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Characterizations of Positive Linear Functional Differential Equations

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Abstract. In this paper, we first prove that if a linear neutral functional differential equation is positive then it must degrade into a linear functional differential equation of retarded type. Then, we give some explicit criteria for positive linear functional differential equations. Consequently, we obtain a novel criterion for exponential stability of positive linear functional differential equations.

Key Words and Phrases. Linear functional differential equation, Positive system, Stability.

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1. Introduction

A dynamical system is called *positive* if for any nonnegative initial condition, the corresponding solution of the system is also nonnegative. In particular, a dynamical system with state space $\boldsymbol{R}^n$ is positive if any trajectory of the system starting at an initial state in the positive orthant $\boldsymbol{R}^n_+$ remains forever in $\boldsymbol{R}^n_+$. Positive dynamical systems play an important role in the modelling of dynamical phenomena whose variables are restricted to be nonnegative. This model class is used in many areas such as economics, populations dynamics and ecology, see [1], [11]. The mathematical theory of positive systems is based on the theory of nonnegative matrices founded by Perron and Frobenius. As references we mention [1], [6]. Positive systems are objects for many interesting problems in Mathematics, Physics, Economics, Biology, ... Moreover, in general, obtained results of problems for class of positive systems are often very interesting, see e.g. [1], [6], [8]–[9], [11], [12]–[14], [18], [19].

In the literature, there are some criteria for familiar positive linear systems such as positive linear invariant-time differential (difference) system, positive linear invariant time-delay system of retarded type. For example, it is well-known that a linear time-delay system of the form $\dot{x}(t) = A_0x(t) + A_1x(t-h)$,

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$t \geq 0$, is positive if and only if A_0 is a Metzler matrix and A_1 is a nonnegative matrix and a linear discrete system of the form $x(k+1) = A_0 x(k) + A_1 x(k-h)$, $k \in \boldsymbol{N}$, $k \geq h$ is positive if and only if A_0, A_1 are nonnegative matrices, see e.g. [13], [18], [14]. However, to the best of our knowledge, there is not any criterion for positive linear neutral delay-differential systems of the form

$$\frac{d}{dt}(x(t) - Bx(t-h)) = A_0 x(t) + A_1 x(t-h), \qquad t \geq 0, \tag{1}$$

in the literature. So a natural question arising here is: *When is a linear neutral differential system positive*?

In this paper, we first prove that if a linear neutral functional differential equation of the form

$$\frac{d}{dt}\left(x(t) - \sum_{i=1}^{N} B_i x(t-h_i)\right) = A_0 x(t) + \int_{-h}^{0} d\eta(\theta) x(t+\theta), \qquad t \geq 0, \tag{2}$$

is positive then it must degrade into a linear functional differential equation of retarded type (that is, $B_i = 0$, $i \in \underline{N} := \{1, 2, \ldots . N\}$). Then, we give some explicit criteria for positive linear functional differential equations. Finally, we offer a novel criterion for exponential stability of linear functional differential equations.

The organization of the paper is as follows. In the next section, we give some notations and preliminary results which will be used in the sequel. In Section 3, we prove that if a linear neutral functional differential equation of the form (2) is positive then it must degrade into a linear functional differential equation of retarded type. Then, we give some explicit criteria for positive linear functional differential equations. In Section 4, we present a novel criterion for exponential stability of linear functional differential equations. As a direct consequence of this, it will be shown that for positive linear time-delay differential systems with discrete time-delays, the asymptotic stability, delay-independent stability, strong delay-independent stability of the system are equivalent.

2. Preliminaries

Let $\boldsymbol{K} = \boldsymbol{C}$ or $\boldsymbol{R}$ and n, l, q be positive integers. For a complex number s, denote by $\Re s$ the real part of s. Inequalities between real matrices or vectors will be understood componentwise, i.e. for two real $l \times q$-matrices $A = (a_{ij})$ and $B = (b_{ij})$, the inequality $A \geq B$ means $a_{ij} \geq b_{ij}$ for $i = 1, \ldots, l$, $j = 1, \ldots, q$. The set of all nonnegative $l \times q$-matrices is denoted by $\boldsymbol{R}_+^{l \times q}$. If $x = (x_1, x_2, \ldots, x_n) \in \boldsymbol{K}^n$ and $P = (p_{ij}) \in \boldsymbol{K}^{l \times q}$ we define $|x| = (|x_i|)$ and $|P| = (|p_{ij}|)$.

It is easy to see that $|CD| \le |C|\,|D|$. For any matrix $A \in \boldsymbol{K}^{n\times n}$ the *spectral radius*, *spectral abscissa* of A is denoted respectively, by $\rho(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}$, $\mu(A) = \max\{\operatorname{Re}\lambda : \lambda \in \sigma(A)\}$, where $\sigma(A) := \{z \in \boldsymbol{C} : \det(zI_n - A) = 0\}$ is the set of all eigenvalues of A. A norm $\|\cdot\|$ on $\boldsymbol{K}^n$ is said to be *monotonic* if $|x| \le |y|$ implies $\|x\| \le \|y\|$ for all $x, y \in \boldsymbol{K}^n$. Every p-norm on $\boldsymbol{K}^n$, $1 \le p \le \infty$, is monotonic. In this paper, the norm $\|M\|$ of a matrix $M \in \boldsymbol{K}^{l\times q}$ is always understood as the operator norm defined by $\|M\| = \max_{\|y\|=1}\|My\|$ where $\boldsymbol{K}^q$ and $\boldsymbol{K}^l$ are provided with some monotonic vector norms. Then, the operator norm $\|\cdot\|$ has the following monotonicity property, see e.g. [8],

$$(3) \qquad P \in \boldsymbol{K}^{l\times q}, \qquad Q \in \boldsymbol{R}_+^{l\times q}, \qquad |P| \le Q \Rightarrow \|P\| \le \|\,|P|\,\| \le \|Q\|.$$

A matrix $A \in \boldsymbol{R}^{n\times n}$ is called a Metzler matrix if all the off-diagonal entries of A are nonnegative. It is obvious that $A \in \boldsymbol{R}^{n\times n}$ is a Metzler matrix if and only if $tI_n + A \ge 0$, for some $t \ge 0$. The next theorem summarizes some basic properties of Metzler matrices which will be used in the next section.

