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Bounded Solutions of Parabolic Equations in Continuous Function Spaces

By

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Abstract. This paper is concerned with the existence of bounded mild solutions to equations of the form $u'(t) = Au(t) + f(t)$, where A generates a holomorphic semigroup that is not necessarily strongly continuous, and f is a bounded function. This problem arises when one considers a parabolic equation in spaces of continuous functions. The obtained results, that are stated in terms of spectral properties of the spectrum of A and the uniform spectrum of f, extend previous ones.

Key Words and Phrases. Parabolic equation, Continuous function space, Complete second order evolution equation, Mild solution.

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1. Introduction

In this paper we consider the existence of bounded solutions to first order parabolic equations of the form

$$u'(t) = Au(t) + f(t), \qquad u(t) \in \boldsymbol{X}, \tag{1.1}$$

and to complete second order equations of the form

$$u''(t) = Bu'(t) + Au(t) + f(t), \qquad u(t) \in \boldsymbol{X}, \tag{1.2}$$

where A, B are generally unbounded sectorial operators on a Banach space $\boldsymbol{X}$, f is an $\boldsymbol{X}$-valued bounded function on $\boldsymbol{R}$. Our study is motivated by the following typical example:

Example 1.1. Consider the partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}u(t,x) = \frac{\partial^2}{\partial x^2}u(t,x) + f(t,x), & (t,x) \in (0,\infty) \times (0,1), \\ u(t,0) = u(t,1) = 0, \quad t \geq 0, \\ u(0,x) = \Phi(x), \quad x \in [0,1], \end{cases} \tag{1.3}$$

where $u(t,x)$, $f(t,x)$ are scalar. Let us study Eq. (1.3) in $\boldsymbol{X} := C[0,1]$ (the space of all continuous functions on $[0,1]$ with the sup-norm), and define

$$Au = u'', \qquad D(A) = \{u \in C^2[0,1] : u(0) = u(1) = 0\}. \tag{1.4}$$

Then, the closure of $D(A)$ is

$$\overline{D(A)} = \{u \in C[0,1] : u(0) = u(1) = 0\} \neq C[0,1], \tag{1.5}$$

so, A is not densely defined on $C[0,1]$. Therefore, if we assume that $f(t,\cdot) \in C[0,1]$ depends continuously on t and bounded on $\boldsymbol{R}$, then we will be concerned with the evolution equation (1.1) with the non-densely defined operator A (as shown in [9, 18]) that generates a (non-strongly continuous) analytic semigroup in $\boldsymbol{X}$.

Notice that if we study Eq. (1.3) in $\boldsymbol{X} := L^p[0,1]$, then many available results can be applied depending upon the behavior of f. For instance, since in this case A generates a strongly continuous semigroup in $\boldsymbol{X}$, if f is uniformly continuous or so, the problem can be treated in the frameworks of [15, 20, 11, 6, 12]. If f is not necessarily uniformly continuous, but almost automorphic, the problem has been treated in [3] using the newly introduced concept of uniform spectrum.

The motivation for studying parabolic equations in continuous function spaces was well written in the pioneering works of B. Stewart, X. Mora, E. Sinestrari, A. Lunardi. We refer the reader to [19, 10, 18, 9] for more information. One advantage of using the continuous function spaces instead of the L^p function spaces is that the obtained abstract results on the behavior of solutions for (1.1) imply the pointwise behavior for parabolic systems.

The purpose of this paper is to develop a general framework in which the above example can be treated regardless of the choice of the function space $\boldsymbol{X}$ (that is either $C[0,1]$ or $L^p[0,1]$), and of the behavior of the forcing term f. To this end, we will extend the method of sums of commuting operators to the case where all operators are not densely defined. Next, using the concept of uniform spectrum of a bounded function we can formulate a general sufficient (that is necessary in some sense) condition for the existence and uniqueness of a bounded solution of the same profile as f. The obtained result extends previous ones on the subject. Moreover, we will show that some classes of complete second order evolution equations can be reduced to first order evolution equations to which our obtained results apply.

This paper is organized as follows: In the next section, we explain several notations, and then summarize the concepts of spectra of a bounded function. The main results of the paper (Theorems 3.2, 4.4 and their corollaries) are given in Sections 3 and 4. For the reader's convenience, we add an appendix with a result of Arendt, Rabiger and Sourour on an estimate of the spectrum of a sum of two commuting operators.

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2. Preliminaries

2.1. Notations

For a linear operator T on a complex Banach space $\boldsymbol{X}$, $D(T)$, $\sigma(T)$ and $\rho(T)$ denote the domain, spectrum and the resolvent set of T, respectively. In particular, $\sigma_i(A)$ stands for $\sigma(A) \cap i\boldsymbol{R}$. The field of complex numbers is denoted by $\boldsymbol{C}$. $\Re z$ and $\Im z$ denote the real and imaginary parts of a given complex number z, respectively. Given positive numbers c, θ, R and a real ω, in this paper we will use the following notations

$$\Sigma_c := \{z \in \boldsymbol{C} \,|\, \Re z \geq -c(1 + |\Im z|)\},$$

$$\Sigma(\theta, R) := \{z \in \boldsymbol{C} : |z| \geq R, |\arg z| \leq \theta\},$$

$$S_{\theta,\omega} := \{\lambda \in \boldsymbol{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}.$$

So, for every $c > 0$, $\Sigma_c \supset S_{\pi/2+\varepsilon,0} \supset \Sigma(\pi/2 + \varepsilon, R)$ for some sufficiently small $\varepsilon > 0$ and all positive R. The notation $BC(\boldsymbol{R}, \boldsymbol{X})$ stands for the space of all $\boldsymbol{X}$-valued bounded and continuous functions on $\boldsymbol{R}$, and $BC^1(\boldsymbol{R}, \boldsymbol{X}) := \{f \in BC(\boldsymbol{R}, \boldsymbol{X}) \,|\, \exists f' \in BC(\boldsymbol{R}, \boldsymbol{X})\}$, $BC^2(\boldsymbol{R}, \boldsymbol{X}) := \{f \in BC^1(\boldsymbol{R}, \boldsymbol{X}) \,|\, \exists f'' \in BC(\boldsymbol{R}, \boldsymbol{X})\}$. We will denote by $BUC(\boldsymbol{R}, \boldsymbol{X})$ the subspace of $BC(\boldsymbol{R}, \boldsymbol{X})$ consisting of all uniformly continuous and bounded functions.

