Funkcialaj Ekvacioj, **51** (2008) 1–15

On the Proof of the Capelli Identities

By

Tôru UMEDA

(Kyoto University, Japan)

Abstract. Using exterior calculus, we present detailed proofs of the classical Capelli identities in a purely computational manner. The proof of the fact that the Capelli elements are central is also given in a similar way. In the course of proofs of these two facts, one can easily see the mechanism of the multiplication formula of determinant with non-commutative entries. Simple treatments of the related facts from the Appendix of [HU] are also given.

Key Words and Phrases. Center of universal enveloping algebra, Capelli identity.
2000 *Mathematics Subject Classification Numbers.* 17B35; 15A33.

Introduction

The celebrated Capelli identities [Ca1–3] played important roles in classical invariant theory (see e.g. [My], [Wy]), and have been generalized in several directions. This makes it desirable to have simple proofs of them available. In this article, we give an exposition of the complete set of proofs for the classical Capelli identities utilizing exterior calculus. This approach is based on ideas in [NUW], which proves a quantum group version of the Capelli identity. (Similar treatments are also found in the papers [Kz], [Na].) However, a separate treatment, as it does not require any knowledge of R-matrix for this case, provides more direct access to the proof, and, indeed, yields a short, simple, computaional proof. In the same spirit, we also present a reason why the Capelli elements are central elements of the enveloping algebra of $\mathfrak{gl}_n$. In the proof of this fact as well as in the proof of Capelli identity, one can recognize the multiplication formula for determinant. As our framework gives a clear view point for the proofs given in the Appendix of [HU], we discuss centrality also.

In this paper, we work over $\boldsymbol{C}$ for simplicity, though this restriction is not needed in any crucial points.

Notes (Added in 2006 following the suggestion by the referee and the editor. The references asterisked like [I1*] are supplemented accordingly.): This paper was written in 1997 to give an explicit exposition on the proof of the Capelli identities using the exterior calculus, as explained in the Introduction above. While the manuscript are circulated as a preprint, the technique presented here has been widely recognized as a basic tool in the study of Capelli type identities, and also its reinforced variants have been developed in several directions. In below we point out some examples of the papers and the results done more or less under the influence of this paper as their prototype.

Similar and/or direct applications of the exterior algebra technique are found in [U2], [I1*], [I2*], [Wa1*], [Wa2*], [Wa3*]. Among them, [Wa3*] constructs a remarkable example, the determinant type central element for the universal enveloping algebra of the orthogonal Lie algebra with split realization. A little bit more sophisticated technique on the exterior algebra with doubled formal variables are developed in [IU*], [I4*], [I5*], [I6*]. This type of refinement is strong enough to establish the Capelli identities for the general dual pair $(\mathrm{O}_M, \mathrm{Sp}_N)$. To treat the Capelli type element expressed with permanent, which was introduced by Nazarov [Na], the commutative formal variables instead of anti-commutative ones are used in [U3*], [I4*], [I6*] for example. Also the mixture of these formal variables, commutative and anti-commutative, are utilized in [U4*] to show an explicit relation between the two types of central elements expressed with determinant and permanent.

As for the use of formal variables to make suitable generating functions, Yangian technique are also known and systematically developed (e.g. [MNO]). The comparison of these tools may be interesting for a wider approach to the study of Capelli type identities.

1. Capelli identity as the multiplication formula for determinant

Consider the space $\mathrm{Mat}(m,n)$ of $m \times n$ matrices, on which the two general linear groups GL_m and GL_n acts respectively from the left and the right. Then the space $\mathscr{P}(\mathrm{Mat}(m,n))$ of polynomial functions on $\mathrm{Mat}(m,n)$ is naturally endowed with the structures of a right GL_m-module and a left GL_n-module. We denote these two actions by λ and ρ, and use the same notation for the infinitesimal actions of the Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$ and for the extensions of these actions to the universal enveloping algebras $U(\mathfrak{gl}_m)$ and $U(\mathfrak{gl}_n)$. For the matrix space $\mathrm{Mat}(m,n)$ or the Lie algebras $\mathfrak{gl}_m$ and $\mathfrak{gl}_n$, the matrix units, which form the standard basis, are denoted by E_{ij}. The coordinate functions of $\mathrm{Mat}(m,n)$ with respect to E_{ij} is denoted by t_{ij} and the partial differential operator $\partial/\partial t_{ij}$ is often abbreviated as ∂_{ij}. It is easy to check that for $E_{ij} \in \mathfrak{gl}_n$ or $\mathfrak{gl}_m$, its image under λ and ρ are respectively expressed as

$$\rho(E_{ij}) = \sum_{\alpha=1}^{m} t_{\alpha i}\partial_{\alpha j}, \qquad \lambda(E_{ij}) = \sum_{\beta=1}^{n} t_{j\beta}\partial_{i\beta}. \tag{1.1}$$

Introducing the four matrices

$$T = (t_{ij})_{1 \le i \le m, 1 \le j \le n}, \qquad D = (\partial_{ij})_{1 \le i \le m, 1 \le j \le n},$$

$$\Pi = (\rho(E_{ij}))_{1 \le i, j \le n}, \qquad \Pi^\circ = (\lambda(E_{ij}))_{1 \le i, j \le m},$$

we can write the relations above in matrix form

$$\Pi = {}^tTD, \qquad {}^t\Pi^\circ = T\,{}^tD. \tag{1.2}$$

Here ^{t} stands for the transpose of a matrix. Roughly speaking, the Capelli identities are the multiplication formulas (or more generally the Binet-Cauchy theorem in the rectangular case) of the determinant of these matrices.