Theorem 2.1 ([16]). *Let $A \in \boldsymbol{R}^{n\times n}$ be a Metzler matrix. Then*

- (i) (*Perron-Frobenius Theorem*) *$\mu(A)$ is an eigenvalue of A and there exists a nonnegative eigenvector $x \ge 0$, $x \ne 0$ such that $Ax = \mu(A)x$.*
- (ii) *Given $\alpha \in \boldsymbol{R}$, there exists a nonzero vector $x \ge 0$ such that $Ax \ge \alpha x$ if and only if $\mu(A) \ge \alpha$.*
- (iii) *$(tI_n - A)^{-1}$ exists and is nonnegative if and only if $t > \mu(A)$.*
- (iv) *Given $B \in \boldsymbol{R}_+^{n\times n}$, $C \in \boldsymbol{C}^{n\times n}$. Then*

$$|C| \le B \Rightarrow \mu(A + C) \le \mu(A + B).$$

A matrix function $\eta(\cdot) : [\alpha, \beta] \to \boldsymbol{R}^{l\times q}$ is called a increasing matrix function, if

$$\eta(\theta_2) \ge \eta(\theta_1) \qquad \text{for } \alpha \le \theta_1 \le \theta_2 \le \beta.$$

A matrix function $\eta(\cdot) : [\alpha, \beta] \to \boldsymbol{K}^{m\times n}$ is said to be of bounded variation if

$$(4) \qquad \operatorname{Var}(\eta; \alpha, \beta) := \sup_{P[\alpha,\beta]} \sum_k \|\eta(\theta_k) - \eta(\theta_{k-1})\| < +\infty,$$

where the supremum is taken over the set of all finite partitions of the interval $[\alpha, \beta]$. The set $BV([\alpha, \beta], \boldsymbol{K}^{m\times n})$ of all matrix functions $\eta(\cdot)$ of bounded variation on $[\alpha, \beta]$ satisfying $\eta(\alpha) = 0$ is a Banach space endowed with the norm $\|\eta\| = \operatorname{Var}(\eta; \alpha, \beta)$. Since all matrix norms on $\boldsymbol{K}^{m\times n}$ are equivalent, it follows that the matrix function $\eta(\cdot) = (\eta_{ij}(\cdot)) \in \boldsymbol{K}^{m\times n}$ is of bounded variation if and only if each $\eta_{ij}(\cdot)$ is of bounded variation.

Given $\eta(\cdot) \in BV([\alpha,\beta], \boldsymbol{K}^{m\times n})$ then for any continuous functions $\gamma \in C([\alpha,\beta], \boldsymbol{K})$ and $\phi \in C([\alpha,\beta], \boldsymbol{K}^n)$, the integrals

$$\int_\alpha^\beta \gamma(\theta) d\eta(\theta) \qquad \text{and} \qquad \int_\alpha^\beta d\eta(\theta)\phi(\theta)$$

exist and are defined respectively as the limits of $S_1(P) := \sum_{k=1}^p \gamma(\zeta_k)(\eta(\theta_k) - \eta(\theta_{k-1}))$ and $S_2(P) := \sum_{k=1}^p (\eta(\theta_k) - \eta(\theta_{k-1}))\phi(\zeta_k)$ as $d(P) := \max_k |\theta_k - \theta_{k-1}| \to 0$, where $P = \{\theta_1 = \alpha \le \theta_2 \le \cdots \le \theta_p = \beta\}$ is any finite partition of the interval $[\alpha,\beta]$ and $\zeta_k \in [\theta_{k-1}, \theta_k]$. It is immediate from the definition that

$$\left\| \int_\alpha^\beta \gamma(\theta) d\eta(\theta) \right\| \le \max_{\theta\in[\alpha,\beta]} |\gamma(\theta)| \, \|\eta\|, \tag{5}$$
$$\left\| \int_\alpha^\beta d\eta(\theta)\phi(\theta) \right\| \le \max_{\theta\in[\alpha,\beta]} \|\phi(\theta)\| \, \|\eta\|.$$

Let $\boldsymbol{K}^n$ be endowed with a vector norm $\|\cdot\|$ and $C([-h,0], \boldsymbol{K}^n)$ be a Banach space of all continuous functions on $[-h,0]$ with values in $\boldsymbol{K}^n$ normed by the maximum norm $\|\phi\| = \max_{\theta\in[-h,0]} \|\phi(\theta)\|$. Let $L : C([-h,0], \boldsymbol{K}^n) \to \boldsymbol{K}^n$ be a linear bounded operator. Then, by the Riesz representation theorem, there exists unique matrix function $\eta = (\eta_{ij}(\cdot)) \in BV([-h,0], \boldsymbol{K}^{n\times n})$ which is *continuous from the left* (or briefly c.f.l.) on $(-h,0)$ such that

$$L\phi = \int_{-h}^0 d\eta(\theta)\phi(\theta), \qquad \forall \phi \in C([-h,0], \boldsymbol{K}^n). \tag{6}$$

In the next section the following subspace of $BV([-h,0], \boldsymbol{K}^{m\times n})$ will be used frequently:

$$NBV([-h,0], \boldsymbol{K}^{m\times n}) := \{\eta \in BV([-h,0], \boldsymbol{K}^{m\times n}) : \eta \text{ is c.f.l on } (-h,0)\}, \tag{7}$$

$$NBV_0([-h,0], \boldsymbol{K}^{m\times n}) := \{\eta \in NBV([-h,0], \boldsymbol{K}^{m\times n}) : \eta \text{ is c.f.l on } [-h,0]\}. \tag{8}$$

It is clear that the spaces (7) and (8) are closed in $BV([-h,0], \boldsymbol{K}^{m\times n})$ and thus it is a Banach space with the norm $\|\eta\| = \mathrm{Var}(\eta; -h, 0)$. We notice that, if $\eta \in NBV_0([-h,0], \boldsymbol{K}^{m\times n})$, then $\lim_{r\to 0+} \mathrm{Var}(\eta; -r, 0) = 0$. Denote by $\mathscr{C} := C([-h,0], \boldsymbol{R}^n)$, $C_0([-h,0], \boldsymbol{R}^n) := \{\phi \in \mathscr{C} : \phi(0) = 0\}$ and $C_0^1([-h,0], \boldsymbol{R}^n) := \{\phi \in \mathscr{C} : d\phi/d\theta \in \mathscr{C}, \phi(0) = 0\}$. Finally, for $\phi \in \mathscr{C}$, the notation $\phi \ge 0$ means that $\phi(\theta) \ge 0$ for every $\theta \in [-h,0]$.