Let $\mathscr{M}$ be a closed subspace of $BC(\boldsymbol{R}, \boldsymbol{X})$. The operator $\mathscr{A}_{\mathscr{M}}$ of multiplication by A is defined on $D(\mathscr{A}_{\mathscr{M}}) := \{g \in \mathscr{M} : g(t) \in D(A) \ \forall t \in \boldsymbol{R}, Ag(\cdot) \in \mathscr{M}\}$, and $\mathscr{A}_{\mathscr{M}} g := Ag(\cdot)$ for all $g \in D(\mathscr{A}_{\mathscr{M}})$.

2.2. Spectral theory of bounded functions

In the present paper, for $u \in BC(\boldsymbol{R}, \boldsymbol{X})$, $\mathrm{sp}(u)$ stands for the Carleman spectrum, which consists of all $\xi \in \boldsymbol{R}$ such that the Carleman transform of u, defined by

$$\hat{u}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt & (\mathrm{Re}\,\lambda > 0), \\ -\int_0^\infty e^{\lambda t} u(-t)dt & (\mathrm{Re}\,\lambda < 0), \end{cases}$$

has no holomorphic extension to any neighborhoods of $i\xi$. For each $u \in BC(\boldsymbol{R}, \boldsymbol{X})$ we denote $\mathscr{M}_u := \overline{\mathrm{span}\{S(\tau)u, \tau \in \boldsymbol{R}\}}$ (here $S(\tau)$ is the translation $BC(\boldsymbol{R}, \boldsymbol{X}) \ni f(\cdot) \mapsto f(\tau + \cdot) \in BC(\boldsymbol{R}, \boldsymbol{X})$) which is a closed subspace of

$BC(\boldsymbol{R}, \boldsymbol{X})$. If $u \in BUC(\boldsymbol{R}, \boldsymbol{X})$, the Carleman spectrum of u coincides with its Arveson spectrum, defined by (see [2, Lemma 4.6.8])

$$i\,\mathrm{sp}(u) = \sigma(\mathscr{D}_u). \tag{2.1}$$

where $\mathscr{D}_u$ is the infinitesimal generator of the restriction of the group of translations $(S(t)|_{\mathscr{M}_u})_{t \in \boldsymbol{R}}$ to the closed subspace $\mathscr{M}_u$.

Below we list some properties of the spectra of functions which we will need in the sequel.

Proposition 2.1. *Let* $u, u_n, v \in BC(\boldsymbol{R}, \boldsymbol{X})$ *such that* $\lim_{n\to\infty} \|u_n - u\| = 0$. *Then*

- (i) $\mathrm{sp}(u)$ *is closed;*
- (ii) $\mathrm{sp}(u+v) \subset \mathrm{sp}(u) \cup \mathrm{sp}(v)$*;*
- (iii) *If* $\mathrm{sp}(u) = \emptyset$, *then* $u = 0$*;*
- (iv) *If* $B \in L(\boldsymbol{X})$, *then* $\mathrm{sp}(Bu(\cdot)) \subset \mathrm{sp}(u)$*;*
- (v) *If* $\mathrm{sp}(u_n) \subset \Lambda$, $\forall n$, *then* $\mathrm{sp}(u) \subset \bar{\Lambda}$.

Proof. We refer the reader to [2] for more details and information on other properties of the Carleman spectrum. □

2.2.1. Uniform spectrum of a function in $BC(\boldsymbol{R}, \boldsymbol{X})$

Let us consider the following simple ordinary differential equation in a complex Banach space $\boldsymbol{X}$

$$x'(t) - \lambda x = f(t), \tag{2.2}$$

where $f \in BC(\boldsymbol{X})$. If $\Re\lambda \neq 0$, the homogeneous equation associated with this has an exponential dichotomy, so, (2.2) has a unique bounded solution which we denote by $x_{f,\lambda}(\cdot)$. Moreover, from the theory of ordinary differential equations, it follows that for every fixed $\xi \in \boldsymbol{R}$,

$$x_{f,\lambda}(\xi) := \begin{cases} \int_{-\infty}^{\xi} e^{\lambda(\xi - t)} f(t)dt & (\text{if } \operatorname{Re}\lambda < 0), \\ -\int_{\xi}^{\infty} e^{\lambda(\xi - t)} f(t)dt & (\text{if } \operatorname{Re}\lambda > 0), \end{cases} \tag{2.3}$$

$$= \begin{cases} \int_{-\infty}^{0} e^{-\lambda\eta} f(\xi + \eta)d\eta & (\text{if } \operatorname{Re}\lambda < 0), \\ -\int_{0}^{\infty} e^{-\lambda\eta} f(\xi + \eta)d\eta & (\text{if } \operatorname{Re}\lambda > 0). \end{cases} \tag{2.4}$$

As is well known, the differentiation operator $\mathscr{D}$ is a closed operator on $BC(\boldsymbol{R}, \boldsymbol{X})$. The above argument shows that $\rho(\mathscr{D}) \supset \boldsymbol{C} \backslash i\boldsymbol{R}$ and $x_{f,\lambda} = (\mathscr{D} - \lambda)^{-1} f$ for every $\lambda \in \boldsymbol{C} \backslash i\boldsymbol{R}$ and $f \in BC(\boldsymbol{R}, \boldsymbol{X})$.