In this paper we understand the determinant $\det(a)$ of an $n \times n$ matrix $a = (a_{ij})_{i,j=1}^n$, whose entries are in a (possibly non-commutative) algebra $\mathscr{A}$, as the alternating sum

$$\det(a) = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sign}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \ldots a_{\sigma(n)n}.$$

The exterior calculus is useful for manipulating the determinant of this type even with non-commutative entries. The exterior algebra Λ_n is an associative algebra generated by the n elements $e_1, e_2, \ldots, e_n$ subject to the relations $e_i e_j + e_j e_i = 0$. We work in an extended algebra $\Lambda_n \otimes \mathscr{A}$, in which the two subalgebras Λ_n and $\mathscr{A}$ commute. The determinant $\det(a)$ then comes in the following way: forming the elements η_i from the columns of a by $\eta_i = \sum_{\alpha=1}^n e_\alpha a_{\alpha i}$ and multiplying them, we find that $e_1 e_2 \ldots e_n \det(a) = \eta_1 \eta_2 \ldots \eta_n$, as in the commutative case. In fact,

$$\begin{aligned}
\eta_1 \eta_2 \ldots \eta_n &= \sum_{1 \le \alpha_1, \alpha_2, \ldots, \alpha_n \le n} e_{\alpha_1} e_{\alpha_2} \ldots e_{\alpha_n} a_{\alpha_1 1} a_{\alpha_2 2} \ldots a_{\alpha_n n} \\
&= \sum_{\sigma \in \mathfrak{S}_n} e_{\sigma(1)} e_{\sigma(2)} \ldots e_{\sigma(n)} a_{\sigma(1)1} a_{\sigma(2)2} \ldots a_{\sigma(n)n} \\
&= \sum_{\sigma \in \mathfrak{S}_n} e_1 e_2 \ldots e_n \operatorname{sign}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} \ldots a_{\sigma(n)n} \\
&= e_1 e_2 \ldots e_n \det(a).
\end{aligned}$$

For our Capelli identities, we use this framework with $\mathscr{A} = \operatorname{End}(\mathscr{P}(\operatorname{Mat}(m, n)))$. According to the multiplication of matrices $\Pi = {}^tTD$, we form the following two sets of elements in $\Lambda_n \otimes \mathscr{A}$:

$$\xi_i = \sum_{\alpha=1}^n e_\alpha t_{i\alpha} \quad (1 \le i \le m), \qquad \zeta_j = \sum_{i=1}^m \xi_i \partial_{ij} \quad (1 \le j \le n). \tag{1.3}$$

Then from the relation (1.1), we see

$$\zeta_j = \sum_{\alpha=1}^{n} e_\alpha \rho(E_{\alpha j}). \tag{1.4}$$

Noting that the variables $\{t_{ij}\}$ commutate with each other, we see easily that the commutation relations among ξ_i are $\xi_i\xi_j + \xi_j\xi_i = 0$. For the computation of a product like $\zeta_1\zeta_2\dots\zeta_n$, we need to know the commutation relations between ξ_i and ∂_{pq}.

Lemma 1. *The following commutation relations hold*:

$$[\partial_{pq}, \xi_i] = e_q\delta_{pi}, \tag{1}$$

$$\zeta_q\xi_i + \xi_i\zeta_q = \xi_i e_q. \tag{2}$$

Proof. The assertion (1) is easy to see by a direct computation.

$$[\partial_{pq}, \xi_i] = \sum_\alpha e_\alpha[\partial_{pq}, t_{i\alpha}] = \sum_\alpha e_\alpha \delta_{pi}\delta_{q\alpha} = e_q\delta_{pi}.$$

For (2), multiply ξ_p from the left on both sides of (1) and sum up with respect to p. Then noting that ξ_p and ξ_i anti-commute, we come to the assertion. □

Suggested by the formula (2) in Lemma 1, we introduce the elements $\zeta_j(u)$ with parameter u as

$$\zeta_j(u) = \zeta_j + ue_j = \sum_{\alpha=1}^{n} e_\alpha\rho(E_{\alpha j} + u\delta_{\alpha j}). \tag{1.5}$$

Then we see from Lemma 1 (2)

$$\zeta_q(u+1)\xi_i + \xi_i\zeta_q(u) = 0, \tag{1.6}$$

because ξ_i and e_q anti-commute.

To describe submatrices made from a given matrix, we introduce some notation here. For subsets $I \subset M = \{1,\dots,m\}$ and $J \subset N = \{1,\dots,n\}$, making their arrangements under the natural order, we denote by a_{IJ} the submatrices of a matrix $a = (a_{ij})$ formed from those a_{ij} such that $i \in I$ and $j \in J$. Also as we will soon see below, we need some shift in the diagonal of Π for the Capelli identity. For this shift, we introduce the following convention: for any matrix a of size k, we put $a^\natural = a + \mathrm{diag}(k-1, k-2, \dots, 0)$.

The following is the Capelli identity for rectangular matrices, which can be regarded as a non-commutative version of the Binet-Cauchy theorem.

Theorem 2 (Capelli identity for rectangular matrices). *The following equality holds*:

$$\det(\Pi^{\natural}_{NN}) = \sum_{\#I=n} \det({}^tT_{NI}) \det(D_{IN}).$$

When $m < n$, we understand that the empty sum in the right-hand side represents 0.

