15}, \lambda_{1345} \notin \boldsymbol{Z}$, *then*

$$\operatorname{reg} D_2 = -\frac{s(\lambda_{345})}{s(\lambda_{34})} \operatorname{reg} D_1 - \frac{s(\lambda_3)}{s(\lambda_{34})} \operatorname{reg} D_3,$$

$$\operatorname{reg} D_4 = \frac{s(\lambda_5)}{s(\lambda_{34})} \operatorname{reg} D_1 - \frac{s(\lambda_4)}{s(\lambda_{34})} \operatorname{reg} D_3,$$

$$\operatorname{reg} D_5 = -\frac{s(\lambda_{12345})}{s(\lambda_{1345})} \operatorname{reg} D_1 - \frac{s(\lambda_3)}{s(\lambda_{1345})} \operatorname{reg} D_{11},$$

$$\begin{aligned}\operatorname{reg} D_6 = {} & \frac{s(\lambda_{345})s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})} \operatorname{reg} D_1 + \frac{s(\lambda_3)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})} \operatorname{reg} D_3 \\ & - \frac{s(\lambda_1)s(\lambda_3)}{s(\lambda_{15})s(\lambda_{1345})} \operatorname{reg} D_{11},\end{aligned}$$

$$\operatorname{reg} D_7 = -\frac{s(\lambda_{125})}{s(\lambda_{15})} \operatorname{reg} D_1 - \frac{s(\lambda_5)}{s(\lambda_{15})} \operatorname{reg} D_{11},$$

$$\begin{aligned}\operatorname{reg} D_8 = {} & \frac{s(\lambda_5)s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})} \operatorname{reg} D_1 + \frac{s(\lambda_4)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})} \operatorname{reg} D_3 \\ & - \frac{s(\lambda_5)s(\lambda_{145})}{s(\lambda_{15})s(\lambda_{1345})} \operatorname{reg} D_{11},\end{aligned}$$

$$\begin{aligned}\operatorname{reg} D_9 = {} & -\frac{s(\lambda_2)s(\lambda_5)}{s(\lambda_{34})s(\lambda_{1345})} \operatorname{reg} D_1 - \frac{s(\lambda_2)s(\lambda_3)}{s(\lambda_{34})s(\lambda_{15})} \operatorname{reg} D_3 \\ & - \frac{s(\lambda_3)s(\lambda_5)}{s(\lambda_{15})s(\lambda_{1345})} \operatorname{reg} D_{11},\end{aligned}$$

$$\operatorname{reg} D_{10} = -\frac{s(\lambda_2)}{s(\lambda_{15})} \operatorname{reg} D_3 - \frac{s(\lambda_1)}{s(\lambda_{15})} \operatorname{reg} D_{11},$$

$$\begin{aligned}\operatorname{reg} D_{12} = {} & \frac{s(\lambda_2)s(\lambda_{345})}{s(\lambda_{34})s(\lambda_{1345})} \operatorname{reg} D_1 - \frac{s(\lambda_2)s(\lambda_4)}{s(\lambda_{34})s(\lambda_{15})} \operatorname{reg} D_3 \\ & + \frac{s(\lambda_1)s(\lambda_{145})}{s(\lambda_{15})s(\lambda_{1345})} \operatorname{reg} D_{11},\end{aligned}$$

$$\operatorname{reg} D_{13} = -\frac{s(\lambda_2)}{s(\lambda_{1345})} \operatorname{reg} D_1 - \frac{s(\lambda_{145})}{s(\lambda_{1345})} \operatorname{reg} D_{11}.$$

These expressions of $\operatorname{reg} D_6$, $\operatorname{reg} D_8$, $\operatorname{reg} D_{12}$ induce the following:

Theorem 2.4. *If* $\operatorname{Re}(\alpha_i) > 0$ $(i = 1, 2)$, $\operatorname{Re}(\beta_i - \alpha_i) > 0$ $(i = 1, 2)$, $\operatorname{Re}(\beta_1 + \beta_2 - \alpha_1 - \alpha_2) > 0$, $1 > \operatorname{Re}(\beta_1 - \alpha_j)$ $(j = 2, 3)$, $1 > \operatorname{Re}(\beta_2 - \alpha_j)$ $(j = 1, 3)$, $1 > \operatorname{Re}(\alpha_3)$ *and* $\beta_1, \beta_2, \beta_2 - \beta_1 \notin \boldsymbol{Z}$, *we have*

$$\begin{aligned}&(-z)^{-\alpha_1} {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_1 - \beta_1 + 1, \alpha_1 - \beta_2 + 1 \\ \alpha_1 - \alpha_2 + 1, \alpha_1 - \alpha_3 + 1\end{matrix}; \frac{1}{z}\right) \\ &\quad = \frac{\Gamma(1 + \alpha_1 - \alpha_2)\Gamma(1 + \alpha_1 - \alpha_3)\Gamma(1 - \beta_1)\Gamma(1 - \beta_2)}{\Gamma(1 + \alpha_1 - \beta_1)\Gamma(1 + \alpha_1 - \beta_2)\Gamma(1 - \alpha_2)\Gamma(1 - \alpha_3)} \times {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2\end{matrix}; z\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_3)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(1+\alpha_1-\beta_2)\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_3)\Gamma(\alpha_1)}\\
&\times(-z)^{1-\beta_1}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_1+1,\alpha_2-\beta_1+1,\alpha_3-\beta_1+1\\ 2-\beta_1,\beta_2-\beta_1+1\end{matrix};z\right)\\
&+\frac{\Gamma(1+\alpha_1-\alpha_2)\Gamma(1+\alpha_1-\alpha_3)\Gamma(\beta_2-1)\Gamma(\beta_2-\beta_1)}{\Gamma(1+\alpha_1-\beta_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_3)\Gamma(\alpha_1)}\\
&\times(-z)^{1-\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_2+1,\alpha_2-\beta_2+1,\alpha_3-\beta_2+1\\ \beta_1-\beta_2+1,2-\beta_2\end{matrix};z\right),
\end{aligned}
$$

$$
\begin{aligned}
&(-z)^{-\alpha_2}{}_3F_2\left(\begin{matrix}\alpha_2,\alpha_2-\beta_1+1,\alpha_2-\beta_2+1\\ \alpha_2-\alpha_1+1,\alpha_2-\alpha_3+1\end{matrix};\frac{1}{z}\right)\\
&=\frac{\Gamma(1+\alpha_2-\alpha_1)\Gamma(1+\alpha_2-\alpha_3)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1+\alpha_2-\beta_1)\Gamma(1+\alpha_2-\beta_2)\Gamma(1-\alpha_1)\Gamma(1-\alpha_3)}\times{}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};z\right)\\
&+\frac{\Gamma(1+\alpha_2-\alpha_1)\Gamma(1+\alpha_2-\alpha_3)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(1+\alpha_2-\beta_2)\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_3)\Gamma(\alpha_2)}\\
&\times(-z)^{1-\beta_1}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_1+1,\alpha_2-\beta_1+1,\alpha_3-\beta_1+1\\ 2-\beta_1,\beta_2-\beta_1+1\end{matrix};z\right)\\
&+\frac{\Gamma(1+\alpha_2-\alpha_1)\Gamma(1+\alpha_2-\alpha_3)\Gamma(\beta_2-1)\Gamma(\beta_2-\beta_1)}{\Gamma(1+\alpha_2-\beta_1)\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_3)\Gamma(\alpha_2)}\\
&\times(-z)^{1-\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_2+1,\alpha_2-\beta_2+1,\alpha_3-\beta_2+1\\ \beta_1-\beta_2+1,2-\beta_2\end{matrix};z\right),
\end{aligned}
$$