Hence, for every $\lambda \in \boldsymbol{C}$ with $\Re\lambda \neq 0$ and $f \in BC(\boldsymbol{R}, \boldsymbol{X})$ the function $[(\lambda - \mathscr{D})^{-1} f](t) = \widehat{S(t)f}(\lambda) \in BC(\boldsymbol{R}, \boldsymbol{X})$. Moreover, $(\lambda - \mathscr{D})^{-1} f$ is analytic on $\boldsymbol{C} \backslash i\boldsymbol{R}$.

Definition 2.2. Let f be in $BC(\boldsymbol{R}, \boldsymbol{X})$. Then,

(i) $\alpha \in \boldsymbol{R}$ is said to be *uniformly regular* with respect to f if there exists a neighborhood $\mathscr{U}$ of $i\alpha$ in $\boldsymbol{C}$ such that the function $(\lambda - \mathscr{D})^{-1} f$, as a complex function of λ with $\Re\lambda \neq 0$, has an analytic continuation into $\mathscr{U}$.

(ii) The set of $\xi \in \boldsymbol{R}$ such that ξ is not uniformly regular with respect to $f \in BC(\boldsymbol{R}, \boldsymbol{X})$ is called *uniform spectrum* of f and is denoted by $\mathrm{sp}_u(f)$.

It turns out that the uniform spectrum of a function $f \in BC(\boldsymbol{R}, \boldsymbol{X})$ coincides with its Carleman spectrum, as shown below:

Proposition 2.3. *Let* $f \in BC(\boldsymbol{R}, \boldsymbol{X})$. *Then*

$$\mathrm{sp}_u(f) = \mathrm{sp}(f). \tag{2.5}$$

Proof. First we show that

$$\mathrm{sp}_u(x_{f,\lambda_0}) = \mathrm{sp}_u(f), \tag{2.6}$$

where $x_{f,\lambda}$ is defined by (2.3), and λ_0 is a given complex number such that $\Re\lambda_0 \neq 0$. In fact, we have, for every $\Re\lambda \neq 0$

$$(\lambda - \mathscr{D})^{-1} x_{f,\lambda_0} = -(\lambda - \mathscr{D})^{-1}(\lambda_0 - \mathscr{D})^{-1} f = -(\lambda_0 - \mathscr{D})^{-1}(\lambda - \mathscr{D})^{-1} f.$$

So, $(\lambda - \mathscr{D})^{-1} x_{f,\lambda_0}$ has an analytic continuation into a neighborhood of $i\beta$, where β is a real number, if and only if so does $(\lambda - \mathscr{D})^{-1} f$. That is (2.6) holds. Note that since the derivative of x_{f,λ_0} is bounded, this function is uniformly continuous, so,

$$\mathrm{sp}(f) \subset \mathrm{sp}_u(f) = \mathrm{sp}_u(x_{f,\lambda_0}) = \mathrm{sp}(x_{f,\lambda_0}).$$

To complete the proof of this proposition, it suffices to show that

$$(\boldsymbol{R} \backslash \mathrm{sp}(f)) \subset (\boldsymbol{R} \backslash \mathrm{sp}(x_{f,\lambda_0})). \tag{2.7}$$

To this end, we will use the Beurling spectrum as an alternative of the Carleman spectrum. That is, $\xi \in (\boldsymbol{R} \backslash \mathrm{sp}(f))$, if and only if there is a positive ε such that if $\phi \in L^1(\boldsymbol{R})$ with the support of its Fourier transform $\mathrm{supp}(\tilde{\phi})$ is contained in $(\xi - \varepsilon, \xi + \varepsilon)$, then $\phi * f = 0$. Next, it can be easily checked that

$$\phi * x_{f,\lambda_0} = \phi * (\lambda_0 - \mathscr{D})^{-1} f = (\lambda_0 - \mathscr{D})^{-1}(\phi * f) = 0.$$

This shows that if $\xi \in (\boldsymbol{R} \backslash \mathrm{sp}(f))$, then $\xi \in (\boldsymbol{R} \backslash \mathrm{sp}(x_{f,\lambda_0}))$. That is (2.7) holds. This completes the proof of the proposition. □

Corollary 2.4. *For any closed subset* $\Lambda \subset \boldsymbol{R}$, *the set* $\Lambda(\boldsymbol{X}) := \{f \in BC(\boldsymbol{R}, \boldsymbol{X}) : \mathrm{sp}(f) \subset \Lambda\}$ *is a closed subspace of* $BC(\boldsymbol{R}, \boldsymbol{X})$ *which is invariant under translations.*

The following result will be needed in the sequel whose proof is given below with some correction of the one in [3] for the reader's convenience.

Lemma 2.5. *Let* Λ *be a closed subset of* $\boldsymbol{R}$ *and let* $\mathscr{D}_\Lambda$ *be the differentiation operator acting on* $\Lambda(\boldsymbol{X})$. *Then we have*

$$\sigma(\mathscr{D}_\Lambda) = i\Lambda. \tag{2.8}$$

Proof. Since the function g_α defined by $g_\alpha(t) := e^{i\alpha t}x$, $\alpha \in \Lambda$, $t \in \boldsymbol{R}$, $x \neq 0$, is in $\Lambda(\boldsymbol{X})$ and $\mathrm{sp}(g_\alpha) = \{\alpha\}$ we see that $i\alpha \in \sigma(\mathscr{D}_\Lambda)$, that is, $i\Lambda \subset \sigma(\mathscr{D}_\Lambda)$. Now we prove the converse. For $\beta \in \boldsymbol{R} \backslash \Lambda$ we consider the equation

$$i\beta g - g' = f, \qquad f \in \Lambda(\boldsymbol{X}). \tag{2.9}$$

We will prove that (2.9) is uniquely solvable for every $f \in \Lambda(\boldsymbol{X})$. This equation has at most one solution. In fact, if g_1, g_2 are two solutions, then $g = g_1 - g_2$ is a solution of the homogeneous equation, that is for $f = 0$. Taking Carlemann transform of both sides of the corresponding equation we may see that $\mathrm{sp}(g) \subset \{\beta\}$. Since $g \in \Lambda(\boldsymbol{X})$ we have $\mathrm{sp}(g) \subset \Lambda$. Combining these facts we have $\mathrm{sp}(g) = \varnothing$, that is $g = 0$.