Corollary 3 (Capelli identity for square matrices). *For the case $m = n$, we have the following equality*:

$$\det(\Pi^{\natural}) = \det({}^tT) \det(D).$$

Proof of Theorem 2. We compute the product $\zeta_1(n-1)\zeta_2(n-2)\dots\zeta_n(0)$ in two ways. Using the definition (1.5), we see on the one hand

$$\zeta_1(n-1)\zeta_2(n-2)\dots\zeta_n(0) = e_1e_2\dots e_n \det(\rho(E_{ij} + (n-i)\delta_{ij})).$$

On the other hand, by the relation (1.6) we have

$$\begin{aligned}
&\zeta_1(n-1)\zeta_2(n-2)\dots\zeta_n(0) \\
&\quad = \sum_{1\le k_n\le m} \zeta_1(n-1)\zeta_2(n-2)\dots\zeta_{n-1}(1)\xi_{k_n}\partial_{k_n n} \\
&\quad = (-)^{n-1} \sum_{1\le k_n\le m} \xi_{k_n}\zeta_1(n-2)\zeta_2(n-3)\dots\zeta_{n-1}(0)\partial_{k_n n} \\
&\quad = (-)^{n-1} \sum_{1\le k_{n-1},k_n\le m} \xi_{k_n}\zeta_1(n-2)\zeta_2(n-3)\dots\zeta_{n-2}(1)\xi_{k_{n-1}}\partial_{k_{n-1}n-1}\partial_{k_n n} \\
&\quad = (-)^{2(n-1)} \sum_{1\le k_{n-1},k_n\le m} \xi_{k_{n-1}}\xi_{k_n}\zeta_1(n-3)\zeta_2(n-4)\dots\zeta_{n-2}(0)\partial_{k_{n-1}n-1}\partial_{k_n n} \\
&\quad = \cdots\cdots\cdots \quad \cdots\cdots\cdots\cdots \quad \cdots\cdots\cdots \\
&\quad = (-)^{n(n-1)} \sum_{1\le k_1,k_2,\dots,k_n\le m} \xi_{k_1}\xi_{k_2}\dots\xi_{k_n}\partial_{k_1 1}\partial_{k_2 2}\dots\partial_{k_n n} \\
&\quad = \sum_{\substack{I=\{i_1<\cdots<i_n\}\\ \sigma\in\mathfrak{S}(I)}} \xi_{i_{\sigma(1)}}\xi_{i_{\sigma(2)}}\dots\xi_{i_{\sigma(n)}}\partial_{i_{\sigma(1)}1}\partial_{i_{\sigma(2)}2}\dots\partial_{i_{\sigma(n)}n} \\
&\quad = \sum_{\substack{I=\{i_1<\cdots<i_n\}\\ \sigma\in\mathfrak{S}(I)}} \mathrm{sign}(\sigma)\xi_{i_1}\xi_{i_2}\dots\xi_{i_n}\partial_{i_{\sigma(1)}1}\partial_{i_{\sigma(2)}2}\dots\partial_{i_{\sigma(n)}n} \\
&\quad = \sum_{I=\{i_1<\cdots<i_n\}} \xi_{i_1}\xi_{i_2}\dots\xi_{i_n} \det(D_{IN}) \\
&\quad = e_1e_2\dots e_n \sum_{I=\{i_1<\cdots<i_n\}} \det({}^tT_{NI}) \det(D_{IN}).
\end{aligned}$$

Thus by comparing these two calculations, we obtain our formula. $\square$

We can extend Theorem 2 slightly, as follows. Starting from the $n \times n$ square matrix space, we choose k columns according to a subset $J = \{j_1, j_2, \dots, j_k\} \subset N$. Then we have the relations $\Pi_{JJ} = {}^t T_{JN} D_{NJ}$ as a subset of the relations $\Pi = {}^t TD$. Applying Theorem 2 to this new $n \times k$ rectangular matrix with the replacement $(m, n) \mapsto (n, k)$, we see

$$\det(\Pi^{\natural}_{JJ}) = \sum_{\#I=k} \det({}^t T_{JI}) \det(D_{IJ}). \tag{1.7}$$

We will use this form of Theorem 2 for the proof of the lower order Capelli identities in the next section.

Remark. We have similar formula for the left action λ as follows.

Theorem 2′ (Capelli identity for rectangular matrices). *We have the following equality*:

$$\det(\Pi^{\circ\natural}_{MM}) = \sum_{\#J=m} \det(T_{MJ}) \det({}^t D_{JM}).$$

When $n < m$, we understand that the empty sum in the right-hand side represents 0.

2. The lower order Capelli identities

In the above, we have proved the Capelli identity as a multiplication formula of non-commutaive determinant. The formula itself seems very nice. However, from the representation-theoretic point of view, the significance of the Capelli identity lies in the fact that it presents an equality between the two invariant differential operators on the space $\mathrm{Mat}(m, n)$ (see [H], [HU], [U1]). In this sense, what we proved under the name of "the Capelli identity for rectangular matrices" is *not* the *real* Capelli identity except for $m = n$. Let us explain what should be done to get the real Capelli identities. First, if $m = n$, it is easy to see that the right-hand side of Corollary 3 is invariant under the actions of GL_n both from the right and the left. Admitting that the representations ρ and λ of $U(\mathfrak{gl}_n)$ are faithful in this case, we see that the element $C = \det(E_{ij} + (n - i)\delta_{ij})$ is in the center of $U(\mathfrak{gl}_n)$. Consider here the conjugation of the operators by the relative invariant $(\det T)^z$. Then since $(\det T)^z \rho(E_{ij}) (\det T)^{-z} = \rho(E_{ij}) - z\delta_{ij}$, we have $(\det T)^z \rho(C) (\det T)^{-z} = \det(\rho(E_{ij} + (n - i - z)\delta_{ij}))$ on one hand. On the other hand, under this conjugation the resulting operator remains invariant. Thus we can conclude that the element $C(z) = \det(E_{ij} + (n - i - z)\delta_{ij})$ is central also for any z. As this $C(z)$ is polynomial in z of degree n, we will get n central elements from $C(z)$ as the coefficients, once we develope it in z in some way. To be more specific, we expand $C(z)$ in a form