$$
\begin{aligned}
&(-z)^{-\alpha_3}{}_3F_2\left(\begin{matrix}\alpha_3,\alpha_3-\beta_1+1,\alpha_3-\beta_2+1\\ \alpha_3-\alpha_1+1,\alpha_3-\alpha_2+1\end{matrix};\frac{1}{z}\right)\\
&=\frac{\Gamma(1+\alpha_3-\alpha_2)\Gamma(1+\alpha_3-\alpha_1)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1+\alpha_3-\beta_1)\Gamma(1+\alpha_3-\beta_2)\Gamma(1-\alpha_2)\Gamma(1-\alpha_1)}\times{}_3F_2\left(\begin{matrix}\alpha_1,\alpha_2,\alpha_3\\ \beta_1,\beta_2\end{matrix};z\right)\\
&+\frac{\Gamma(1+\alpha_3-\alpha_2)\Gamma(1+\alpha_3-\alpha_1)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(1+\alpha_3-\beta_2)\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_1)\Gamma(\alpha_3)}\\
&\times(-z)^{1-\beta_1}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_1+1,\alpha_2-\beta_1+1,\alpha_3-\beta_1+1\\ 2-\beta_1,\beta_2-\beta_1+1\end{matrix};z\right)\\
&+\frac{\Gamma(1+\alpha_3-\alpha_2)\Gamma(1+\alpha_3-\alpha_1)\Gamma(\beta_2-1)\Gamma(\beta_2-\beta_1)}{\Gamma(1+\alpha_3-\beta_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_1)\Gamma(\alpha_3)}\\
&\times(-z)^{1-\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_2+1,\alpha_2-\beta_2+1,\alpha_3-\beta_2+1\\ \beta_1-\beta_2+1,2-\beta_2\end{matrix};z\right).
\end{aligned}
$$

Proof. Combining Propositions 2.1, 2.2 and 2.3, we obtain

$$
\begin{aligned}
&(-z)^{\lambda_3}{}_3F_2\left(\begin{matrix}-\lambda_3, \lambda_{145}+2, \lambda_4+1\\ \lambda_{15}+2, \lambda_{1245}+3\end{matrix}; \frac{1}{z}\right)\\
&\quad=\frac{\Gamma(\lambda_{34}+1)\Gamma(\lambda_{45}+2)\Gamma(\lambda_{1345}+2)\Gamma(\lambda_{1245}+3)}{\Gamma(\lambda_4+1)\Gamma(\lambda_{145}+2)\Gamma(\lambda_{345}+2)\Gamma(\lambda_{12345}+3)}\\
&\qquad\times {}_3F_2\left(\begin{matrix}-\lambda_{345}-1, -\lambda_{12345}-2, -\lambda_3\\ -\lambda_{34}, -\lambda_{1345}-1\end{matrix}; z\right)\\
&\qquad+\frac{\Gamma(-\lambda_{34}-1)\Gamma(\lambda_{15}+1)\Gamma(\lambda_{45}+2)\Gamma(\lambda_{1245}+3)}{\Gamma(-\lambda_3)\Gamma(\lambda_5+1)\Gamma(\lambda_{145}+2)\Gamma(\lambda_{125}+2)}\\
&\qquad\times (-z)^{\lambda_{34}+1}{}_3F_2\left(\begin{matrix}-\lambda_5, -\lambda_{125}-1, \lambda_4+1\\ \lambda_{34}+2, -\lambda_{15}\end{matrix}; z\right)\\
&\qquad+\frac{\Gamma(-\lambda_{1345}-1)\Gamma(-\lambda_{15}-1)\Gamma(\lambda_{45}+2)\Gamma(\lambda_{1245}+3)}{\Gamma(-\lambda_1)\Gamma(-\lambda_3)\Gamma(\lambda_2+1)\Gamma(\lambda_4+1)}\\
&\qquad\times (-z)^{\lambda_{1345}+2}{}_3F_2\left(\begin{matrix}-\lambda_2, \lambda_{145}+2, \lambda_1+1\\ \lambda_{15}+2, \lambda_{1345}+3\end{matrix}; z\right),
\end{aligned}
$$

$$
\begin{aligned}
&(-z)^{\lambda_{345}+1}{}_3F_2\left(\begin{matrix}-\lambda_5, -\lambda_{345}-1, \lambda_1+1\\ -\lambda_{45}, \lambda_{12}+2\end{matrix}; \frac{1}{z}\right)\\
&\quad=\frac{\Gamma(\lambda_{34}+1)\Gamma(\lambda_{45}+2)\Gamma(\lambda_{1345}+2)\Gamma(\lambda_{1245}+3)}{\Gamma(\lambda_4+1)\Gamma(\lambda_{145}+2)\Gamma(\lambda_{345}+2)\Gamma(\lambda_{12345}+3)}\\
&\qquad\times {}_3F_2\left(\begin{matrix}-\lambda_{345}-1, -\lambda_{12345}-2, -\lambda_3\\ -\lambda_{34}, -\lambda_{1345}-1\end{matrix}; z\right)\\
&\qquad+\frac{\Gamma(-\lambda_{34}-1)\Gamma(\lambda_{15}+1)\Gamma(\lambda_{45}+2)\Gamma(\lambda_{1245}+3)}{\Gamma(-\lambda_3)\Gamma(\lambda_5+1)\Gamma(\lambda_{145}+2)\Gamma(\lambda_{125}+2)}\\
&\qquad\times (-z)^{\lambda_{34}+1}{}_3F_2\left(\begin{matrix}-\lambda_5, -\lambda_{125}-1, \lambda_4+1\\ \lambda_{34}+2, -\lambda_{15}\end{matrix}; z\right)\\
&\qquad+\frac{\Gamma(-\lambda_{1345}-1)\Gamma(-\lambda_{15}-1)\Gamma(\lambda_{45}+2)\Gamma(\lambda_{1245}+3)}{\Gamma(-\lambda_1)\Gamma(-\lambda_3)\Gamma(\lambda_2+1)\Gamma(\lambda_4+1)}\\
&\qquad\times (-z)^{\lambda_{1345}+2}{}_3F_2\left(\begin{matrix}-\lambda_2, \lambda_{145}+2, \lambda_1+1\\ \lambda_{15}+2, \lambda_{1345}+3\end{matrix}; z\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&(-z)^{\lambda_{12345}+2}{}_3F_2\left(\begin{matrix}-\lambda_2, -\lambda_{125}-1, -\lambda_{12345}-2\\ -\lambda_{1245}-1, -\lambda_{12}\end{matrix}; \frac{1}{z}\right)\\
&\quad=\frac{\Gamma(-\lambda_{12})\Gamma(-\lambda_{1245}-1)\Gamma(\lambda_{34}+1)\Gamma(\lambda_{1345}+2)}{\Gamma(\lambda_3+1)\Gamma(-\lambda_{125}-1)\Gamma(-\lambda_2)\Gamma(\lambda_{345}+2)}
\end{aligned}
$$

$$
\begin{aligned}
&\times {}_3F_2\left(\begin{matrix} -\lambda_{345}-1, -\lambda_{12345}-2, -\lambda_3 \\ -\lambda_{34}, -\lambda_{1345}-1 \end{matrix}; z\right) \\
&+\frac{\Gamma(-\lambda_{34}-1)\Gamma(\lambda_{15}+1)\Gamma(-\lambda_{12})\Gamma(-\lambda_{1245}-1)}{\Gamma(-\lambda_2)\Gamma(\lambda_5+1)\Gamma(-\lambda_4)\Gamma(-\lambda_{12345}-2)} \\
&\times (-z)^{\lambda_{34}+1}{}_3F_2\left(\begin{matrix} -\lambda_5, -\lambda_{125}-1, \lambda_4+1 \\ \lambda_{34}+2, -\lambda_{15} \end{matrix}; z\right) \\
&+\frac{\Gamma(-\lambda_{1345}-2)\Gamma(-\lambda_{15}-1)\Gamma(-\lambda_{12})\Gamma(-\lambda_{1245}-1)}{\Gamma(-\lambda_1)\Gamma(-\lambda_{145}-1)\Gamma(-\lambda_{125}-1)\Gamma(-\lambda_{12345}-2)} \\
&\times (-z)^{\lambda_{1345}+2}{}_3F_2\left(\begin{matrix} -\lambda_2, \lambda_{145}+2, \lambda_1+1 \\ \lambda_{15}+2, \lambda_{1345}+3 \end{matrix}; z\right).
\end{aligned}
$$