Now we prove the existence of at least one solution to Eq. (2.9). In fact, for $\Re\lambda \neq 0$ the equation $\lambda g - g' = f$ has a unique solution which is nothing but $(\lambda - \mathscr{D})^{-1}f$. Since $i\beta \notin \mathrm{sp}(f)$, by definition, the function $(\lambda - \mathscr{D})^{-1}f$, defined on $\boldsymbol{C} \backslash i\boldsymbol{R}$, has an analytic continuation into a neighborhood of $i\beta$. In particular, the following limit exists $\lim_{\lambda \to i\beta}(\lambda - \mathscr{D})^{-1}f := g_0$. We are going to show that g_0 is a solution of (2.9) and $g_0 \in \Lambda$. Indeed, since

$$\begin{aligned}(i\beta - \mathscr{D})(\lambda - \mathscr{D})^{-1}f &= ((i\beta - \lambda) + (\lambda - \mathscr{D}))(\lambda - \mathscr{D})^{-1}f \\ &= (i\beta - \lambda)(\lambda - \mathscr{D})^{-1}f + (\lambda - \mathscr{D})(\lambda - \mathscr{D})^{-1}f \\ &= (i\beta - \lambda)(\lambda - \mathscr{D})^{-1}f + f,\end{aligned}$$

we have

$$\lim_{\lambda \to i\beta}(i\beta - \mathscr{D})(\lambda - \mathscr{D})^{-1}f = f. \tag{2.10}$$

Using the closedness of the operator $(i\beta - \mathscr{D})$, we come up with g_0 being in the domain of $i\beta - \mathscr{D}$ and $(i\beta - \mathscr{D})g_0 = f$. Next, to prove that $g_0 \in \Lambda(\boldsymbol{X})$, in view of Corollary 2.4 it suffices to show that for every $\Re\lambda_0 \neq 0$, the function $(\lambda_0 - \mathscr{D})^{-1}f$ is in $\Lambda(\boldsymbol{X})$. Since both λ and λ_0 are in $\rho(\mathscr{D})$, and

$$(\lambda - \mathscr{D})^{-1}(\lambda_0 - \mathscr{D})^{-1}f = (\lambda_0 - \mathscr{D})^{-1}(\lambda - \mathscr{D})^{-1}f$$

we see that $(\lambda - \mathscr{D})^{-1}(\lambda_0 - \mathscr{D})^{-1}f$ has an analytic continuation into a neighborhood of $i\beta$. This completes the proof of the lemma. □

2.3. Condition H1 and H2

Below we will need the following conditions:

Condition H1: Two operators A and B on a Banach space $\boldsymbol{X}$ satisfy the following conditions:

(a) B is invertible and $D(B) \subset D(A)$;
(b) A and B are sectorial operators with $\rho(A)$ and $\rho(B)$ containing $\Sigma(\omega, R)$ for some real number ω and positive number R.

Condition H2: A subspace $\mathscr{M}$ of $BC(\boldsymbol{R}, \boldsymbol{X})$ with sup-norm satisfies the following conditions:

(a) $\mathscr{M}$ is closed;
(b) $\mathscr{M}$ contains all functions of the form $Bf(\cdot)$, whenever $f \in \mathscr{M}$ and $B \in L(\boldsymbol{X})$;
(c) For every complex number λ with $\Re\lambda \neq 0$, the following inclusion holds:

$$(\lambda - \mathscr{D})^{-1}\mathscr{M} \subset \mathscr{M}.$$

Remark 2.6. If $\mathscr{M}$ is a closed subspace of $BUC(\boldsymbol{R}, \boldsymbol{X})$, then the condition (c) in the above definition of Condition H2 is nothing but the translation invariance of the function space $\mathscr{M}$. In fact, in this case, by the identity

$$(\lambda - \mathscr{D})^{-1}f = \begin{cases} \int_0^\infty e^{-\lambda\xi} S(\xi) f \, d\xi, & (\Re\lambda > 0, f \in \mathscr{M}), \\ -\int_{-\infty}^0 e^{-\lambda\xi} S(\xi) f \, d\xi & (\Re\lambda < 0, f \in \mathscr{M}), \end{cases}$$

the translation invariance of $\mathscr{M}$ yields the condition (c) of Condition H2. Conversely, if the condition (c) of Condition H2 is satisfied, then, the exponential formula

$$S(t)f = \lim_{n\to\infty} \left(I - \frac{t}{n}\mathscr{D} \right)^{-n} f, \qquad (f \in \mathscr{M})$$

yields the translation invariance.

Example 2.7. The following classes of function spaces satisfy Condition H2:

(i) The function space $\Lambda(\boldsymbol{X}) := \{ f \in BC(\boldsymbol{R}, \boldsymbol{X}) : \mathrm{sp}(f) \subset \Lambda \}$, where Λ is a closed subset of $\boldsymbol{R}$, satisfies Condition H2. In fact, it is closed, so condition (a) is satisfied. By (iv) of Proposition 2.1, condition (b) is satisfied. From the last lines of the proof of Lemma 2.5, condition (c) is satisfied;

(ii) Any function spaces $\mathscr{M}$ satisfying the condition (11) in [20], that is, $\mathscr{M} \subset BUC(\boldsymbol{R}, \boldsymbol{X})$, and $CS(\cdot)f \in \mathscr{M}$ whenever $f \in \mathscr{M}$ and $C \in L(\mathscr{M}, \boldsymbol{X})$;

(iii) Any function spaces satisfying Condition H1 in [11], that is, $\mathscr{M} \subset BUC(\boldsymbol{R}, \boldsymbol{X})$, and $\mathscr{M}$ satisfies the two conditions listed in the definition of Condition H2.