3. Criteria for positive linear functional differential equations

Consider a linear neutral functional differential equation of the form

$$\frac{d}{dt} Dx_t = L(x_t), \qquad t \ge 0,\ x(t) \in \boldsymbol{R}^n, \tag{9}$$

where, for each $t \geq 0$, $x_t \in \mathscr{C} := C([-h, 0], \boldsymbol{R}^n)$ is defined by $x_t(\theta) = x(t + \theta)$, $\theta \in [-h, 0]$ and $D, L : \mathscr{C} \to \boldsymbol{R}^n$ are given linear bounded operators. We assume that

$$D\phi = \phi(0) - \sum_{i=1}^{N} B_i \phi(-h_i), \tag{10}$$

where $B_i \in \boldsymbol{R}^{n \times n}$, $i \in \underline{N} := \{1, 2, \ldots .N\}$, $h_0 := 0 < h_1 < h_2 < \cdots < h_N := h$.

Definition 3.1. A continuous function $x : [-h, \infty) \to \boldsymbol{R}^n$ is called a solution of (9) with the initial value

$$x_0 = \phi \in \mathscr{C}, \tag{11}$$

if the function $t \to Dx_t$ is continuously differentiable on $(0, \infty)$ with a continuous right hand derivative at $t = 0$ and (9) is satisfied for $t \geq 0$.

It is well-known that the initial value problem (9)–(11) has a unique solution $x(\cdot, \phi)$ on $[-h, \infty)$, see e.g. [7]. Then we associate with (9)–(11) a semigroup of solution operator in $\mathscr{C}$. The semigroup is strongly continuous and given by translation along the solution of (9)–(11)

$$T(t)\phi := x_t(\cdot; \phi),$$

where $x(\cdot; \phi)$ denotes the solution of (9)–(11). See [7] for further detail and more information. The infinitesimal generator of the semigroup $(T(t))_{t \geq 0}$ is given by

$$\mathscr{A}\phi = \frac{d\phi}{d\theta}, \qquad \mathscr{D}(\mathscr{A}) = \left\{ \phi \in \mathscr{C} \,\middle|\, \frac{d\phi}{d\theta} \in \mathscr{C}, D\frac{d\phi}{d\theta} = L\phi \right\}. \tag{12}$$

Definition 3.2. The linear neutral functional differential equation (9) is said to be positive if its solution semigroup is a positive semigroup.

Recall that the semigroup $(T(t))_{t \geq 0}$ is called positive if, by definition, $T(t)\phi \geq 0$, for every $\phi \in \mathscr{C}$, $\phi \geq 0$.

Remark 3.3. By the definition, it is obvious that the equation (9) is positive if and only if for any initial function $\phi \in C([-h, 0], \boldsymbol{R}^n_+)$ the corresponding solution $x(\cdot, \phi)$ of (9)–(11) satisfies $x(t, \phi) \in \boldsymbol{R}^n_+$ for every $t \geq 0$.

Theorem 3.4. *If the equation* (9) *is positive then* $B_i = 0$, $\forall i \in \underline{N}$.

Proof. Let the equation (9) be positive and $x(t) := x(t, \phi)$, $t \in [-h, \infty)$ be the solution of (9)–(11). Then, $x(t)$ satisfies the following

$$\frac{d}{dt}\left(x(t) - \sum_{i=1}^{N} B_i x(t - h_i) \right) = L(x_t), \qquad t \geq 0. \tag{13}$$

First, we will show that $B_i \geq 0$, $\forall i \in \underline{N}$. To do this, we fix $j \in \underline{N}$, $x \in \boldsymbol{R}^n_+$, and an integer k such that $k > 1/(h_j - h_{j-1})$, and consider the function $\phi_k \in \mathscr{C}$ defined by

$$\phi_k(\theta) := \begin{cases} 0 & \text{if } \theta \in [-h, -h_j] \\ (\theta + h_j)x & \text{if } \theta \in [-h_j, -h_j + 1/k] \\ \frac{(\theta + h_{j-1})x}{1 - k(h_j - h_{j-1})} & \text{if } \theta \in (-h_j + 1/k, -h_{j-1}] \\ 0 & \text{if } \theta \in [-h_{j-1}, 0]. \end{cases} \tag{14}$$

Then, it follows from (13) with $t = 0$ that

$$\frac{dx}{dt}(0) = B_j \frac{d\phi_k}{d\theta}(-h_j + 0) + L(\phi_k) = B_j x + L(\phi_k).$$

Since the equation (9) is positive and $\phi_k \geq 0$, we have

$$\lim_{t \to 0^+} \frac{x(t)}{t} = \frac{dx}{dt}(0) = B_j x + L(\phi_k) \geq 0.$$