When the parameters are given by

$$
\begin{aligned}
&\lambda_1 = \alpha_1 - \beta_2, \qquad \lambda_2 = \beta_2 - \alpha_2 - 1, \qquad \lambda_3 = -\alpha_3, \\
&\lambda_4 = \alpha_3 - \beta_1, \qquad \lambda_5 = \beta_1 - \alpha_1 - 1,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&-\lambda_{345} - 1 = \alpha_1, \quad -\lambda_{12345} - 2 = \alpha_2, \quad -\lambda_{125} - 1 = 1 + \alpha_2 - \beta_1, \\
&\lambda_{145} + 2 = 1 + \alpha_3 - \beta_2, \quad \lambda_{34} = \beta_1, \quad -\lambda_{1345} = 1 + \beta_2, \quad -\lambda_{15} = 1 - \beta_1 + \beta_2.
\end{aligned}
$$

Therefore we reach the desired result. □

Remark. The condition on the exponents can be relaxed if the meaning of ${}_3F_2$ is considered as its analytic continuation.

3. The solution around 1 in terms of the solutions around 0

In this section, let $\mathscr{L}_z$ be a local system determined by a function

$$
u(t) = t_1^{\lambda_1}(t_1 - 1)^{\lambda_2} t_2^{\lambda_3}(t_2 - z)^{\lambda_4}(t_2 - t_1)^{\lambda_5} \tag{3.1}
$$

on

$$
T_z = \boldsymbol{C}^2 \backslash \{t_1 - t_2 = 0\} \cup \{t_1 = 0\} \cup \{t_2 = 0\} \cup \{t_1 - 1 = 0\} \cup \{t_2 - z = 0\}.
$$

In this case, the genericity condition for the exponents λ_i $(1 \le i \le 5)$ is that

$$
\lambda_1,\ \lambda_2,\ \lambda_3,\ \lambda_4,\ \lambda_5,\ \lambda_{135},\ \lambda_{125},\ \lambda_{345},\ \lambda_{12345} \notin \boldsymbol{Z}. \tag{3.2}
$$

We assume this condition and the map

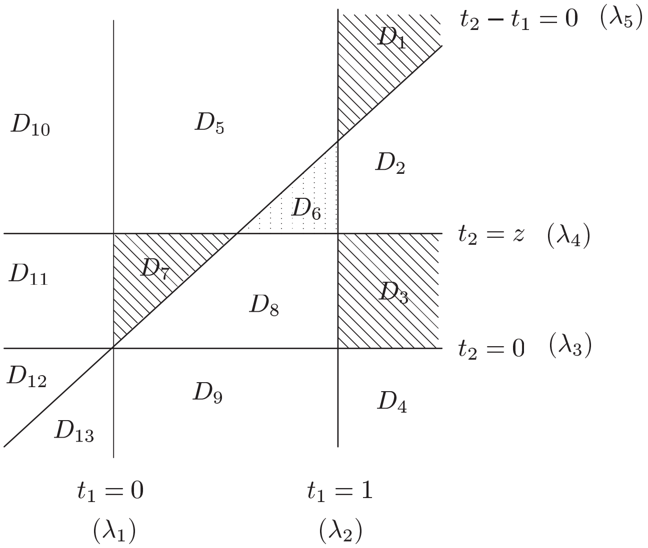


Fig. 2

$$\mathrm{reg} : H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z) \to H_2(T_z, \mathscr{L}_z)$$

is freely used as in the previous Section.

We fix a complex variable z to be real such that $0 < z < 1$ and assign the name D_j to each domain of the real manifold $T_{\boldsymbol{R}}$ as in Fig. 2, which also expresses the loaded cycle $D_j \otimes u_{D_j}(t)$ for brevity.

Then the functions over the cycles reg D_1, reg D_3, reg D_7 are the same as the function over reg D_1, reg D_3, reg D_{11} in Proposition 2.1. Hence, they give a fundamental set of solutions around the origin.

Proposition 3.1. (1) *If* $\mathrm{Re}(-\lambda_{345} - 1), \mathrm{Re}(\lambda_5 + 1), \mathrm{Re}(-\lambda_{12345} - 2), \mathrm{Re}(\lambda_2 + 1) > 0$, *we have*

$$\begin{aligned}
\iint_{\mathrm{reg}\, D_1} u_{D_1}\, dt_1 dt_2 &= \iint_{D_1} u_{D_1}\, dt_1 dt_2 \\
&= \int_1^{+\infty} \int_{t_1}^{+\infty} t_1^{\lambda_1} (t_1 - 1)^{\lambda_2} t_2^{\lambda_3} (t_2 - z)^{\lambda_4} (t_2 - t_1)^{\lambda_5} dt_2 dt_1 \\
&= B(-\lambda_{345} - 1, \lambda_5 + 1) B(-\lambda_{12345} - 2, \lambda_2 + 1) \\
&\quad \times {}_3F_2\left(\begin{matrix} -\lambda_{345} - 1, -\lambda_{12345} - 2, -\lambda_4 \\ -\lambda_{34}, -\lambda_{1345} - 1 \end{matrix} ; z \right).
\end{aligned}$$

(2) *If* $\mathrm{Re}(\lambda_3 + 1), \mathrm{Re}(\lambda_4 + 1), \mathrm{Re}(-\lambda_{125} - 1), \mathrm{Re}(\lambda_2 + 1) > 0$, *we have*

$$\begin{aligned}\iint_{\mathrm{reg}\, D_3} u_{D_3}\, dt_1 dt_2 &= \iint_{D_3} u_{D_3}\, dt_1 dt_2 \\ &= \int_1^{+\infty}\int_0^z t_1^{\lambda_1}(t_1-1)^{\lambda_2} t_2^{\lambda_3}(z-t_2)^{\lambda_4}(t_1-t_2)^{\lambda_5} dt_2 dt_1 \\ &= B(\lambda_3+1,\lambda_4+1)B(-\lambda_{125}-1,\lambda_2+1) \\ &\quad \times z^{\lambda_{34}+1}{}_3F_2\left(\begin{matrix}-\lambda_5,-\lambda_{125}-1,\lambda_3+1\\ \lambda_{34}+2,-\lambda_{15}\end{matrix};z\right).\end{aligned}$$