Recall that

Definition 2.8. An operator A on a Banach space $\boldsymbol{X}$ is said to be *sectorial* if there are constants $\omega \in \boldsymbol{R}$, $\theta \in (\pi/2, \pi)$ and $M > 0$ such that

$$\begin{cases} \rho(A) \supset S_{\theta,\omega} := \{\lambda \in \boldsymbol{C} : \lambda \neq \omega, |\arg(\lambda - \omega)| < \theta\}, \\ \|R(\lambda, A)\| \leq \frac{M}{|\lambda - \omega|}, \quad \forall \lambda \in S_{\theta,\omega}. \end{cases} \tag{2.11}$$

Lemma 2.9. *Let $\mathscr{M}$ be a function space that satisfies Condition* H2, *and let A be a sectorial operator on $\boldsymbol{X}$. Then the operator $\mathscr{A}_{\mathscr{M}}$ of multiplication by A is a sectorial operator on $\mathscr{M}$.*

Proof. First we prove that $\sigma(\mathscr{A}_{\mathscr{M}}) \subset \sigma(A)$. In fact, let $\mu \in \rho(A)$. To prove that $\mu \in \rho(\mathscr{A}_{\mathscr{M}})$ we show that for each $h \in \mathscr{M}$ the equation $\mu g - \mathscr{A}_{\mathscr{M}} g = h$ has a unique solution in $\mathscr{M}$. In fact, since, by Condition H2, $(\mu - A)^{-1} h(\cdot) \in \mathscr{M}$, the function $g := (\mu - A)^{-1} h(\cdot)$ is a solution of the equation $\mu g - \mathscr{A}_{\mathscr{M}} g = h$. This equation has no more than one solution because for every fixed t_0 the equation $\mu x - Ax = h(t_0)$ has a unique solution. This shows that $\mu \in \rho(\mathscr{A}_{\mathscr{M}})$ and $(\mu - \mathscr{A}_{\mathscr{M}})^{-1} h = g := (\mu - A)^{-1} h(\cdot)$ for every $h \in \mathscr{M}$. This also yields the same estimate on $\|R(\mu, \mathscr{A}_{\mathscr{M}})\|$ in (2.11), proving the lemma. □

As shown in [9, Chapter 2], a sectorial operator A generates an analytic semigroup that is not necessarily strongly continuous.

3. First order equations

In this section we investigate first order equations of the form

$$u'(t) = Au(t) + f(t), \qquad u(t) \in \boldsymbol{X}, \tag{3.1}$$

where A generates an analytic semigroup (that is not necessarily strongly continuous) and f is an $\boldsymbol{X}$-valued bounded and continuous function. Before proceeding we recall the concepts of classical and mild solutions of (3.1).

Definition 3.1. (i) A function $u \in BC(\boldsymbol{R}, \boldsymbol{X})$ is said to be a mild solution of (3.1) on $\boldsymbol{R}$ if for all $t \geq s$, $\int_s^t u(\xi) d\xi \in D(A)$, and

$$u(t) - u(s) = A \int_s^t u(\xi) d\xi + \int_s^t f(\xi) d\xi. \tag{3.2}$$

(ii) A function $u \in BC^1(\boldsymbol{R}, \boldsymbol{X})$ is said to be a classical solution of (3.1) on $\boldsymbol{R}$ if for all $t \in \boldsymbol{R}$, $u(t)$ is in $D(A)$, and (3.1) holds.

We notice that a classical solution is a mild solution. However, as is well known, a mild solution may not be a classical solution. In case where A generates a C_0-semigroup $(T(t))_{t \geq s}$, the concept of mild solutions defined above coincides with the well-known one (see e.g. [14]), that is, in this case mild solutions on $\boldsymbol{R}$ of (3.1) are continuous solutions to the following integral equation

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\xi)f(\xi)d\xi, \qquad \forall t \geq s. \tag{3.3}$$

The following result improves in some respect one of the main results in [15, 20, 11].

Theorem 3.2. *Let A satisfy the following conditions for some positive numbers ε, R and a real number α:*

(i)

$$\rho(A+\alpha) \supset \Sigma\left(\frac{\pi}{2} + \varepsilon, R\right),$$

(ii)

$$\sup_{\lambda \in \Sigma(\pi/2+\varepsilon, R)} \|\lambda R(\lambda, A+\alpha)\| < \infty.$$

Assume further that $\mathscr{M}$ be a function space satisfying Condition H2. *Then, for* (3.1) *to have at least a mild solution in $\mathscr{M}$ for every given $f \in \mathscr{M}$, it is sufficient that*

$$\sigma(\mathscr{D}_{\mathscr{M}}) \cap \sigma(A) = \varnothing. \tag{3.4}$$

Proof. Without loss of generality we may assume $\alpha = 0$. The main idea follows the one in [11], that is, we will apply the method of sums of commuting operators. However, we will not use the explicit expression for the generator of the associated evolution semigroups because the analytic semigroup, say $(T(t))_{t \geq 0}$, may not be strongly continuous.

Before proceeding, we check that $\mathscr{A}_{\mathscr{M}}$ and $\mathscr{D}_{\mathscr{M}}$ satisfy the condition P of Definition 5.2 (ii). By Lemma 2.9 it suffices to show that $\mathscr{A}_{\mathscr{M}}$ and $\mathscr{D}_{\mathscr{M}}$ are commuting. In fact, let λ be such that $\Re\lambda \neq 0$, and let $\mu \in \rho(A)$. Then $\lambda \in \rho(\mathscr{D})$ and $\mu \in \rho(\mathscr{A})$, where $\mathscr{D}$ is the differentiation operator on $BC(\boldsymbol{R}, \boldsymbol{X})$, and $\mathscr{A}$ is the operator of multiplication by A on $BC(\boldsymbol{R}, \boldsymbol{X})$. By the conditions (c) and (b) of Condition H2, this yields that $\lambda \in \rho(\mathscr{D}_{\mathscr{M}})$ and $\mu \in \rho(\mathscr{A}_{\mathscr{M}})$. Next, since