$$C(z) = \sum_{k=0}^{n} (-)^k z^{(k)} C_{n-k}$$

with $z^{(k)} = z(z-1)\dots(z-k+1)$, and call C_k the kth Capelli element. The explicit form of these C_k in terms of minor determinants will be given in Proposition 4 below. According to the transition from the ring of differential operators on $\mathscr{P}(\mathrm{Mat}(m,n))$ to $U(\mathfrak{gl}_n)$, we introduce an $n \times n$ matrix $\boldsymbol{E} = (E_{ij})_{1 \le i,j \le n}$, so that we have $C(z) = \det(\boldsymbol{E}^\natural - z)$ by definition. As the preimage in $\Lambda_n \otimes U(\mathfrak{gl}_n)$ of ζ_j and $\zeta_j(u)$ defined by (1.4) and (1.5), we put

$$\omega_j = \sum_{\alpha=1}^{n} e_\alpha E_{\alpha j}; \qquad \omega_j(u) = \omega_j + u e_j = \sum_{\alpha=1}^{n} e_\alpha (E_{\alpha j} + u\delta_{\alpha j}). \tag{2.1}$$

Then we see clearly $\omega_1(n-1-z)\omega_2(n-2-z)\dots\omega_n(-z) = e_1 e_2 \dots e_n C(z)$.

Proposition 4. *The kth Capelli element is expressed as the sum of $k \times k$ minor determinants*:

$$C_k = \sum_{\#J=k} \det(\boldsymbol{E}^\natural_{JJ}).$$

Proof. Let us denote by Δ the difference operator defined by $\Delta\varphi(z) = \varphi(z+1) - \varphi(z)$. Note that $\Delta z^{(k)} = k z^{(k-1)}$. Then r times applications of Δ to the expansion for $C(z)$ yield

$$\Delta^r C(z) = \sum_{k=0}^{n} (-)^k \binom{k}{r} r! z^{(k-r)} C_{n-k},$$

so that

$$\Delta^r C(z)|_{z=0} = (-)^r r! C_{n-r}.$$

Our task is thus now to compute the difference of the determinant $C(z) = \det(\boldsymbol{E}^\natural - z)$. For this we recall the formula $\Delta(\varphi(z)\psi(z)) = \Delta\varphi(z) \cdot \psi(z) + \varphi(z+1) \cdot \Delta\psi(z)$ for the difference of a product of functions. Note that this is valid even for the case where $\varphi(z)$ and $\psi(z)$ do not necessarily commute. We apply this formula k times in succession on the left-hand side of $\omega_1(n-1-z)\omega_2(n-2-z)\dots\omega_n(-z) = e_1 e_2 \dots e_n C(z)$. Then we get

$$\Delta^r C(z) = (-)^r r! \sum_{\#I=r} \det(\boldsymbol{E}^\natural_{I^c I^c} - z)$$

Here I^c stands for the complement of the r-set $I = \{i_1, i_2, \dots, i_r\}$ in $N = \{1, \dots, n\}$. The factor $r!$ for the term $\det(\boldsymbol{E}^{\natural}_{I^cI^c} - z)$ can be counted as follows. Under an operation of the difference operator Δ on a product made from $\omega_i(\bullet - z)$'s with $\bullet$ a suitable constant, it will get replacements $\omega_i(\bullet - z) \mapsto -e_i$ from the product formula above. Thus the term $\det(\boldsymbol{E}^{\natural}_{I^cI^c} - z)$ will be produced from $\det(\boldsymbol{E}^{\natural} - z)$ under the successive r operations of Δ as many times as the number of the processes that I appears in the end, each of which process can be identified with the numbering of I by $\{1, 2, \dots, r\}$. The number $r!$ of permutations of the r-set I hence comes as the factor. Putting $z = 0$ in this formula and comparing it with the other formula above, we see our assertion with $k = n - r$. $\square$

Remark. If we introduce a $U(\mathfrak{gl}_n)$-valued polynomial $C_k(w)$ by the formula

$$C(z + w) = \sum_{k=0}^{n} (-)^k z^{(k)} C_{n-k}(w),$$

then we have similarly its expression as

$$C_k(w) = \sum_{\#I=k} \det(\boldsymbol{E}^{\natural}_{II} - w).$$

It is also easy to see that $\Delta C_k(w) = -(n - k + 1) C_{k-1}(w)$.

With Proposition 4 in hand, we can now show the kth Capelli identity.

Theorem 5 (The kth Capelli identity). *The image of the kth Capelli element under ρ is expressed as*:

$$\rho(C_k) = \sum_{\#I=k,\, \#J=k} \det({}^tT_{JI}) \det(D_{IJ}).$$

Proof. For each k-set $J \subset N$, we have the relation (1.7) deduced from Theorem 2:

$$\det(\Pi^{\natural}_{JJ}) = \sum_{\#I=k} \det({}^tT_{JI}) \det(D_{IJ}).$$

Summing up over k-set J, we get the proof of our assertion, because the resulting left-hand side is just $\rho(C_k)$ by Propositon 4. $\square$

3. Centrality of the Capelli elements

In the previous section, we stressed the fact that $C(z)$ is central, and gave an explanation for it from the representation-theoretic point of view, especially by the Capelli identity itself. It should be natural, however, that this can be proved independently of the Capelli identity, and we shall do so here.

We observe the following basic fact.