(3) *If* $\mathrm{Re}(\lambda_1+1),\mathrm{Re}(\lambda_5+1),\mathrm{Re}(\lambda_{135}+2),\mathrm{Re}(\lambda_4+1)>0,$ *we have*

$$\begin{aligned}\iint_{\mathrm{reg}\, D_7} u_{D_7}\, dt_1 dt_2 &= \iint_{D_7} u_{D_7}\, dt_1 dt_2 \\ &= \int_0^z\int_0^{t_2} t_1^{\lambda_1}(1-t_1)^{\lambda_2} t_2^{\lambda_3}(z-t_2)^{\lambda_4}(t_2-t_1)^{\lambda_5} dt_1 dt_2 \\ &= B(\lambda_1+1,\lambda_5+1)B(\lambda_{135}+2,\lambda_4+1) \\ &\quad \times z^{\lambda_{1345}+2}{}_3F_2\left(\begin{matrix}-\lambda_2,\lambda_{135}+2,\lambda_1+1\\ \lambda_{15}+2,\lambda_{1345}+3\end{matrix};z\right).\end{aligned}$$

The cycle $\mathrm{reg}\, D_6$ gives the non-holomorphic solution around the point $z=1$. Indeed,

Theorem 3.2. *If* $\mathrm{Re}(\lambda_2+1),\mathrm{Re}(\lambda_5+1),\mathrm{Re}(\lambda_{25}+2),\mathrm{Re}(\lambda_4+1)>0,$ *we have*

$$\begin{aligned}\iint_{\mathrm{reg}\, D_6} u_{D_6}\, dt_1 dt_2 &= \iint_{D_6} u_{D_6}\, dt_1 dt_2 \\ &= \int_z^1\int_0^{t_2} t_1^{\lambda_1}(1-t_1)^{\lambda_2} t_2^{\lambda_3}(t_2-z)^{\lambda_4}(t_1-t_2)^{\lambda_5} dt_1 dt_2 \\ &= B(\lambda_2+1,\lambda_5+1)B(\lambda_{25}+2,\lambda_4+1)\times(1-z)^{\lambda_{245}+2} \\ &\quad \times \sum_{n_1,n_2\geq 0}\frac{(-\lambda_1)_{n_1}(\lambda_2+1)_{n_1}(-\lambda_3)_{n_2}(\lambda_{25}+2)_{n_1+n_2}}{n_1!(\lambda_{25}+2)_{n_1}n_2!(\lambda_{245}+3)_{n_1+n_2}}(1-z)^{n_1+n_2}.\end{aligned}$$

Proof. The change of integration variables such as $t_1=1+u_1u_2(z-1)$, $t_2=1+u_2(z-1)$ makes

$$\frac{\partial(t_1,t_2)}{\partial(u_1,u_2)}=(1-z)^2u_2$$

and

$$\begin{aligned}
&\int_z^1\int_0^{t_2} t_1^{\lambda_1}(1-t_1)^{\lambda_2}t_2^{\lambda_3}(t_2-z)^{\lambda_4}(t_1-t_2)^{\lambda_5}dt_1dt_2\\
&\qquad=(1-z)^{2+\lambda_{245}}\int_0^1\int_0^1 u_1^{\lambda_2}u_2^{\lambda_{25}+1}(1-u_1)^{\lambda_5}(1-u_2)^{\lambda_4}\\
&\qquad\quad\times\{1-(1-z)u_1u_2\}^{\lambda_1}\{1-(1-z)u_2\}^{\lambda_3}du_1du_2.
\end{aligned}$$

The condition $\mathrm{Re}(\lambda_2+1),\mathrm{Re}(\lambda_5+1),\mathrm{Re}(\lambda_{25}+2),\mathrm{Re}(\lambda_4+1)>0$ guarantees the convergence of the integral. Moreover, by means of the formula

$$\int_0^1 v^{\mu_1}(1-v)^{\mu_2}(1-zv)^{\mu_3}dv=B(\mu_1+1,\mu_2+1)\sum_{n\geq 0}\frac{(-\mu_3)_n(\mu_1+1)_n}{n!(\mu_1+\mu_2)_n}z^n,$$

we have

$$\begin{aligned}
&\int_0^1\int_0^1 u_1^{\lambda_2}u_2^{\lambda_{25}+1}(1-u_1)^{\lambda_5}(1-u_2)^{\lambda_4}\\
&\qquad\times\{1-(1-z)u_1u_2\}^{\lambda_1}\{1-(1-z)u_2\}^{\lambda_3}du_1du_2\\
&=B(\lambda_2+1,\lambda_5+1)\sum_{n\geq 0}\frac{(-\lambda_1)_n(\lambda_2+1)_n}{n!(\lambda_{25}+2)_n}\\
&\qquad\times\int_0^1 u_2^{\lambda_{25}+1+n}(1-u_2)^{\lambda_4}\{1-(1-z)u_2\}^{\lambda_3}du_2\\
&=B(\lambda_2+1,\lambda_5+1)\\
&\qquad\times\sum_{n_1\geq 0}\frac{(-\lambda_1)_{n_1}(\lambda_2+1)_{n_1}}{n_1!(\lambda_{25}+2)_{n_1}}(1-z)^{n_1}B(\lambda_{25}+2+n_1,\lambda_4+1)\\
&\qquad\times\sum_{n_2\geq 0}\frac{(-\lambda_3)_{n_2}(\lambda_{25}+2+n_1)_{n_2}}{n_2!(\lambda_{245}+3+n_1)_{n_2}}(1-z)^{n_2}\\
&=B(\lambda_2+1,\lambda_5+1)B(\lambda_{25}+2,\lambda_4+1)\\
&\qquad\times\sum_{n_1\geq 0}\frac{(-\lambda_1)_{n_1}(\lambda_2+1)_{n_1}}{n_1!(\lambda_{245}+3)_{n_1}}(1-z)^{n_1}\\
&\qquad\times\sum_{n_2\geq 0}\frac{(-\lambda_3)_{n_2}(\lambda_{25}+2+n_1)_{n_2}}{n_2!(\lambda_{245}+3+n_1)_{n_2}}(1-z)^{n_2}.
\end{aligned}$$

This reaches the expression we need to derive. □

The same argument as in the previous section gives the following.

$$\mathrm{reg}\ D_1 + e_5\ \mathrm{reg}\ D_2 + e_{45}\ \mathrm{reg}\ D_3 + e_{345}\ \mathrm{reg}\ D_4 = 0,$$

$$\mathrm{reg}\ D_5 + e_5\ \mathrm{reg}\ D_6 + e_4\ \mathrm{reg}\ D_7 + e_{45}\ \mathrm{reg}\ D_8 + e_{345}\ \mathrm{reg}\ D_9 = 0,$$

$$\mathrm{reg}\ D_{10} + e_4\ \mathrm{reg}\ D_{11} + e_{34}\ \mathrm{reg}\ D_{12} + e_{345}\ \mathrm{reg}\ D_{13} = 0,$$

$$\mathrm{reg}\ D_{10} + e_1\ \mathrm{reg}\ D_5 + e_{15}\ \mathrm{reg}\ D_6 + e_{12}\ \mathrm{reg}\ D_1 + e_{125}\ \mathrm{reg}\ D_2 = 0,$$

$$\mathrm{reg}\ D_{11} + e_1\ \mathrm{reg}\ D_7 + e_{15}\ \mathrm{reg}\ D_8 + e_{125}\ \mathrm{reg}\ D_3 = 0,$$

$$\mathrm{reg}\ D_{12} + e_5\ \mathrm{reg}\ D_{13} + e_{15}\ \mathrm{reg}\ D_9 + e_{125}\ \mathrm{reg}\ D_4 = 0,$$

and

$$\mathrm{reg}\ D_1 + e_5^{-1}\ \mathrm{reg}\ D_2 + e_{45}^{-1}\ \mathrm{reg}\ D_3 + e_{345}^{-1}\ \mathrm{reg}\ D_4 = 0,$$