$g = (\lambda - \mathscr{D}_{\mathscr{M}})^{-1} f$ is the unique bounded solution of the equation $\lambda x - x' = f$ for a given $f \in \mathscr{M}$, it is obvious that $(\mu - A)^{-1} g(\cdot) = (\lambda - \mathscr{D}_{\mathscr{M}})^{-1}(\mu - \mathscr{A}_{\mathscr{M}})^{-1} f$ is the unique bounded solution of the equation $\lambda x - x' = (\mu - A)^{-1} f(\cdot)$, so,

$$(\mu - \mathscr{A}_{\mathscr{M}})^{-1}(\lambda - \mathscr{D}_{\mathscr{M}})^{-1} f = (\lambda - \mathscr{D}_{\mathscr{M}})^{-1}(\mu - \mathscr{A}_{\mathscr{M}})^{-1} f$$

for every $f \in \mathscr{M}$. That is, $\mathscr{A}_{\mathscr{M}}$ and $\mathscr{D}_{\mathscr{M}}$ are commuting.

By Theorem 5.3, that applies to the pair of operators $\mathscr{A}_{\mathscr{M}}$ and $\mathscr{D}_{\mathscr{M}}$, we have

$$\sigma(\overline{(\mathscr{A}_{\mathscr{M}} - \mathscr{D}_{\mathscr{M}})}^{\mathscr{A}_{\mathscr{M}}}) \subset \sigma(\mathscr{A}_{\mathscr{M}}) + \sigma(-\mathscr{D}_{\mathscr{M}}). \tag{3.5}$$

By (3.4), and (3.5) we see that

$$0 \notin \sigma(\overline{(\mathscr{A}_{\mathscr{M}} - \mathscr{D}_{\mathscr{M}})}^{\mathscr{A}_{\mathscr{M}}}).$$

Therefore, there is a unique $u \in \mathscr{M}$ such that

$$\overline{(\mathscr{A}_{\mathscr{M}} - \mathscr{D}_{\mathscr{M}})}^{\mathscr{A}_{\mathscr{M}}} u = f.$$

This means that there is a sequence of classical solutions $\{u_n\}$ to the equations

$$u_n'(t) = A u_n(t) + f_n(t)$$

such that $u_n \to u$ and $f_n \to f$ as $n \to \infty$ in $\mathscr{T}_{\mathscr{A}_{\mathscr{M}}}$-topology, that is, $\|R(\lambda_0, \mathscr{A}_{\mathscr{M}})(u_n - u)\| \to 0$ and $\|R(\lambda_0, \mathscr{A}_{\mathscr{M}})(f_n - f)\| \to 0$ as $n \to \infty$, for some (and thus, for all $\lambda_0 \in \rho(\mathscr{A}_{\mathscr{M}})$). Notice that we can take λ_0 in $\rho(A)$. Since $\mathscr{M}$ satisfies Condition H2, both $R(\lambda_0, A) u_n(\cdot)$ and $R(\lambda_0, A) u(\cdot)$ are in $\mathscr{M}$, so they are $R(\lambda_0, \mathscr{A}_{\mathscr{M}}) u_n$ and $R(\lambda_0, \mathscr{A}_{\mathscr{M}}) u$. And hence,

$$\|R(\lambda_0, \mathscr{A}_{\mathscr{M}})(u_n - u)\| = \sup_{t \in \boldsymbol{R}} \|R(\lambda_0, A)(u_n(t) - u(t))\|.$$

Next, since

$$u_n(t) - u_n(s) = A \int_s^t u_n(\xi) d\xi + \int_s^t f_n(\xi) d\xi, \qquad \forall t \geq s,$$

we have

$$\begin{aligned} AR(\lambda_0, A) \int_s^t u_n(\xi) d\xi &= R(\lambda_0, A) A \int_s^t u_n(\xi) d\xi \\ &= R(\lambda_0, A)(u_n(t) - u_n(s)) - R(\lambda_0, A) \int_s^t f_n(\xi) d\xi. \end{aligned}$$

The right hand side approaches

$$R(\lambda_0, A)(u(t) - u(s)) - \int_s^t R(\lambda_0, A) f(\xi) d\xi,$$

as $n \to \infty$. Also, we have

$$\lim_{n\to\infty} \int_s^t R(\lambda_0, A) u_n(\xi) d\xi = \int_s^t R(\lambda_0, A) u(\xi) d\xi = R(\lambda_0, A) \int_s^t u(\xi) d\xi.$$

Therefore, in view of the closedness of A we get

$$AR(\lambda_0, A) \int_s^t u(\xi) d\xi = R(\lambda_0, A)(u(t) - u(s)) - \int_s^t R(\lambda_0, A) f(\xi) d\xi,$$

so,

$$-\int_s^t u(\xi) d\xi + \lambda_0 R(\lambda_0, A) \int_s^t u(\xi) d\xi = R(\lambda_0, A)(u(t) - u(s)) - \int_s^t R(\lambda_0, A) f(\xi) d\xi.$$

This yields

$$\int_s^t u(\xi) d\xi = R(\lambda_0, A)\left(\lambda_0 \int_s^t u(\xi) d\xi - u(t) + u(s) + \int_s^t f(\xi) d\xi\right) \in D(A)$$

and

$$(\lambda_0 - A) \int_s^t u(\xi) d\xi = \lambda_0 \int_s^t u(\xi) d\xi - u(t) + u(s) + \int_s^t f(\xi) d\xi.$$

Therefore,

$$u(t) - u(s) = A \int_s^t u(\xi) d\xi + \int_s^t f(\xi) d\xi,$$

that is u is a mild solution of (3.1). This completes the proof of the theorem. □

Remark 3.3. In Theorem 3.2 we have discussed only the existence of a mild solution with the same profile as f. However, we do not know if it is unique. With additional condition on $\mathscr{M}$ we can show the uniqueness of the mild solution.