Lemma 6. *The commutation relations for $\omega_i(u)$ are given by*

$$\omega_i(u+1)\omega_j(u) + \omega_j(u+1)\omega_i(u) = 0.$$

In particular, we have

$$\omega_i(u+1)\omega_i(u) = 0.$$

Proof. This can be shown by an easy calculation:

$$\begin{aligned}\omega_i(u)\omega_j(v) + \omega_j(v)\omega_i(u) &= \sum_{\alpha,\beta=1}^{n} e_\alpha e_\beta [E_{\alpha i} + u\delta_{\alpha i}, E_{\beta j} + v\delta_{\beta j}] \\ &= \sum_{\alpha,\beta=1}^{n} e_\alpha e_\beta (E_{\alpha j}\delta_{i\beta} - E_{\beta i}\delta_{j\alpha}) \\ &= \sum_{\alpha=1}^{n} e_\alpha e_i E_{\alpha j} - \sum_{\beta=1}^{n} e_j e_\beta E_{\beta i} \\ &= -e_i\omega_j - e_j\omega_i.\end{aligned}$$

From this, we see that $\omega_i(u+1)\omega_j(v) + \omega_j(v+1)\omega_i(u) = e_i\omega_j(v) + e_j\omega_i(u) - e_i\omega_j - e_j\omega_i = (v-1)e_ie_j + (u-1)e_je_i$. Put here $u = v$. Then the last term vanishes and we come to the conclusion. □

Recall that GL_n acts on $\mathfrak{gl}_n$ by the adjoint action, so that it acts on the enveloping algebra $U(\mathfrak{gl}_n)$ as an automorphism group. We will first show that the element $C(z)$ is invariant under this action. This statement can be paraphrased as follows. Take any $g = (g_{ij}) \in GL_n$, and write its inverse as $g^{-1} = h = (h_{ij})$. From $\boldsymbol{E} = (E_{ij})$, we form $g\boldsymbol{E}g^{-1} = \boldsymbol{E}^* = (E^*_{ij})$, so that

$$E^*_{ij} = \sum_{k,\ell=1}^{n} g_{ik}E_{k\ell}h_{\ell j}.$$

Under the algebra automorphism of $U(\mathfrak{gl}_n)$ extending this $\boldsymbol{E} \mapsto \boldsymbol{E}^*$, the polynomial $C(z) = \det(\boldsymbol{E}^\natural - z)$ is transformed to $C^*(z) = \det(\boldsymbol{E}^{*\natural} - z)$. The invariancy of $C(z)$ then amounts to the equality $C(z) = C^*(z)$, which is what we will prove below.

Theorem 7. *The Capelli elements are invariant under the adjoint action of GL_n, i.e., $C(z) \in U(\mathfrak{gl}_n)^{GL_n}$. In particular, $C(z)$ is central in $U(\mathfrak{gl}_n)$.*

Proof. As remarked above, it suffices to prove the equality $C(z) = C^*(z)$. For this calculation, let us introduce some more suitably transformed elements corresponding to $\omega_j(u)$ and e_i:

$$\omega_j^*(u) = \sum_{\alpha=1}^{n} e_\alpha(E_{\alpha j}^* + u\delta_{\alpha j}), \qquad e_i' = \sum_{\alpha=1}^{n} e_\alpha g_{\alpha i}, \qquad \omega_j'(u) = \sum_{\alpha=1}^{n} e_\alpha'(E_{\alpha j} + u\delta_{\alpha j}).$$

Then we have

$$\begin{aligned}
\omega_j^*(u) &= \sum_{\alpha} e_\alpha(E_{\alpha j}^* + u\delta_{\alpha j}) = \sum_{\alpha,k,\ell} e_\alpha(g_{\alpha k}E_{k\ell}h_{\ell j} + ug_{\alpha k}\delta_{k\ell}h_{\ell j}) \\
&= \sum_{\alpha,k,\ell} e_\alpha g_{\alpha k}(E_{k\ell} + u\delta_{k\ell})h_{\ell j} \\
&= \sum_{k,\ell} e_k'(E_{k\ell} + u\delta_{k\ell})h_{\ell j} = \sum_{\ell} \omega_\ell'(u)h_{\ell j}
\end{aligned}$$

Note that Lemma 6 above is also valid for $\omega_j'(u)$, because e_i' is just transformed linearly from e_i with scalar coefficients. Put $u_i = n - i - z$ for brevity. Then we see from the relation above and Lemma 6

$$\begin{aligned}
e_1e_2\dots e_nC^*(z) &= \omega_1^*(u_1)\omega_2^*(u_2)\dots\omega_n^*(u_n) \\
&= \sum_{1\le\ell_1,\ell_2,\dots,\ell_n\le n} \omega_{\ell_1}'(u_1)\omega_{\ell_2}'(u_2)\dots\omega_{\ell_n}'(u_n)h_{\ell_1 1}h_{\ell_2 2}\dots h_{\ell_n n} \\
&= \sum_{\sigma\in\mathfrak{S}_n} \omega_{\sigma(1)}'(u_1)\omega_{\sigma(2)}'(u_2)\dots\omega_{\sigma(n)}'(u_n)h_{\sigma(1)1}h_{\sigma(2)2}\dots h_{\sigma(n)n} \\
&= \sum_{\sigma\in\mathfrak{S}_n} \mathrm{sign}(\sigma)\omega_1'(u_1)\omega_2'(u_2)\dots\omega_n'(u_n)h_{\sigma(1)1}h_{\sigma(2)2}\dots h_{\sigma(n)n} \\
&= \omega_1'(u_1)\omega_2'(u_2)\dots\omega_n'(u_n)\ \det(h) \\
&= e_1'e_2'\dots e_n'\ \det(\boldsymbol{E}^\natural - z)\ \det(g)^{-1} \\
&= e_1e_2\dots e_n\ \det(g)C(z)\ \det(g)^{-1} = e_1e_2\dots e_nC(z)
\end{aligned}$$

Thus verified our assertion. □

In the mechanism of the proof above, we can easily recognize the (two) multiplication formulas of determinant as in the classical counterpart $\det(gag^{-1}) = \det a$. We can of course prove that the Capelli elements are central in $U(\mathfrak{gl}_n)$ within the framework of the Lie algebra, not using the group GL_n. For this we first prepare a lemma, which is more fundamental than Lemma 6. Let us write $E_{pq}(u) = E_{pq} + u\delta_{pq}$ for short.