$$\mathrm{reg}\ D_5 + e_5^{-1}\ \mathrm{reg}\ D_6 + e_4^{-1}\ \mathrm{reg}\ D_7 + e_{45}^{-1}\ \mathrm{reg}\ D_8 + e_{345}^{-1}\ \mathrm{reg}\ D_9 = 0,$$

$$\mathrm{reg}\ D_{10} + e_4^{-1}\ \mathrm{reg}\ D_{11} + e_{34}^{-1}\ \mathrm{reg}\ D_{12} + e_{345}^{-1}\ \mathrm{reg}\ D_{13} = 0,$$

$$\mathrm{reg}\ D_{10} + e_1^{-1}\ \mathrm{reg}\ D_5 + e_{15}^{-1}\ \mathrm{reg}\ D_6 + e_{12}^{-1}\ \mathrm{reg}\ D_1 + e_{125}^{-1}\ \mathrm{reg}\ D_2 = 0,$$

$$\mathrm{reg}\ D_{11} + e_1^{-1}\ \mathrm{reg}\ D_7 + e_{15}^{-1}\ \mathrm{reg}\ D_8 + e_{125}^{-1}\ \mathrm{reg}\ D_3 = 0,$$

$$\mathrm{reg}\ D_{12} + e_5^{-1}\ \mathrm{reg}\ D_{13} + e_{15}^{-1}\ \mathrm{reg}\ D_9 + e_{125}^{-1}\ \mathrm{reg}\ D_4 = 0.$$

Hence we have

Proposition 3.3. *If* λ_i $(1 \le i \le 5)$, $\lambda_{15}, \lambda_{34}, \lambda_{125}, \lambda_{135}, \lambda_{345}, \lambda_{1345}, \lambda_{12345} \notin \boldsymbol{Z}$, *then*

$$\mathrm{reg}\ D_2 = -\frac{s(\lambda_{345})}{s(\lambda_{34})}\ \mathrm{reg}\ D_1 - \frac{s(\lambda_3)}{s(\lambda_{34})}\ \mathrm{reg}\ D_3,$$

$$\mathrm{reg}\ D_4 = \frac{s(\lambda_5)}{s(\lambda_{34})}\ \mathrm{reg}\ D_1 - \frac{s(\lambda_4)}{s(\lambda_{34})}\ \mathrm{reg}\ D_3,$$

$$\mathrm{reg}\ D_5 = -\frac{s(\lambda_{12345})}{s(\lambda_{1345})}\ \mathrm{reg}\ D_1 - \frac{s(\lambda_{135})}{s(\lambda_{1345})}\ \mathrm{reg}\ D_7,$$

$$\mathrm{reg}\ D_6 = \frac{s(\lambda_{345})s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})}\ \mathrm{reg}\ D_1 + \frac{s(\lambda_3)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})}\ \mathrm{reg}\ D_3 + \frac{s(\lambda_1)s(\lambda_{135})}{s(\lambda_{15})s(\lambda_{1345})}\ \mathrm{reg}\ D_7,$$

$$\mathrm{reg}\ D_8 = -\frac{s(\lambda_{125})}{s(\lambda_{15})}\ \mathrm{reg}\ D_3 - \frac{s(\lambda_1)}{s(\lambda_{15})}\ \mathrm{reg}\ D_7,$$

$$\begin{aligned}
\mathrm{reg}\, D_9 &= -\frac{s(\lambda_5)s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})}\ \mathrm{reg}\, D_1 + \frac{s(\lambda_4)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})}\ \mathrm{reg}\, D_3 \\
&\quad - \frac{s(\lambda_4)s(\lambda_5)}{s(\lambda_{15})s(\lambda_{1345})}\ \mathrm{reg}\, D_7, \\
\mathrm{reg}\, D_{10} &= -\frac{s(\lambda_2)s(\lambda_5)}{s(\lambda_{34})s(\lambda_{1345})}\ \mathrm{reg}\, D_1 - \frac{s(\lambda_2)s(\lambda_3)}{s(\lambda_{34})s(\lambda_{15})}\ \mathrm{reg}\, D_3 \\
&\quad + \frac{s(\lambda_5)s(\lambda_{135})}{s(\lambda_{15})s(\lambda_{1345})}\ \mathrm{reg}\, D_7, \\
\mathrm{reg}\, D_{11} &= \frac{s(\lambda_2)}{s(\lambda_{15})}\ \mathrm{reg}\, D_3 - \frac{s(\lambda_5)}{s(\lambda_{15})}\ \mathrm{reg}\, D_7, \\
\mathrm{reg}\, D_{12} &= \frac{s(\lambda_2)s(\lambda_{345})}{s(\lambda_{34})s(\lambda_{1345})}\ \mathrm{reg}\, D_1 - \frac{s(\lambda_2)s(\lambda_4)}{s(\lambda_{34})s(\lambda_{15})}\ \mathrm{reg}\, D_3 \\
&\quad - \frac{s(\lambda_1)s(\lambda_4)}{s(\lambda_{15})s(\lambda_{1345})}\ \mathrm{reg}\, D_7, \\
\mathrm{reg}\, D_{13} &= -\frac{s(\lambda_2)}{s(\lambda_{1345})}\ \mathrm{reg}\, D_1 + \frac{s(\lambda_4)}{s(\lambda_{1345})}\ \mathrm{reg}\, D_7.
\end{aligned}$$

These induce the following connection formula, which corresponds to (4.5) with Theorem 3 in [16].

Theorem 3.4. *If* $\mathrm{Re}(\alpha_i) > 0$ $(i = 1, 2)$, $\mathrm{Re}(\beta_i - \alpha_i) > 0$ $(i = 1, 2)$, $\mathrm{Re}(\beta_1 + \beta_2 - \alpha_1 - \alpha_2) > 0$, $1 > \mathrm{Re}(\beta_1 - \alpha_j)$ $(j = 2, 3)$, $1 > \mathrm{Re}(\beta_2 - \alpha_j)$ $(j = 1, 3)$, $1 > \mathrm{Re}(\alpha_3)$ *and* $\beta_1, \beta_2, \beta_2 - \beta_1 \notin \boldsymbol{Z}$, *we have*

$$\begin{aligned}
&(1-z)^{\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3} \\
&\times \sum_{n_1, n_2 \geq 0} \frac{(\beta_2-\alpha_1)_{n_1}(\beta_2-\alpha_2)_{n_1}(\beta_1-\alpha_3)_{n_2}(\beta_1+\beta_2-\alpha_1-\alpha_2)_{n_1+n_2}}{n_1!(\beta_1+\beta_2-\alpha_1-\alpha_2)_{n_1} n_2!(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3+1)_{n_1+n_2}} (1-z)^{n_1+n_2} \\
&= \frac{\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3+1)\Gamma(1-\beta_1)\Gamma(1-\beta_2)}{\Gamma(1-\alpha_1)\Gamma(1-\alpha_2)\Gamma(1-\alpha_3)} \times {}_3F_2\left(\begin{matrix}\alpha_1, \alpha_2, \alpha_3 \\ \beta_1, \beta_2\end{matrix}; z\right) \\
&\quad + \frac{\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3+1)\Gamma(\beta_1-1)\Gamma(\beta_1-\beta_2)}{\Gamma(\beta_1-\alpha_1)\Gamma(\beta_1-\alpha_2)\Gamma(\beta_1-\alpha_3)} \\
&\quad \times z^{1-\beta_1} {}_3F_2\left(\begin{matrix}\alpha_1-\beta_1+1, \alpha_2-\beta_1+1, \alpha_3-\beta_1+1 \\ 2-\beta_1, \beta_2-\beta_1+1\end{matrix}; z\right)
\end{aligned}$$

$$+\frac{\Gamma(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3+1)\Gamma(\beta_2-1)\Gamma(\beta_2-\beta_1)}{\Gamma(\beta_2-\alpha_1)\Gamma(\beta_2-\alpha_2)\Gamma(\beta_2-\alpha_3)}$$

$$\times z^{1-\beta_2}{}_3F_2\left(\begin{matrix}\alpha_1-\beta_2+1,\alpha_2-\beta_2+1,\alpha_3-\beta_2+1\\ \beta_1-\beta_2+1,2-\beta_2\end{matrix};z\right).$$