Moreover, since L is continuous, and since $\|\phi_k\| \leq \|x\|/k$, letting $k \to \infty$, we get

$$B_j x \geq 0,$$

for arbitrary $x \in \boldsymbol{R}^n_+$. Hence, $B_j \geq 0$, $\forall j \in \underline{N}$. We now show that actually, $B_j = 0$, $\forall j \in \underline{N}$. Indeed, for every $k \in \boldsymbol{N}$ such that $k > 1/(h_N - h_{N-1})$ and $x \in \boldsymbol{R}^n_+$, we consider the function $\psi_k \in C^1_0([-h, 0], \boldsymbol{R}^n)$ given by

$$\psi_k(\theta) := \begin{cases} [k^2(\theta - (-h + (1/k)))^2]x & \text{if } \theta \in [-h, -h + (1/k)] \\ 0 & \text{if } \theta \in (-h + (1/k), 0]. \end{cases} \tag{15}$$

Since ψ_k is continuously differentiable on $[-h, 0]$ and nonnegative, it follows from (13) that

$$\frac{dx}{dt}(0) = \sum_{i=1}^{N} B_i \frac{d\psi_k}{dt}(-h_i) + L(\psi_k) = -2kB_N x + L(\psi_k) \geq 0, \tag{16}$$

for $k \in \boldsymbol{N}$. Hence,

$$L(\psi_k) \geq 2kB_N x \geq 0, \qquad k \in \boldsymbol{N}.$$

Take a monotone norm in $\boldsymbol{R}^n$. Then we have $2k\|B_N x\| \leq \|L(\psi_k)\|$. Since $\|\psi_k\| \leq \|x\|$, this implies $B_N x = 0$, for arbitrary $x \in \boldsymbol{R}^n_+$. Therefore, $B_N = 0$. By a similar way, we can show that $B_j = 0$ for every $j \in \underline{N}$. This completes our proof. □

By Theorem 3.4, *there does not exist a positive linear neutral functional differential equation of the form* (9). So in the rest of this paper, we consider restrictively to the linear functional differential equations of the form

$$\frac{d}{dt}x(t) = A_0x(t) + M(x_t). \tag{17}$$

where $A_0 \in \boldsymbol{R}^{n\times n}$ and the operator M is represented by $\eta \in NBV_0([-h,0],\boldsymbol{R}^{n\times n})$ as

$$M(\phi) = \int_{-h}^{0} d\eta(\theta)\phi(\theta), \qquad \phi \in C([-h,0],\boldsymbol{R}^n). \tag{18}$$

We give some characterizations of positive equations of this type.

Let us define the characteristic quasi-polynomial of the equation (17) by

$$\Delta(s) = sI_n - A_0 - \int_{-h}^{0} e^{s\theta}\, d\eta(\theta) \qquad s \in \boldsymbol{C}, \tag{19}$$

If we define $\eta(\theta) = \eta(-h) = 0$ for $\theta < -h$, then

$$\int_{-h}^{0} e^{s\theta}\, d\eta(\theta) = \int_{0}^{h} e^{-st}\, d\eta(-t) = \int_{0}^{\infty} e^{-st}\, d\eta(-t) = \mathscr{L}\check{\eta}(s),$$

the Laplace-Stieltjes transform of $\check{\eta}(t) = \eta(-t)$, and

$$\Delta(s) = sI_n - A_0 - \mathscr{L}\check{\eta}(s).$$

Let $(T(t))_{t\geq 0}$ be the solution semigroup of (17) and $\mathscr{A}$ be its infinitesimal generator. It is also seen as a particular case of the solution semigroup of (9) where $B_i = 0$, $\forall i \in \underline{N}$.

Recall that the equation (17) is called positive if its solution semigroup $(T(t))_{t\geq 0}$ is positive. Denote by

$$\omega(\mathscr{A}) := \lim_{t\to\infty} \frac{1}{t} \ln\|T(t)\|, \tag{20}$$

the *growth bound* of the semigroup $(T(t))_{t\geq 0}$. We need the following explicit formula for the resolvent of $\mathscr{A}$. To state it, we introduce a notation of functions; for $s \in \boldsymbol{C}$, $x \in \boldsymbol{C}^n$, let

$$(\varepsilon_s \otimes x)(\theta) = e^{s\theta}x, \qquad \theta \in [-h,0].$$

Moreover, we introduce an integral transform M_s, $s \in \boldsymbol{C}$, on $C([-h,0],\boldsymbol{C}^n)$ by

$$(M_s\phi)(\theta) = \int_{\theta}^{0} e^{s(\theta-t)}\phi(t)dt, \qquad \phi \in C([-h,0],\boldsymbol{C}^n).$$

It is clear that M_s is a continuous linear operator.

Lemma 3.5 ([7]). *If* $\mathscr{A}$ *is defined by* (12) *then the resolvent* $(\lambda I - \mathscr{A})^{-1}$ *of* $\mathscr{A}$ *is given by*

$$(\lambda I - \mathscr{A})^{-1}\phi = M_\lambda \phi + \varepsilon_\lambda \otimes \Delta^{-1}(\lambda)(\phi(0) + M(M_\lambda \phi)) \tag{21}$$

where $\Re\lambda > \omega(\mathscr{A})$.

Before proving main results, we prove two technical lemmas.

Lemma 3.6. *If the linear functional differential equation* (17) *is positive then* $\Delta(s)^{-1} \geq 0$ *for* $s \in \boldsymbol{R}$, $s > \omega(\mathscr{A})$.

Proof. Since $(T(t))_{t\geq 0}$ is positive and

$$(\lambda I - \mathscr{A})^{-1}\phi = \int_0^\infty e^{-\lambda\tau} T(\tau)\phi \, d\tau, \qquad \lambda \in \boldsymbol{C}, \Re\lambda > \omega(\mathscr{A})$$

(see e.g. [15]), it follows that $(sI - \mathscr{A})^{-1}\phi$, $s \in \boldsymbol{R}$, $s > \omega(\mathscr{A})$ is nonnegative for every $\phi \in \mathscr{C}$, $\phi \geq 0$. Then, by (21) we have

$$(sI - \mathscr{A})^{-1}\phi(0) = \Delta(s)^{-1}[\phi(0) + M(M_s)\phi] \geq 0, \qquad s > \omega(\mathscr{A}) \tag{22}$$

for every $\phi \in \mathscr{C}$, $\phi \geq 0$. Fix $x \in \boldsymbol{R}^n_+$, $k \in \boldsymbol{N}$, $1/k < h$ and consider the function $\phi_k \in \mathscr{C}$ defined by

$$\phi_k(\theta) := \begin{cases} 0 & \text{if } \theta \in [-h, -1/k] \\ (k\theta + 1)x & \text{if } \theta \in (-1/k, 0]. \end{cases} \tag{23}$$