As a consequence we have

Corollary 3.4. *Let A be a sectorial operator, and let Λ be a closed subset of the real line. Then, for* (3.1) *to have a unique mild solution in $\Lambda(\boldsymbol{X})$ for every given $f \in \Lambda(\boldsymbol{X})$, it is necessary and sufficient that*

$$i\Lambda \cap \sigma(A) = \emptyset. \tag{3.6}$$

Proof. *Sufficiency*: By Lemma 2.5, $i\Lambda = \sigma(\mathscr{D}_\Lambda)$. Since $\Lambda(\boldsymbol{X})$ satisfies Condition H2, the existence of a mild solution in $\Lambda(\boldsymbol{X})$ follows from Theorem 3.2. We now show that this solution is unique. This can be done via the concept of Carleman spectrum. In fact, let v and w in $\Lambda(\boldsymbol{X})$ be two bounded mild solutions of (3.1) on $\boldsymbol{R}$. We will show that $u = v - w = 0$. In fact, for $\Re\lambda > 0$ and $\lambda \in \rho(A)$, taking the Carleman transforms of both sides of (3.2) with $s = 0$, we have $\hat{u}(\lambda) \in D(A)$ and

$$\hat{u}(\lambda) - \frac{1}{\lambda}u(0) = \frac{1}{\lambda}A\hat{u}(\lambda).$$

So,

$$\hat{u}(\lambda) = R(\lambda, A)u(0). \tag{3.7}$$

Similarly, if $\Re\lambda < 0$ and $\lambda \in \rho(A)$ we can show that (3.7) holds. Therefore, since $R(\lambda, A)$ is holomorphic, one has $i\,\mathrm{sp}(u) \subset \sigma(A)$. On the other hand, since v and w are in $\Lambda(\boldsymbol{X})$, $\mathrm{sp}(u) \subset \Lambda$. And hence, $i\,\mathrm{sp}(u) \subset i\Lambda \cap \sigma(A) = \varnothing$. So, by Proposition 2.1, $u = 0$, that is, $v = w$, and the bounded mild solution in $\Lambda(\boldsymbol{X})$ is unique.

Necessity: For every fixed real $\lambda_0 \in \Lambda$ take $f(t) = ae^{i\lambda_0 t}$, where a is any element of $\boldsymbol{X}$. By the above theorem, there exists a unique mild solution u_f of (3.1) whose uniform spectrum is contained in $\{\lambda_0\}$. In particular, the Carleman spectrum u_f is contained in $\{\lambda_0\}$. Hence, u_f is of the form $u_f(t) = be^{i\lambda_0 t}$. So, u_f is continuously differentiable. We have

$$\frac{u(t) - u(s)}{t - s} = A\frac{1}{t - s}\int_s^t u(\xi)d\xi + \frac{1}{t - s}\int_s^t f(\xi)d\xi, \qquad \forall t > s.$$

Letting $t \to s$ implies that $u(s) \in D(A)$ for all $s \in \boldsymbol{R}$, and

$$u'(s) = Au(s) + f(s) \tag{3.8}$$

for all $s \in \boldsymbol{R}$, that is, $u(\cdot)$ is a classical solution of (3.1). On the other hand, (3.8) is nothing but

$$\begin{aligned} bi\lambda_0 e^{\lambda_0 s} &= (be^{i\lambda_0 s})' \\ &= Abe^{\lambda_0 s} + ae^{i\lambda_0 s}. \end{aligned}$$

Dividing both sides by $e^{\lambda_0 s}$ we get $i\lambda_0 b = Ab + a$. So, for every $a \in \boldsymbol{X}$, there exists a unique b such that $i\lambda_0 b - Ab = a$. This means that $i\lambda_0 \notin \sigma(A)$. That is, $\sigma(A) \cap i\Lambda = \varnothing$. The proof is completed. □

Remark 3.5. Even if A generates a strongly continuous analytic semigroup, the operator $\mathscr{A}$ of multiplication by A may not generate a strongly continuous analytic semigroup in $BC(\boldsymbol{R}, \boldsymbol{X})$. Similarly for the operators of multiplication by A on other subspaces of $BC(\boldsymbol{R}, \boldsymbol{X})$. Therefore, it is natural to consider the general case where A generates an analytic semigroup that is not necessarily strongly continuous.

Corollary 3.6. *Let A be a sectorial operator, and let f be in $BC(\boldsymbol{R}, \boldsymbol{X})$. Then,* (3.1) *has a unique mild solution $u_f \in BC(\boldsymbol{R}, \boldsymbol{X})$ such that* $\mathrm{sp}(u_f) \subset \mathrm{sp}(f)$ *provided*

$$i\,\mathrm{sp}(f) \cap \sigma(A) = \varnothing. \tag{3.9}$$

Proof. It suffices to take $\Lambda = \mathrm{sp}(f)$ and apply the above corollary. ☐

Remark 3.7. If f is almost automorphic function, then the function space $AA_\Lambda(\boldsymbol{X})$ (the space of all almost automorphic functions with uniform spectrum contained in Λ) satisfies Condition H2, where $\Lambda := \mathrm{sp}(f)$. Since the spectrum of the differentiation operator on this function space is exactly $i\Lambda$, by Theorem 3.2 and Corollary 3.6 we get a main result of [3].

Remark 3.8. In the proof of Theorem 3.2 we have actually used the closedness of the following operator $\mathscr{L}$ defined on the function space $\mathscr{M}$ with the domain $D(\mathscr{L})$ consists of all functions $u \in \mathscr{M}$ such that there is a function $f \in \mathscr{M}$ for which u is a mild solution of Eq. (3.1), and $\mathscr{L}u := f$ if $u \in D(\mathscr{L})$ with such a function f. The following proposition may be of independent interest.