Proof. Combining Proposition 3.1, Theorem 3.2 and Proposition 3.3 leads to the following:

$$(1-z)^{\lambda_{245}+2}\times\sum_{n_1,n_2\geq 0}\frac{(-\lambda_1)_{n_1}(\lambda_2+1)_{n_1}(-\lambda_3)_{n_2}(\lambda_{25}+2)_{n_1+n_2}}{n_1!(\lambda_{25}+2)_{n_1}n_2!(\lambda_{245}+3)_{n_1+n_2}}(1-z)^{n_1+n_2}$$

$$=\frac{\Gamma(\lambda_{34}+1)\Gamma(\lambda_{1345}+2)\Gamma(\lambda_{245}+3)}{\Gamma(\lambda_{345}+2)\Gamma(\lambda_{12345}+3)\Gamma(\lambda_4+1)}$$

$$\times {}_3F_2\left(\begin{matrix}-\lambda_{345}-1,-\lambda_{12345}-2,-\lambda_4\\ -\lambda_{34},-\lambda_{1345}-1\end{matrix};z\right)$$

$$+\frac{\Gamma(-\lambda_{34}-1)\Gamma(\lambda_{15}+1)\Gamma(\lambda_{245}+3)}{\Gamma(-\lambda_3)\Gamma(\lambda_{125}+2)\Gamma(\lambda_5+1)}$$

$$\times z^{\lambda_{34}+1}{}_3F_2\left(\begin{matrix}-\lambda_5,-\lambda_{125}-1,\lambda_3+1\\ \lambda_{34}+2,-\lambda_{15}\end{matrix};z\right)$$

$$+\frac{\Gamma(-\lambda_{15}-1)\Gamma(-\lambda_{1345}-2)\Gamma(\lambda_{245}+3)}{\Gamma(-\lambda_1)\Gamma(-\lambda_{135}-1)\Gamma(\lambda_2+1)}$$

$$\times z^{\lambda_{1345}+2}{}_3F_2\left(\begin{matrix}-\lambda_2,\lambda_{135}+2,\lambda_1+1\\ \lambda_{15}+2,\lambda_{1345}+3\end{matrix};z\right).$$

When the parameters are given by

$$\lambda_1=\alpha_1-\beta_2,\qquad \lambda_2=\beta_2-\alpha_2-1,\qquad \lambda_3=\alpha_3-\beta_1,$$
$$\lambda_4=-\alpha_3,\qquad \lambda_5=\beta_1-\alpha_1-1,$$

it follows that

$$-\lambda_{345}-1=\alpha_1,\qquad -\lambda_{12345}-2=\alpha_2,\qquad -\lambda_{125}-1=1+\alpha_2-\beta_1,$$
$$\lambda_{135}+2=1+\alpha_3-\beta_2,\qquad \lambda_{25}+2=\beta_1+\beta_2-\alpha_1-\alpha_2$$

and

$$\lambda_{34}=-\beta_1,\qquad \lambda_{15}=-1+\beta_1-\beta_2,\qquad \lambda_{1345}=-1-\beta_2.$$

It leads to the theorem. □

4. Monodromy representation

In this section, T_z and $\mathscr{L}_z$ are taken as in Section 3: namely, the function

$$u(t) = t_1^{\lambda_1}(t_1-1)^{\lambda_2} t_2^{\lambda_3}(t_2-z)^{\lambda_4}(t_2-t_1)^{\lambda_5}$$

with the genericity condition

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_{135}, \lambda_{125}, \lambda_{345}, \lambda_{12345} \notin \boldsymbol{Z}$$

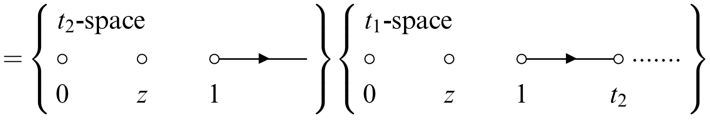
defines $\mathscr{L}_z$. For each $i = 0, 1$, let γ_i be a simple loop which starts and ends at a base point z such as $0 < z < 1$ and surrounds the singular point i with counterclockwise direction. It corresponds to a generator of $\pi_1(\boldsymbol{C}\backslash\{0,1\})$. This loop γ_i induces an action γ_i^* (a monodromy action) of $\pi_1(\boldsymbol{C}\backslash\{0,1\})$ on the family of the homology group $H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z)$ or $H_2(T_z, \mathscr{L}_z)$. As a basis of $H_2^{\mathrm{lf}}(T_z, \mathscr{L}_z)$, we take

$$D_1 = \{1 < t_1 < t_2\}, \qquad D_3 = \{0 < t_2 < z, 1 < t_1\}, \qquad D_7 = \{0 < t_1 < t_2 < z\}$$

as in Section 3. On these cycles, γ_0 acts diagonally:

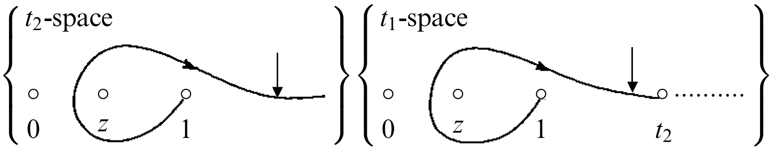
(4.1) $$\gamma_0^*(D_1) = D_1, \qquad \gamma_0^*(D_3) = e_{34}^2 D_3, \qquad \gamma_0^*(D_7) = e_{1345}^2 D_7.$$

The action of γ_1 is derived in what follows. Since

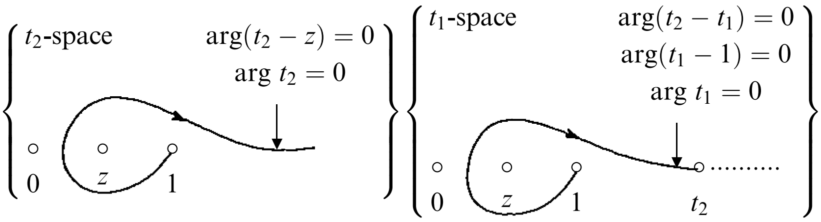
$D_1 =$  ,

we have

(4.2)

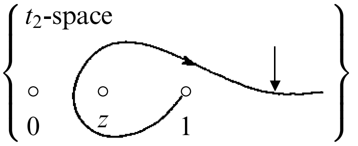
$\gamma_1^*(D_1) =$  .