From (22), it follows that

$$\Delta(s)^{-1}[\phi_k(0) + M(M_s \phi_k)] \geq 0, \qquad s > \omega(\mathscr{A}).$$

By an easy computation, we have

$$\|(M_s\phi_k)(\theta)\| \leq \|\varepsilon_s\| \int_{-1/k}^0 \|\phi_k(\theta)\| d\theta \leq \|\varepsilon_s\| \frac{\|x\|}{k},$$

where $\|\varepsilon_s\| = \max_{\theta\in[-h,0]}|e^{s\theta}|$. Since $\|M(M_s(\phi_k))\| \leq \|L\| \, \|M_s\| \, \|x\|/k \to 0$ as $k \to \infty$, we get $\Delta(s)^{-1}x \geq 0$ for every $x \in \boldsymbol{R}^n_+$ and $s \in \boldsymbol{R}$, $s > \omega(\mathscr{A})$. Therefore, $\Delta(s)^{-1} \geq 0$ for every $s \in \boldsymbol{R}$, $s > \omega(\mathscr{A})$. □

Lemma 3.7. *For every* $\eta \in NBV_0([-h, 0], \boldsymbol{R}^{n\times n})$, *we have*

$$\lim_{s\to+\infty} \int_{-h}^0 e^{s\theta} \, d\eta(\theta) = 0. \tag{24}$$

Proof. For every $k \in \boldsymbol{N}$, and $s > 0$, we have

$$\left\| \int_{-h}^0 e^{s\theta} \, d\eta(\theta) \right\| \leq \left\| \int_{-h}^{-1/k} e^{s\theta} \, d\eta(\theta) \right\| + \left\| \int_{-1/k}^0 e^{s\theta} \, d\eta(\theta) \right\|.$$

Taking (5) into account, this gives

$$\left\|\int_{-h}^{0} e^{s\theta}\, d\eta(\theta)\right\| \le e^{-s/k}\|\eta\| + \mathrm{Var}\left(\eta; -\frac{1}{k}, 0\right), \qquad k \in \boldsymbol{N}.$$

For given $\epsilon > 0$, since η is continuous from the left at 0,

$$\left\|\int_{-h}^{0} e^{s\theta}\, d\eta(\theta)\right\| \le e^{-s/k_0}\|\eta\| + \epsilon,$$

for $k_0 \in \boldsymbol{N}$ large enough. Thus,

$$\lim_{s\to+\infty} \int_{-h}^{0} e^{s\theta}\, d\eta(\theta) = 0. \qquad \square$$

Definition 3.8. Let X be a vector subspace of $C([-h,0],\boldsymbol{R}^n)$. A bounded linear operator $M : C([-h,0],\boldsymbol{R}^n) \to \boldsymbol{R}^n$ is called positive on X if $M\phi \ge 0$ for every $\phi \in X$, $\phi \ge 0$.

We are now in the position to prove the main result of this paper.

Theorem 3.9. *The equation* (17) *is positive if and only if* A_0 *is a Metzler matrix and the operator* M *is positive on* $C_0([-h,0],\boldsymbol{R}^n)$.

Proof. Let the equation (17) be positive. By Lemma 3.7, we have $\lim_{s\to\infty} \mathscr{L}\check{\eta}(s) = 0$. Therefore, for $s > 0$ large enough, we get

$$\begin{aligned} \Delta(s)^{-1} &= s^{-1}(I_n - s^{-1}(A_0 + \mathscr{L}\check{\eta}(s)))^{-1} \\ &= s^{-1}I_n + s^{-2}(A_0 + \mathscr{L}\check{\eta}(s)) + \sum_{k=2}^{\infty} s^{-(k+1)}(A_0 + \mathscr{L}\check{\eta}(s))^k. \end{aligned}$$

Since $\Delta(s)^{-1} \ge 0$ by Lemma 3.6,

$$sI_n + (A_0 + \mathscr{L}\check{\eta}(s)) + \sum_{k=2}^{\infty} s^{-(k-1)}(A_0 + \mathscr{L}\check{\eta}(s))^k \ge 0 \tag{25}$$

for $s > 0$ large enough. Let $A_0 = (a_{ij})$ and assume contrary that $a_{i_0 j_0} < 0$ for some $i_0 \ne j_0$. It is important to note that

$$\lim_{s\to+\infty} \sum_{k=2}^{\infty} s^{-(k-1)}(A_0 + \mathscr{L}\check{\eta}(s))^k = 0$$

Then, from (25) it follows that the entry $b_{i_0 j_0}$ of the matrix on the left side of (25) is negative for $s > 0$ large enough. It is a contradiction. Hence, A_0 must be a Metzler matrix. Let $x(t) := x(\cdot,\phi)$ be the solution of (17) with the initial function $\phi \in C_0([-h,0],\boldsymbol{R}^n)$, $\phi \ge 0$. Then $x(t)$ satisfies the following

$$\frac{dx}{dt}(0) = A_0 x(0) + M(x_0) = M(\phi).$$

Since the equation (17) is positive, we have

$$\frac{dx}{dt}(0) = \lim_{t \to 0} \frac{x(t)}{t} \geq 0,$$

which implies that $M(\phi) \geq 0$ for $\phi \in C_0([-h,0], \boldsymbol{R}^n)$, $\phi \geq 0$.