Lemma 8. *The commutation relation for E_{pq} and $\omega_i(u)$ is given by*

$$[E_{pq}, \omega_j(u)] = e_q E_{pj}(u) - \delta_{pj}\omega_q(u).$$

Proof. This is shown also by an easy calculation:

$$[E_{pq}, \omega_j(u)] = \sum_{\alpha} e_\alpha [E_{pq}, E_{\alpha j} + u\delta_{\alpha j}] = \sum_{\alpha} e_\alpha (E_{pj}\delta_{q\alpha} - E_{\alpha q}\delta_{pj})$$
$$= e_q E_{pj} - \delta_{pj}\omega_q = e_q E_{pj}(u) - \delta_{pj}\omega_q(u). \qquad \square$$

Remark. As an easy variant on the proof of Lemma 8, we can obtaine a more general formula as follows: for any u, v, w, we have

$$[E_{pq}(u), \omega_j(v)] = e_q E_{pj}(w) - \delta_{pj}\omega_q(w).$$

Multiplying e_p from the left on both sides of this with $u = v = w$, we will get Lemma 6 again.

The following is essentially the same as the proof in the Appendix A of [HU].

Theorem 9. *The Capelli elements are invariant under the adjoint action of $\mathfrak{gl}_n$. In other words, $C(z)$ is in the center of $U(\mathfrak{gl}_n)$.*

Proof. We will show that $[E_{pq}, C(z)] = 0$ for any $1 \le p, q \le n$. From the expression $e_1 e_2 \dots e_n C(z) = \omega_1(u_1)\omega_2(u_2)\dots\omega_n(u_n)$ with $u_j = n - j + z$, using Lemma 8, we have

$$e_1 e_2 \dots e_n [E_{pq}, C(z)]$$
$$= \sum_{j=1}^{n} \omega_1(u_1)\dots\omega_{j-1}(u_{j-1})\underbrace{(e_q E_{pj}(u_j) - \delta_{pj}\omega_q(u_j))}_{j}\omega_{j+1}(u_{j+1})\dots\omega_n(u_n)$$
$$= \sum_{j=1}^{n} \omega_1(u_1)\dots\omega_{j-1}(u_{j-1})\underbrace{e_q E_{pj}(u_j)}_{j}\omega_{j+1}(u_{j+1})\dots\omega_n(u_n)$$
$$- \omega_1(u_1)\dots\omega_{p-1}(u_{p-1})\underbrace{\omega_q(u_p)}_{p}\omega_{p+1}(u_{p+1})\dots\omega_n(u_n).$$

Here the second term can be calculated by Lemma 6 as

$$\omega_1(u_1)\dots\omega_{p-1}(u_{p-1})\underbrace{\omega_q(u_p)}_{p}\omega_{p+1}(u_{p+1})\dots\omega_n(u_n)$$
$$= \begin{cases} 0, & (\text{for } p \ne q), \\ e_1 e_2 \dots e_n C(z), & (\text{for } p = q). \end{cases}$$

To calculate the first term, we introduce $\omega_j^{(q)}(u) = \omega_j(u) - e_q E_{qj}(u)$ by omitting the qth component of $\omega_j(u)$. Then we have

$$(\omega_1^{(q)}(u_1) + e_q E_{p1}(u_1))(\omega_2^{(q)}(u_2) + e_q E_{p2}(u_2)) \ldots (\omega_n^{(q)}(u_n) + e_q E_{pn}(u_n))$$

$$= \sum_{j=1}^{n} \omega_1^{(q)}(u_1) \ldots \omega_{j-1}^{(q)}(u_{j-1}) e_q \overset{j}{\smile}{E_{pj}}(u_j) \omega_{j+1}^{(q)}(u_{j+1}) \ldots \omega_n^{(q)}(u_n)$$

$$= \sum_{j=1}^{n} \omega_1(u_1) \ldots \omega_{j-1}(u_{j-1}) e_q \overset{j}{\smile}{E_{pj}}(u_j) \omega_{j+1}(u_{j+1}) \ldots \omega_n(u_n),$$

because $e_q^2 = 0$ and $\omega_1^{(q)}(u_1) \ldots \omega_n^{(q)}(u_n) = 0$. Thus the first term is expressed as the product of the elements $\omega_j^{(q)}(u_j) + e_q E_{pj}(u_j)$. Note that $\omega_j^{(q)}(u_j) + e_q E_{pj}(u_j)$ is gotten by the replacement $E_{qj}(u_j) \mapsto E_{pj}(u_j)$ in $\omega_j(u_j)$. Then the product of these elements vanishes if $p \neq q$, because it gives a determinant with two identical rows. For $p = q$, since $\omega_j^{(q)}(u_j) + e_q E_{pj}(u_j) = \omega_j(u_j)$, their product is nothing but $e_1 e_2 \ldots e_n C(z)$. In any case, the first term is cancelled by the second, so that $[E_{pq}, C(z)] = 0$ as desired. □

Remark. Comparing the proof here and that given in [HU], although they are essentially the same, we will get different impressions from them, especially of the length of the treatments of the 'first' and 'second' terms. The introduction of the elements $\omega_j^{(q)}(u) = \omega_j(u) - e_q E_{qj}(u)$ to treat 'first' term corresponds to the simple trick (A.1.3) in [HU]. Also we should compare the proofs of Theorems 7 and 9. We notice that the 'first' and the 'second' terms above correspond respectively to the determinants $\det(g)$ from the right and $\det(g)^{-1}$ from the left in the proof of Theorem 7.