Hereafter, $u(t)$ is loaded standardly near the vertical arrows (we also continue to adopt a convention that if just the chamber is denoted, it means a standardly loaded cycle). For example, the right-hand side of the equality (4.2) means

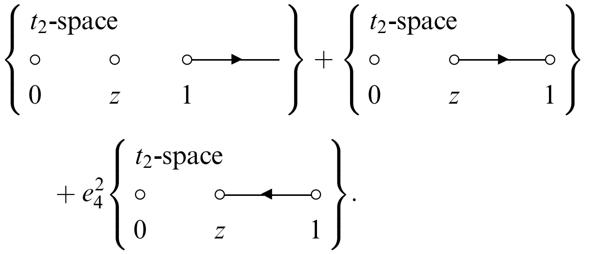


$$\otimes\, t_1^{\lambda_1}(t_1 - 1)^{\lambda_2} t_2^{\lambda_3}(t_2 - z)^{\lambda_4}(t_2 - t_1)^{\lambda_5}.$$

Under this convention, the cycle

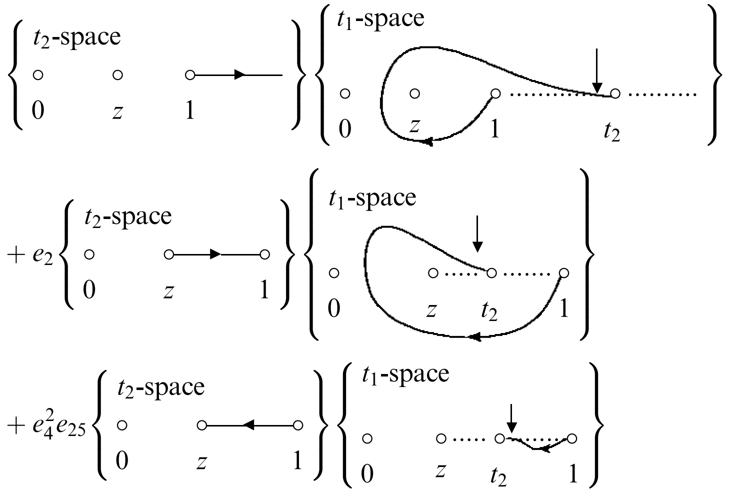


is expressed by the sum of three terms:



Hence, we obtain

(4.3) $\gamma_1^*(D_1) =$



$$
\begin{aligned}
&=\left\{\begin{array}{llll} t_2\text{-space} & & & \\ \circ & \circ & \circ\!\!\longrightarrow & \\ 0 & z & 1 & \end{array}\right\}\left\{\begin{array}{llll} t_1\text{-space} & & & \\ \circ & \circ & \circ\!\!\longrightarrow\!\!\circ & \cdots\cdots \\ 0 & z & 1 & t_2 \end{array}\right\} \\
&\quad + e_2\left\{\begin{array}{lll} t_2\text{-space} & & \\ \circ & \circ\!\!\longrightarrow\!\!\circ & \\ 0 & z & 1 \end{array}\right\}\left\{\begin{array}{llll} t_1\text{-space} & & & \\ \circ & \circ\cdots\cdots & \circ\!\!\longleftarrow\!\!\circ & \\ 0 & z & t_2 & 1 \end{array}\right\} \\
&\quad + e_4^2e_{25}\left\{\begin{array}{lll} t_2\text{-space} & & \\ \circ & \circ\!\!\longleftarrow\!\!\circ & \\ 0 & z & 1 \end{array}\right\}\left\{\begin{array}{llll} t_1\text{-space} & & & \\ \circ & \circ\cdots\cdots & \circ\!\!\longleftarrow\!\!\circ & \\ 0 & z & t_2 & 1 \end{array}\right\} \\
&= D_1 - e_{25}(1-e_4^2)D_6.
\end{aligned}
$$

Similarly it follows that

$$\gamma_1^*(D_3) = D_3 - e_{24}(1-e_5^2)D_6, \tag{4.4}$$

$$\gamma_1^*(D_7) = D_7 - e_{45}(1-e_2^2)D_6. \tag{4.5}$$

Combining (4.3–5) with the connection formula

$$
\begin{aligned}
D_6 &= \frac{(e_{345}^2-1)(e_{12345}^2-1)}{e_{25}(e_{34}^2-1)(e_{1345}^2-1)}D_1 + \frac{e_4(e_3^2-1)(e_{125}^2-1)}{e_2(e_{34}^2-1)(e_{15}^2-1)}D_3 \\
&\quad + \frac{e_{45}(e_1^2-1)(e_{135}^2-1)}{(e_{15}^2-1)(e_{1345}^2-1)}D_7
\end{aligned}
$$

in Proposition 3.3 leads to the following.

$$
\begin{aligned}
\gamma_1^*(D_1) &= \left\{1 + \frac{(e_4^2-1)(e_{345}^2-1)(e_{12345}^2-1)}{(e_{34}^2-1)(e_{1345}^2-1)}\right\}D_1 \\
&\quad + e_{45}\frac{(e_4^2-1)(e_3^2-1)(e_{125}^2-1)}{(e_{34}^2-1)(e_{15}^2-1)}D_3 \\
&\quad + e_{25}e_{45}\frac{(e_1^2-1)(e_4^2-1)(e_{135}^2-1)}{(e_{15}^2-1)(e_{1345}^2-1)}D_7 \\
&= \left\{1 + 2\sqrt{-1}e_{245}\frac{s(\lambda_4)s(\lambda_{345})s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})}\right\}D_1 \\
&\quad + 2\sqrt{-1}e_{245}\frac{s(\lambda_4)s(\lambda_3)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})}D_3 + 2\sqrt{-1}e_{245}\frac{s(\lambda_1)s(\lambda_4)s(\lambda_{135})}{s(\lambda_{15})s(\lambda_{1345})}D_7,
\end{aligned}
$$

$$\begin{aligned}
\gamma_1^*(D_3) &= \frac{e_4(e_5^2-1)(e_{345}^2-1)(e_{12345}^2-1)}{e_5(e_{34}^2-1)(e_{1345}^2-1)} D_1 \\
&\quad + \left\{1 + \frac{e_4^2(e_3^2-1)(e_5^2-1)(e_{125}^2-1)}{(e_{34}^2-1)(e_{15}^2-1)}\right\} D_3 \\
&\quad + \frac{e_{24}e_{45}(e_1^2-1)(e_5^2-1)(e_{135}^2-1)}{(e_{15}^2-1)(e_{1345}^2-1)} D_7 \\
&= 2\sqrt{-1}e_{245}\frac{s(\lambda_5)s(\lambda_{345})s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})} D_1 \\
&\quad + \left\{1 + 2\sqrt{-1}e_{245}\frac{s(\lambda_3)s(\lambda_5)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})}\right\} D_3 + 2\sqrt{-1}e_{245}\frac{s(\lambda_1)s(\lambda_5)s(\lambda_{135})}{s(\lambda_{15})s(\lambda_{1345})} D_7
\end{aligned}$$

and

$$\begin{aligned}
\gamma_1^*(D_7) &= \frac{e_4(e_2^2-1)(e_{345}^2-1)(e_{12345}^2-1)}{e_2(e_{34}^2-1)(e_{1345}^2-1)} D_1 \\
&\quad + \frac{e_{45}e_4(e_2^2-1)(e_3^2-1)(e_{125}^2-1)}{e_2(e_{34}^2-1)(e_{15}^2-1)} D_3 \\
&\quad + \left\{1 + \frac{e_{45}(e_1^2-1)(e_2^2-1)(e_{135}^2-1)}{(e_{15}^2-1)(e_{1345}^2-1)}\right\} D_7 \\
&= 2\sqrt{-1}e_{245}\frac{s(\lambda_2)s(\lambda_{345})s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})} D_1 \\
&\quad + 2\sqrt{-1}e_{245}\frac{s(\lambda_2)s(\lambda_3)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})} D_3 + \left\{1 + 2\sqrt{-1}e_{245}\frac{s(\lambda_1)s(\lambda_2)s(\lambda_{135})}{s(\lambda_{15})s(\lambda_{1345})}\right\} D_7.
\end{aligned}$$