($\Leftarrow$) Denote by $R(\lambda, \mathscr{A}) := (\lambda I - \mathscr{A})^{-1}$, $\Re\lambda > \omega(\mathscr{A})$. By the standard property of a C_0-semigroup

$$T(t)\phi = \lim_{k \to \infty} \left[\frac{k}{t} R\left(\frac{k}{t}, \mathscr{A} \right) \right]^k \phi, \qquad t > 0,$$

for every $\phi \in C([-h,0], \boldsymbol{R}^n)$. We only have to show that $R(s, \mathscr{A}) \geq 0$ for every $s > 0$ large enough. Observe the formula (22). Since M is a positive operator on $C_0([-h,0], \boldsymbol{R}^n)$ and $M_s\phi \geq 0$, if $\phi \geq 0$, for completing the proof, it is sufficient to show that $\Delta(s)^{-1}$ is nonnegative for $s > 0$ large enough. To do so, first we observe that since M is positive on $C_0([-h,0], \boldsymbol{R}^n)$ and $\eta \in NBV_0([-h,0], \boldsymbol{R}^{n \times n})$, it follows that M is also positive on $C([-h,0], \boldsymbol{R}^n)$. Indeed, for a given function $\phi \in C([-h,0], \boldsymbol{R}^n)$, $\phi \geq 0$, consider the approximate functions $\phi_k \in C_0([-h,0], \boldsymbol{R}^n)$, $k \in \boldsymbol{N}$, $k > 1/h$, defined by

$$\phi_k(\theta) := \begin{cases} \phi(\theta) & \text{if } \theta \in [-h, -1/k] \\ (-k\theta)\phi(-1/k) & \text{if } \theta \in [-1/k, 0]. \end{cases} \tag{26}$$

Then, we have

$$M(\phi_k) = \int_{-h}^{0} d\eta(\theta)\phi_k(\theta) = \int_{-h}^{-1/k} d\eta(\theta)\phi(\theta) + \int_{-1/k}^{0} d\eta(\theta)\phi_k(\theta) \geq 0.$$

Since η is continuous from the left at 0, it is easy to see that $\lim_{k \to 0} \int_{-h}^{-1/k} d\eta(\theta)\phi(\theta) = \int_{-h}^{0} d\eta(\theta)\phi(\theta)$ and $\lim_{k \to 0} \int_{-1/k}^{0} d\eta(\theta)\phi_k(\theta) = 0$. This implies that $M(\phi) = \int_{-h}^{0} d\eta(\theta)\phi(\theta) \geq 0$. Using this fact, we have $\mathscr{L}\check{\eta}(s) \geq 0$, for every $s \in \boldsymbol{R}$. So, $(A_0 + \mathscr{L}\check{\eta}(s))$ is also a Metzler matrix for every $s \in \boldsymbol{R}$. Taking (24) into account, by the continuity of the spectral abscissa $\mu(X)$ in X, we have

$$\mu(A_0 + \mathscr{L}\check{\eta}(s)) < \mu(A_0) + 1, \qquad \text{for every } s \geq s_1,$$

for some $s_1 > 0$. Finally, it follows from Theorem 2.1(iii) that

$$\Delta(s)^{-1} = (sI_n - (A_0 + \mathscr{L}\check{\eta}(s)))^{-1} \geq 0,$$

for every $s > \max\{s_1, \mu(A_0) + 1\}$. This completes our proof. $\square$

Remark 3.10. It is important to note that the function $M_s\phi$ used in the proof of the above theorem belongs to $C_0^1([-h,0],\boldsymbol{R}^n)$. On the other hand, by the Weierstrass density theorem (see, e.g. [2]), for every $\phi \in C([-h,0],\boldsymbol{R}^n)$, there exists the sequence of Bernshtein polynomials uniformly converging to ϕ on $[-h,0]$. Moreover, if $\phi \in C_0([-h,0],\boldsymbol{R}^n)$, $\phi \geq 0$ then the corresponding Bernshtein polynomials are also nonnegative and vanish at 0. So the conclusion of Theorem 3.9 still holds if the space $C_0([-h,0],\boldsymbol{R}^n)$ is replaced by its subspace $C_0^1([-h,0],\boldsymbol{R}^n)$.

The following theorem is a sharpened version of Theorem 3.9.

Theorem 3.11. *The equation* (17) *is positive if and only if* A_0 *is a Metzler matrix and* $\eta \in NBV_0([-h,0],\boldsymbol{R}^{n\times n})$ *is an increasing matrix function.*

Proof. It is sufficient to show that the operator M is positive on $C_0([-h,0],\boldsymbol{R}^n)$ if and only if $\eta \in NBV([-h,0],\boldsymbol{R}^{n\times n})$ is an increasing matrix function. In fact, if η is an increasing matrix function then by the definition of Riemann-Stieltjes integral, we have

$$M\phi = \lim_{d(P)\to 0} \sum_{k=1}^{p} (\eta(\theta_k) - \eta(\theta_{k-1}))\phi(\zeta_k) \geq 0,$$

for every $\phi \in C_0([-h,0],\boldsymbol{R}^n)$, $\phi \geq 0$. It means that M is positive.

Conversely, assume that M is positive on $C_0([-h,0],\boldsymbol{R}^n)$. Let $\eta(\cdot) = (\eta_{ij}(\cdot))$. We show that $\eta_{ij}(\cdot) \in NBV([-h,0],\boldsymbol{R})$ is an increasing scalar function for every $i,j \in \{1,2,\dots,n\}$. Since M is positive, it is easy to see that the operator

$$M_{ij} : C_0([-h,0],\boldsymbol{R}) \to \boldsymbol{R}; \qquad \phi \mapsto M_{ij}\phi := \int_{-h}^{0} \phi(\theta)d\eta_{ij}(\theta),$$

is also positive for every $i,j \in \{1,2,\dots,n\}$. Fix $\theta_1, \theta_2 \in (-h,0)$, $\theta_1 < \theta_2$, $k \in \boldsymbol{N}$, $k > \max\{1/(\theta_1+h), 1/(\theta_2-\theta_1)\}$ and consider the continuous function ϕ_k defined by

$$\phi_k(\theta) := \begin{cases} 0 & \text{if } \theta \in [-h, \theta_1 - 1/k] \\ k\theta + 1 - k\theta_1 & \text{if } \theta \in (\theta_1 - 1/k, \theta_1] \\ 1 & \text{if } \theta \in (\theta_1, \theta_2 - 1/k] \\ -k\theta + k\theta_2 & \text{if } \theta \in (\theta_2 - 1/k, \theta_2] \\ 0 & \text{if } \theta \in (\theta_2, 0]. \end{cases} \tag{27}$$