Appendix

Some central elements in $U(\mathfrak{o}_n)$: Here, in analogy with Theorem 9, we will translate the proof of A.2 in [HU]. First we take a realization of the orthogonal Lie algebra $\mathfrak{o}_n$ as

$$\mathfrak{o}_n = \{X \in \mathfrak{gl}_n; X + {}^t X = 0\},$$

and accordingly consider the standard elements $A_{ij} = E_{ij} - E_{ji} \in \mathfrak{o}_n$. Parallel to the preceeding sections, we introduce

$$\boldsymbol{A} = (A_{ij})_{1 \le i, j \le n}, \qquad A_{ij}(u) = A_{ij} + u\delta_{ij},$$

$$\psi_j = \sum_{\alpha=1}^{n} e_\alpha A_{\alpha j}, \qquad \psi_j(u) = \sum_{\alpha=1}^{n} e_\alpha A_{\alpha j}(u) = \psi_j + u e_j$$

Our objective is the determinant $\det(A^{\natural} - z)$, which is to be shown central in the universal enveloping algebra $U(\mathfrak{o}_n)$. We know $e_1 e_2 \dots e_n \det(A^{\natural} - z) = \psi_1(u_1)\psi_2(u_2)\dots\psi_n(u_n)$ with $u_j = n - j - z$. Let us compute the basic commutation relations.

Lemma A1. *We have the following*:

$$[A_{ij}, A_{k\ell}] = A_{i\ell}\delta_{jk} - A_{ik}\delta_{j\ell} - A_{j\ell}\delta_{ik} + A_{jk}\delta_{i\ell}, \tag{1}$$

$$[A_{pq}(u), \psi_j(v)] = e_q A_{pj}(x) - \delta_{pj}\psi_q(x) + \delta_{qj}\psi_p(y) - e_p A_{qj}(y), \tag{2}$$

$$\psi_i(u+1)\psi_j(u) + \psi_j(u+1)\psi_i(u) = -\delta_{ij}\Omega, \tag{3}$$

where $\Omega = \sum_{1 \le \alpha,\beta \le n} e_\alpha e_\beta A_{\alpha\beta} = -\sum_\alpha e_\alpha \psi_\alpha = -\sum_\alpha e_\alpha \psi_\alpha(u)$. *Note that in* (2), *the parameters* u, v, x, y *are independent.*

Proof. The assertion (1) is immediate from the definition. For (2), we compute:

$$\begin{aligned}
[A_{pq}(u), \psi_j(v)] &= \sum_\alpha e_\alpha [A_{pq}(u), A_{\alpha j}(v)] = \sum_\alpha e_\alpha [A_{pq}, A_{\alpha j}] \\
&= \sum_\alpha e_\alpha (A_{pj}\delta_{q\alpha} - A_{p\alpha}\delta_{qj} - A_{qj}\delta_{p\alpha} + A_{q\alpha}\delta_{pj}) \\
&= e_q A_{pj} - \delta_{qj}\sum_\alpha e_\alpha A_{p\alpha} - e_p A_{qj} + \delta_{pj}\sum_\alpha e_\alpha A_{q\alpha} \\
&= e_q A_{pj} + \delta_{qj}\psi_p - e_p A_{qj} - \delta_{pj}\psi_q \\
&= e_q A_{pj}(x) + \delta_{qj}\psi_p(y) - e_p A_{qj}(y) - \delta_{pj}\psi_q(x).
\end{aligned}$$

For (3), multiply e_p from the left on both sides of (2) and sum up over p. Then we have

$$\psi_q(u)\psi_j(v) + \psi_j(v)\psi_q(u) = -e_q\psi_j(x) - e_j\psi_q(x) + \delta_{qj}\sum_p e_p\psi_p(y),$$

so that for $u = v = x$

$$\psi_q(u+1)\psi_j(u) + \psi_j(u+1)\psi_q(u) = -\delta_{qj}\Omega,$$

the conclusion. □

Theorem A. *The determinant* $\det(A^{\natural} - z)$ *is central in* $U(\mathfrak{o}_n)$.

Proof. Using the relation $e_1 e_2 \dots e_n \det(A^{\natural} - z) = \psi_1(u_1)\psi_2(u_2)\dots\psi_n(u_n)$ with $u_j = n - j - z$, we show $[A_{pq}, \det(A^{\natural} - z)] = 0$ for $p < q$. By the formula (2) in Lemma A1 with $u = 0$, $v = x = y = u_j$, we see

$$
\begin{aligned}
&e_1 e_2 \ldots e_n [E_{pq}, \det(A^{\natural} - z)] \\
&\quad = \sum_{j=1}^{n} \psi_1(u_1) \ldots \psi_{j-1}(u_{j-1}) \, [A_{pq}, \overset{\overset{j}{\smile}}{\psi_j(u_j)}] \, \psi_{j+1}(u_{j+1}) \ldots \psi_n(u_n) \\
&\quad = \sum_{j=1}^{n} \psi_1(u_1) \ldots \psi_{j-1}(u_{j-1}) \, e_q \overset{\overset{j}{\smile}}{A_{pj}(u_j)} \, \psi_{j+1}(u_{j+1}) \ldots \psi_n(u_n) \\
&\qquad - \sum_{j=1}^{n} \psi_1(u_1) \ldots \psi_{j-1}(u_{j-1}) \, e_p \overset{\overset{j}{\smile}}{A_{qj}(u_j)} \, \psi_{j+1}(u_{j+1}) \ldots \psi_n(u_n) \\
&\qquad - \psi_1(u_1) \ldots \psi_{p-1}(u_{p-1}) \, \overset{\overset{p}{\smile}}{\psi_q(u_p)} \, \psi_{p+1}(u_{p+1}) \ldots \psi_n(u_n) \\
&\qquad + \psi_1(u_1) \ldots \psi_{q-1}(u_{q-1}) \, \overset{\overset{q}{\smile}}{\psi_p(u_q)} \, \psi_{q+1}(u_{q+1}) \ldots \psi_n(u_n).
\end{aligned}
$$