These are equivalent to

$$\begin{aligned}
(4.6)\quad \gamma_1^*\begin{bmatrix} s(\lambda_4)^{-1}D_1 \\ s(\lambda_5)^{-1}D_3 \\ s(\lambda_2)^{-1}D_7 \end{bmatrix} &= \left\{ I + 2\sqrt{-1}e_{245}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right. \\
&\quad \left. \times \left[\frac{s(\lambda_4)s(\lambda_{345})s(\lambda_{12345})}{s(\lambda_{34})s(\lambda_{1345})}, \frac{s(\lambda_3)s(\lambda_5)s(\lambda_{125})}{s(\lambda_{34})s(\lambda_{15})}, \frac{s(\lambda_1)s(\lambda_2)s(\lambda_{135})}{s(\lambda_{15})s(\lambda_{1345})}\right]\right\} \\
&\quad \times \begin{bmatrix} s(\lambda_4)^{-1}D_1 \\ s(\lambda_5)^{-1}D_3 \\ s(\lambda_2)^{-1}D_7 \end{bmatrix}.
\end{aligned}$$

If we substitute

$$\lambda_1 = \alpha_1 - \beta_2,\ \lambda_2 = \beta_2 - \alpha_2 - 1,\ \lambda_3 = \alpha_3 - \beta_1,\ \lambda_4 = -\alpha_3,\ \lambda_5 = \beta_1 - \alpha_1 - 1$$

into (4.1) and (4.6), we have the following.

Theorem 4.1. *If*

$$D_1 = \{1 < t_1 < t_2\}, \qquad D_3 = \{0 < t_2 < z, 1 < t_1\}, \qquad D_7 = \{0 < t_1 < t_2 < z\}$$

are the twisted cycles determined by the function

$$u(t) = t_1^{\alpha_1-\beta_2}(t_1-1)^{\beta_2-\alpha_2-1}t_2^{\alpha_3-\beta_1}(t_2-z)^{-\alpha_3}(t_2-t_1)^{\beta_1-\alpha_1-1}$$

and

$$\Xi = {}^t(s(-\alpha_3)^{-1}D_1, s(\alpha_1-\beta_1)^{-1}D_3, s(\alpha_2-\beta_2)^{-1}D_7),$$

then we have

$$\gamma_i^*\Xi = \rho(\gamma_i)\Xi \qquad (i = 1, 2),$$

where

$$\rho(\gamma_0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & e(-2\beta_1) & 0 \\ 0 & 0 & e(-2\beta_2) \end{bmatrix}, \tag{4.7}$$

and

$$\begin{aligned}\rho(\gamma_1) = I - 2\sqrt{-1}e(\beta_1+\beta_2-\alpha_1-\alpha_2-\alpha_3)\begin{bmatrix}1\\1\\1\end{bmatrix} \\ \times\left[\frac{s(\alpha_1)s(\alpha_2)s(\alpha_3)}{s(\beta_1)s(\beta_2)}, -\frac{s(\beta_1-\alpha_1)s(\beta_1-\alpha_2)s(\beta_1-\alpha_3)}{s(\beta_1)s(\beta_1-\beta_2)},\right. \\ \left.-\frac{s(\beta_2-\alpha_1)s(\beta_2-\alpha_2)s(\beta_2-\alpha_3)}{s(\beta_2-\beta_1)s(\beta_2)}\right].\end{aligned} \tag{4.8}$$

The expressions (4.7), (4.8) are the same as (3.2) and (3.3) in [6].

We finally give a Hermitian form which is invariant with respect to the monodromy group.

Following [12] [13], the self-intersection numbers for D_1, D_3, D_7 are calculated as follows:

$$\begin{aligned}
D_1 \bullet D_1 &= \frac{\sqrt{-1}}{2}\frac{s(-\lambda_{1345})}{s(\lambda_2)s(-\lambda_{12345})}\frac{\sqrt{-1}}{2}\frac{s(-\lambda_{34})}{s(\lambda_5)s(-\lambda_{345})}\\
&= -\frac{1}{4}\frac{s(\lambda_{1345})}{s(\lambda_2)s(\lambda_{12345})}\frac{s(\lambda_{34})}{s(\lambda_5)s(\lambda_{345})},\\
D_3 \bullet D_3 &= \frac{\sqrt{-1}}{2}\frac{s(\lambda_{34})}{s(\lambda_3)s(\lambda_4)}\frac{\sqrt{-1}}{2}\frac{s(-\lambda_{15})}{s(\lambda_2)s(-\lambda_{125})}\\
&= -\frac{1}{4}\frac{s(\lambda_{34})}{s(\lambda_3)s(\lambda_4)}\frac{s(\lambda_{15})}{s(\lambda_2)s(\lambda_{125})},\\
D_7 \bullet D_7 &= \frac{\sqrt{-1}}{2}\frac{s(\lambda_{1345})}{s(\lambda_4)s(\lambda_{135})}\frac{\sqrt{-1}}{2}\frac{s(\lambda_{15})}{s(\lambda_1)s(\lambda_5)}\\
&= -\frac{1}{4}\frac{s(\lambda_{1345})}{s(\lambda_4)s(\lambda_{135})}\frac{s(\lambda_{15})}{s(\lambda_1)s(\lambda_5)}.
\end{aligned}$$

These imply the Hermitian form $F(z,\bar{z})$ which is invariant with respect to the action of the monodromy group:

$$\begin{aligned}
F(z,\bar{z}) = &-\frac{4}{s(\lambda_{15})s(\lambda_{34})s(\lambda_{1345})}\\
&\times \{s(\lambda_2)s(\lambda_5)s(\lambda_{15})s(\lambda_{345})s(\lambda_{12345})|I_1(z)|^2\\
&\quad + s(\lambda_2)s(\lambda_3)s(\lambda_4)s(\lambda_{125})s(\lambda_{1345})|I_3(z)|^2\\
&\quad + s(\lambda_1)s(\lambda_4)s(\lambda_5)s(\lambda_{34})s(\lambda_{135})|I_7(z)|^2\},
\end{aligned} \tag{4.9}$$

where

$$I_j(z) = \int_{D_j} u_{D_j}\, dt_1 dt_2. \tag{4.10}$$

If we substitute

$$\lambda_1 = \alpha_1 - \beta_2,\ \lambda_2 = \beta_2 - \alpha_2 - 1,\ \lambda_3 = \alpha_3 - \beta_1,\ \lambda_4 = -\alpha_3,\ \lambda_5 = \beta_1 - \alpha_1 - 1$$

into (4.9) and (4.10), we have the following.

Theorem 4.2. *The Hermitian form given by*

$$\begin{aligned}
F(z,\bar{z}) = &-\frac{4}{s(\beta_1)s(\beta_2)s(\beta_1-\beta_2)}\\
&\times \{s(\beta_2-\alpha_2)s(\beta_1-\alpha_1)s(\beta_1-\beta_2)s(\alpha_1)s(\alpha_2)|I_1(z)|^2\\
&\quad + s(\beta_2-\alpha_2)s(\alpha_3-\beta_1)s(\alpha_3)s(\beta_1-\alpha_2)s(\beta_2)|I_3(z)|^2\\
&\quad + s(\alpha_1-\beta_2)s(\alpha_3)s(\beta_1-\alpha_1)s(\beta_1)s(\alpha_3-\beta_2)|I_7(z)|^2\},
\end{aligned}$$

where

$$I_j(z) = \int_{D_j} u_{D_j}\, dt_1 dt_2$$

and

$$u(t) = t_1^{\alpha_1-\beta_2}(t_1-1)^{\beta_2-\alpha_2-1} t_2^{\alpha_3-\beta_1}(t_2-z)^{-\alpha_3}(t_2-t_1)^{\beta_1-\alpha_1-1}$$

is invariant with respect to the monodromy group.

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