Since ϕ_k is a continuous on $[-h,0]$, it follows from a standard property of Riemann-Stieltjes integral that

$$\int_{-h}^{0} \phi_k(\theta) d\eta_{ij}(\theta) = \left(\int_{-h}^{\theta_1 - 1/k} + \int_{\theta_1 - 1/k}^{\theta_1} + \int_{\theta_1}^{\theta_2 - 1/k} + \int_{\theta_2 - 1/k}^{\theta_2} + \int_{\theta_2}^{0} \right) \phi_k(\theta) d\eta_{ij}(\theta),$$

see e.g. [20]. This gives

$$\int_{\theta_1 - 1/k}^{\theta_1} \phi_k(\theta) d\eta_{ij}(\theta) + \eta_{ij}\left(\theta_2 - \frac{1}{k}\right) - \eta_{ij}(\theta_1) + \int_{\theta_2 - 1/k}^{\theta_2} \phi_k(\theta) d\eta_{ij}(\theta) \geq 0,$$

for every $k \in \boldsymbol{N}$ large enough. Taking into account that η_{ij} is continuous from the left at θ_1, θ_2 and letting $k \to \infty$, we have $\eta_{ij}(\theta_2) \geq \eta_{ij}(\theta_1)$ for every $\theta_1, \theta_2 \in (-h, 0)$. In case of $\theta_1 = -h < \theta_2 < 0$, by a similar way, we also get $\eta_{ij}(\theta_2) \geq \eta_{ij}(\theta_1)$. Finally, since η_{ij} is continuous from the left at 0, we have $\eta_{ij}(0) \geq \eta_{ij}(\theta)$ for $\theta \in [-h, 0]$. This completes our proof. □

Combining Theorem 3.11, Remark 3.10 and Theorem 3.11, we have the following.

Theorem 3.12. *The following statements are equivalent*:
(i) *The equation* (17) *is positive.*
(ii) *A_0 is a Metzler matrix and $\eta \in NBV_0([-h, 0], \boldsymbol{R}^{n\times n})$ is an increasing matrix function.*
(iii) *A_0 is a Metzler matrix and the operator M defined by* (18) *is positive on X, where X is one of the following vector spaces $C_0^1([-h, 0], \boldsymbol{R}^n)$, $C_0([-h, 0], \boldsymbol{R}^n)$, $C^1([-h, 0], \boldsymbol{R}^n)$, $C([-h, 0], \boldsymbol{R}^n)$.*

As a direct corollary of the above theorems, we obtain a criterion for positive linear time delay differential systems with discrete time-delays.

Corollary 3.13. *The linear time delay differential system of the form*

$$\frac{dx}{dt}(t) = A_0 x(t) + \sum_{i=1}^{N} A_i x(t - h_i), \qquad t \geq 0, \tag{28}$$

is positive if and only if A_0 is a Metzler matrix and A_i, $i \in \underline{N}$, are nonnegative matrices.

Proof. Without loss of generality, we assume that $0 < h_1 < h_2 < \cdots < h_N$. It is important to note that the system (28) can be represented in the form (17) where $\eta(\cdot)$ is defined uniquely by

$$\eta(\theta) = \begin{cases} 0 & \text{if } \theta = -h_N \\ A_N & \text{if } \theta \in (-h_N, -h_{N-1}] \\ A_N + A_{N-1} & \text{if } \theta \in (-h_{N-1}, -h_{N-2}] \\ \dots & \text{if } \dots. \\ A_N + A_{N-1} + \cdots + A_1 & \text{if } \theta \in (-h_1, 0]. \end{cases} \tag{29}$$

The conclusion of the Corollary 3.13 is now straight-forward from Theorem 3.11. □

4. A novel criterion for exponential stability of positive linear functional differential equations

Consider a linear retarded system described by the following general functional differential equation

$$\dot{x}(t) = A_0 x(t) + \int_{-h}^{0} d\eta(\theta) x(t+\theta), \qquad t \geq 0,\ x(t) \in \boldsymbol{R}^n \tag{30}$$

$$x(\theta) = \phi_0(\theta), \qquad \theta \in [-h, 0].$$

where $A_0 \in \boldsymbol{R}^{n \times n}$ and $\eta(\cdot) \in NBV_0([-h,0], \boldsymbol{R}^{n \times n})$ are given. It is well-known that, for any given $\phi_0 \in \mathscr{C}$, the system (30) has a unique solution $x(\phi_0, \cdot)$ defined and continuous on $[-h, \infty)$, see, e.g. [7].

The system (30) is said to be exponentially asymptotically stable, if there are constants $K > 0$, $\alpha > 0$ such that for all $\phi \in \mathscr{C}$, the solution $x(\phi, \cdot)$ of (30) satisfies

$$\|x(\phi, t)\| \leq K e^{-\alpha t} \|\phi\|, \qquad t \geq 0.$$

Denote the set of all roots of the characteristic equation of the system as

$$\sigma(A_0, \eta) := \{s \in \boldsymbol{C} : \det \Delta(s) = 0\}, \tag{31}$$

where $\Delta(s)$ is the characteristic quasi-polynomial defined by (19). Then the necessary and sufficient condition for the system (30) to be exponentially asymptotically stable is

$$\sigma(A_0, \eta) \subset \boldsymbol{C}^- := \{s \in \boldsymbol{C} : \mathscr{R}s < 0\} \tag{32}$$

By the property of $\sigma(A_0, \eta)$ it can be shown that (32) is equivalent to the condition

$$\mu(A_0, \eta) := \max\{\mathscr{R}s : s \in \sigma(A_0, \eta)\} < 0,$$

see e.g. [5]. Then, $\mu(A_0, \eta)$ is called the *spectral abscissa* of the retarded system (30).