Then the first and the second terms both vanish by the same reasoning as in the proof of Theorem 9, because each of those sums gives a determinant with two identical rows. The products in the third and the fourth terms actually amount to be the same, so that they are cancelled with the opposite signatures. In fact, both of them are calculated by the formula (3) in Lemma A1 as

$$
-\psi_1(u_1) \ldots \psi_{p-1}(u_{p-1}) \overset{\overset{p}{\smile}}{\cdot} \psi_{p+1}(u_p) \ldots \psi_{q-1}(u_{q-2}) \overset{\overset{q}{\smile}}{\cdot} \psi_{q+1}(u_{q-1}) \ldots \psi_n(u_{n-2}) \Omega/2,
$$

because $\psi_p(u+1)\psi_p(u) = \psi_q(u+1)\psi_q(u) = -\Omega/2$. Thus proved our assertion. $\square$

References

[Ca1] Capelli, A., Über die Zurückführung der Cayley'schen Operation Ω auf gewöhnliche Polar-Operationen, Math. Ann., **29** (1887), 331–338.

[Ca2] Capelli, A., Ricerca delle operazioni invariantive fra piu serie di variabili permutabili con ogni altra operazione invariantiva fra le stesse serie, Atti delle Scinze Fis. e Mat. di Napoli, **I** (1888), 1–17.

[Ca3] Capelli, A., Sur les opérations dans la théorie des formes algébriques, Math. Ann., **37** (1890), 1–37.

[H] Howe, R., Remarks on classical invariant theory, Trans. Amer. Math. Soc., **313** (1989), 539–570. Erratum Trans. Amer. Math. Soc., **318** (1990), 823.

[HU] Howe, R. and Umeda, T., The Capelli identity, the double commutant theorem, and multiplicity-free actions, Math. Ann., **290** (1991), 565–619.

[I1*] Itoh, M., Capelli elements for the orthogonal Lie algebras, J. Lie Theory, **10** (2000), 463–489.

[I2*] Itoh, M., A Cayley-Hamilton theorem for the skew Capelli elements, J. Algebra, **242** (2001), 740–761.

[I3*] Itoh, M., Capelli identities for the dual pair (O_M, Sp_N), Math. Z., **246** (2004), 125–154.

[I4*] Itoh, M., Capelli identities for the reductive dual pairs, Adv. Math., **194** (2005), 345–397.

[I5*] Itoh, M., Two determinants in the universal enveloping algebras of the orthogonal Lie algebras, preprint (2006).

[I6*] Itoh, M., Two permanents in the universal enveloping algebras of the symplectic Lie algebras, preprint (2006).

[IU*] Itoh, M. and Umeda, T., On central elements in the universal enveloping algebras of the orthogonal Lie algebras, Compositio Math., **127** (2001), 333–359.

[Kz] Koszul, J.-L., Les algèbre de Lie graduée de type $\mathfrak{sl}(n, 1)$ et l'opérateur de A. Capelli, C.R. Acad. Sc. Paris, **292** (1981), 139–141.

[My] Meyer, F., Bericht über den gegenwärtingen Stand der Invariantentheorie, Jber. d. Dt. Math.-Verein, **1** (1892), 79–292.

[MNO] Molev, A., Nazarov, M. and Olshanskii, G., Yangians and classical Lie algebras, Russian Math. Surveys, **51** (1996), 205–282.

[Na] Nazarov, M., Quantum Berezinian and the classical Capelli identity, Lett. Math. Phys., **21** (1991), 123–131.

[NUW] Noumi, M., Umeda, T. and Wakayama, M., A quantum analogue of the Capelli identity and an elementary differential calculas on $GL_q(n)$, Duke Math. J., **76** (1994), 567–594.

[U1] Umeda, T., The Capelli identities, a century after, Sugaku, **46** (1994), 206–227; (in Japanese); English transl. in "Selected Papers on Harmonic Analysis, Groups, and Invariants", AMS Translations, Series 2, **183** (1998), 51–78, ed. by K. Nomizu.

[U2] Umeda, T., Newton's formula for $\mathfrak{gl}_n$, Proc. Amer. Math. Soc., **126** (1998), 3169–3175.

[U3*] Umeda, T., On Turnbull identity for skew-symmetric matrices, Proc. Edinburgh. Math. Soc., **43** (2000), 379–393.

[U4*] Umeda, T., Application of Koszul complex to Wronski relations for $U(\mathfrak{gl}_n)$, Comment. Math. Helv., **73** (2003), 663–680.

[Wa1*] Wachi, A., Capelli type identities on certain scalar generalized Verma modules, J. Math. Kyoto Univ., **40** (2000), 705–727.

[Wa2*] Wachi, A., Capelli type identities on certain scalar generalized Verma modules II, J. Math. Japan, **56** (2004), 447–473.

[Wa3*] Wachi, A., Central elements in the universal enveloping algebaras for the split realization of the orthogonal Lie algebras, Lett. Math. Phys., **77** (2006), 155–168.

[Wy] Weyl, H., The Classical Groups, their Invariants and Representations, Princeton Univ. Press, 1946.

[Z] Želobenko, D. P., Compact Lie Groups and their Representations, Transl. Math. Monographs **40** (Amer. Math. Soc.), 1973.

nuna adreso:
Department of Mathematics
Faculty of Science
Kyoto University
Kyoto 606-8502
Japan
E-mail: umeda@math.kyoto-u.ac.jp

(Riceivita la 4-an de septembro, 2006)
(Reviziita la 10-an de novembro, 2006)