Theorem 4.1. *Suppose that the system* (30) *is positive. Then, the system* (30) *is exponentially asymptotically stable if and only if* $\mu(A_0 + \eta(0)) < 0$.

Proof. Consider the continuous real function

$$f(t) := t - \mu(A_0 + \mathscr{L}\check{\eta}(t)) \qquad t \in \boldsymbol{R}. \tag{33}$$

Since $\eta \in NBV_0([-h,0], \boldsymbol{R}^{n\times n})$ is an increasing matrix function, it is immediate that

$$0 \le \mathscr{L}\check{\eta}(t_2) \le \mathscr{L}\check{\eta}(t_1) \qquad t_1, t_2 \in \boldsymbol{R},\ t_2 > t_1.$$

Since A_0 is a Metzler matrix, by Theorem 2.1 (iv), we have

$$\mu(A_0 + \mathscr{L}\check{\eta}(t_2)) \le \mu(A_0 + \mathscr{L}\check{\eta}(t_1)), \qquad t_2 > t_1.$$

Therefore, the real function f defined by (33) is strictly increasing on $\boldsymbol{R}$. Set $\mu_0 := \mu(A_0, \eta)$. Then, by the definition of $\mu(A_0, \eta)$, there exists a complex number s such that $\Re s = \mu_0$ and $\det \Delta(s) = 0$. This implies that $\mu_0 = \Re s \le \mu(A_0 + \mathscr{L}\check{\eta}(s))$. On the other hand, from the inequality

$$\left|\int_{-h}^{0} e^{s\theta}\, d\eta(\theta)\right| \le \int_{-h}^{0} e^{\mu_0\theta}\, d\eta(\theta)$$

and from Theorem 2.1 (iv), it follows that $\mu(A_0 + \mathscr{L}\check{\eta}(s)) \le \mu(A_0 + \mathscr{L}\check{\eta}(\mu_0))$. Thus, $\mu_0 \le \mu(A_0 + \mathscr{L}\check{\eta}(\mu_0))$, or equivalently, $f(\mu_0) \le 0$. Assume $f(\mu_0) < 0$. Since, clearly $\lim_{t\to+\infty} f(t) = +\infty$, we have $f(t_0) = 0$ for some $t_0 > \mu_0$, so that $t_0 = \mu(A_0 + \mathscr{L}\check{\eta}(t_0))$. By Theorem 2.1 (i), t_0 is an eigenvalue of the Metzler matrix $A_0 + \mathscr{L}\check{\eta}(t_0)$. Then, we have $\det \Delta(t_0) = 0$ for $t_0 > \mu_0$. However, this conflicts with the definition of μ_0. Thus $f(\mu_0) = 0$.

Taking into account that f is strictly increasing on $\boldsymbol{R}$ and $f(\mu_0) = 0$, we derive that $\mu_0 = \mu(A_0, \eta) < 0$ if and only if $0 = f(\mu_0) < f(0) = -\mu(A_0 + \int_{-h}^{0} d[\eta(\theta)])$. It is equivalent to $\mu(A_0 + \eta(0)) < 0$. This completes our proof. □

Example 4.2. Consider the scalar linear functional differential equation given by

$$\dot{x}(t) = -x(t) + \int_{-1}^{0} e^{\theta} x(t+\theta)d\theta \qquad t \ge 0,\ x(t) \in \boldsymbol{R}. \tag{34}$$

The equation (34) can be represented as the equation of the form (30) with $\eta = e^{\theta} - e^{-1}$, $\theta \in [-1, 0]$. Clearly, η is an increasing function on $[-1, 0]$. Moreover, we have $-1 + \eta(0) = -e^{-1}$. So, by Theorem 4.1, the equation (34) is exponentially asymptotically stable.

We now consider again the linear time delay differential system of the form (28). Since the characteristic polynomial is given by

$$\Delta(s) = sI_n - A_0 - \sum_{i=1}^{N} A_i e^{-h_i s},$$

the system (28) is asymptotically stable if and only if

$$\det\left(sI_n - A_0 - \sum_{i=1}^{N} A_i e^{-h_i s}\right) \neq 0 \qquad \forall s \in \boldsymbol{C},\ \Re s \geq 0.$$

Furthermore, the system (28) is said to be delay-independently stable if

$$\det\left(sI_n - A_0 - \sum_{i=1}^{N} A_i e^{-h_i s}\right) \neq 0 \quad \forall s \in \boldsymbol{C},\ \Re s \geq 0,\ \forall h_i \geq 0,\ i \in \underline{N} := \{1, 2, \ldots, N\}$$

see e.g. [3]. Denote by

$$H(s, z_1, \ldots, z_N) = \left(sI_n - A_0 - \sum_{i=1}^{N} z_i A_i\right), \qquad (s, z_1, \ldots, z_N) \in \boldsymbol{C}^{N+1}. \tag{35}$$

Then, a slightly stronger property was defined in [4] as follows.

Definition 4.3. The system (28) is called strongly delay-independently stable if

$$\det H(s, z_1, \ldots, z_N) \neq 0, \quad \forall (s, z_1, \ldots, z_N) \in \boldsymbol{C}^{N+1},\ \Re s \geq 0,\ |z_i| \leq 1,\ i \in \underline{N}. \tag{36}$$

The following theorem is a direct corollary of Theorem 4.1 and Theorem 2.1 (iv).

Theorem 4.4. *Let the linear time delay system* (28) *be positive. Then the following statements are equivalent*:

(i) *The linear time delay differential system* (28) *is asymptotically stable.*

(ii) *The linear time delay differential system* (28) *is delay-independently stable.*

(iii) *The linear time delay differential system* (28) *is strongly delay-independently stable.*

(iv) $\mu(A_0 + A_1 + \cdots + A_N) < 0$.

Proof. The proof is straight-forward from Theorem 4.1 and Theorem 2.1 (iv) and is omitted here. □

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