Proposition 3.9. *Let $\mathscr{M}$ be a closed subspace of $BC(\boldsymbol{R}, \boldsymbol{X})$ that satisfies Condition* H2, *and let $\mathscr{L}$ be defined as above. Then, $\mathscr{L}$ is a single-valued linear operator which is closed in the $\mathscr{T}_A$ topology, that is, if $u_n \in D(\mathscr{L})$ is a sequence such that* $\sup_t \|R(\lambda_0, A)(u_n(t) - u(t))\| \to 0$, *and* $\sup_t \|R(\lambda_0, A)(\mathscr{L}u_n(t) - f(t))\| \to 0$ *for some $u, f \in \mathscr{M}$, then $u \in D(\mathscr{L})$ and $\mathscr{L}u = f$.*

Proof. First we show that the operator $\mathscr{L}$ is a single-valued linear operator. The linearity is clear. Now, suppose that there are functions $u, f, g \in \mathscr{M}$ such that $\int_s^t u(\xi)d\xi \in D(A)$ for all $t \geq s$; $t, s \in \boldsymbol{R}$, and

$$u(t) - u(s) = A\int_s^t u(\xi)d\xi + \int_s^t f(\xi)d\xi, \tag{3.10}$$

$$u(t) - u(s) = A\int_s^t u(\xi)d\xi + \int_s^t g(\xi)d\xi. \tag{3.11}$$

Therefore, for all $t > s$

$$\frac{1}{t-s}\int_s^t f(\xi)d\xi = \frac{1}{t-s}\int_s^t g(\xi)d\xi.$$

Since f and g are continuous, we come up with $f(s) = g(s)$ for all $s \in \boldsymbol{R}$, that is, $f = g$, yielding the single-valuedness of $\mathscr{L}$.

Now we prove the $\mathscr{T}_A$-closedness of this operator. Let $u_n \in D(\mathscr{L})$ be a sequence such that $\sup_t \|R(\lambda_0, A)(u_n(t) - u(t))\| \to 0$, and $\sup_t \|R(\lambda_0, A)(\mathscr{L}u_n(t) - f(t))\| \to 0$ for some $u, f \in \mathscr{M}$. Then, by the assumption and definition of $\mathscr{L}$, we have

$$u_n(t) - u_n(s) = A\int_s^t u_n(\xi)d\xi + \int_s^t f_n(\xi)d\xi, \qquad \forall t \geq s;\ t, s \in \boldsymbol{R}. \tag{3.12}$$

Now we can follow exactly the lines of the proof of Theorem 3.2. □

4. Second order equations

In this section we will study the existence of bounded solutions to the second order equations of the form

$$u''(t) = Bu'(t) + Au(t) + f(t), \qquad u(t) \in \boldsymbol{X}, \tag{4.1}$$

where A and B are closed linear operators and f is an $\boldsymbol{X}$-valued bounded and continuous function.

Define an operator

$$P(\lambda) = \lambda^2 - \lambda B - A \tag{4.2}$$

and

$$\Lambda_0 := \{\lambda \in i\boldsymbol{R} : \nexists P^{-1}(\lambda) \in L(\boldsymbol{X})\}.$$

Definition 4.1. A function $u(\cdot)$ defined on $\boldsymbol{R}$ is said to be a (bounded) classical solution on $\boldsymbol{R}$ of (1.2) if

(i) $u \in BC^2(\boldsymbol{R}, \boldsymbol{X})$;

(ii) For every $t \in \boldsymbol{R}$, $u(t) \in D(A)$, $u'(t) \in D(B)$ and $Bu' \in BC(\boldsymbol{R}, \boldsymbol{X})$, $Au \in BC(\boldsymbol{R}, \boldsymbol{X})$;

(iii) For every $t \in \boldsymbol{R}$, (1.2) holds.

Definition 4.2. A function $u(\cdot) \in BC(\boldsymbol{R}, \boldsymbol{X})$ is said to be a (bounded) mild solution on $\boldsymbol{R}$ of (1.2) if the following holds

(i) For all $t \geq s$; $t, s \in \boldsymbol{R}$, $\int_s^t u(\xi)d\xi \in D(B)$, $\int_s^t (t-\xi)u(\xi)d\xi \in D(A)$;

(ii) For every given $s \in \boldsymbol{R}$ there is $x \in \boldsymbol{X}$ such that the following holds for all $t \geq s$:

$$u(t) - u(s) = (t-s)x + B\int_s^t u(\xi)d\xi + A\int_s^t (t-\xi)u(\xi)d\xi + \int_s^t (t-\xi)f(\xi)d\xi. \tag{4.3}$$

Remark 4.3. It is clear that every classical solution on $\boldsymbol{R}$ of (1.2) is a mild solution (on $\boldsymbol{R}$) of this equation.

Theorem 4.4. *Assume that A and B satisfy the following conditions*:

(i) *A and B satisfy Condition* H1*;*

(ii) *$\rho(B) \supset \Sigma_c$ for some $c > 0$, and for some $k > 0$,*

$$\|R(\lambda, B)\| \le k(1+|\lambda|)^{-1}, \qquad \forall \lambda \in \Sigma_c.$$

Then Eq. (1.2) *has a unique mild solution with* $\mathrm{sp}(u) \subset \mathrm{sp(f)}$ *provided*

$$i\,\mathrm{sp}(f) \cap \Lambda_0 = \varnothing. \tag{4.4}$$

Proof. By setting

$$x(t) := \begin{pmatrix} u(t) \\ u'(t) \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & I \\ A & B \end{pmatrix}, \qquad F(t) := \begin{pmatrix} 0 \\ f(t) \end{pmatrix}, \tag{4.5}$$

we will reduce (1.2) to a first order equation of the form

$$x'(t) = Lx(t) + F(t), \tag{4.6}$$

where $x(t)$ is in $\boldsymbol{Y} := D(B) \times \boldsymbol{X}$, (here $D(B)$ is equipped with the graph norm), and L is an operator on $D(B) \times \boldsymbol{X}$ with $D(L) = D(B) \times D(B)$.

First, notice that L satisfies the following: There are positive numbers ε and R such that

(i)

$$\rho(L) \supset \Sigma(\pi/2 + \varepsilon, R);$$

(ii)

$$\sup_{\lambda \in \Sigma(\pi/2+\varepsilon, R)} \|\lambda R(\lambda, L)\| < \infty.$$

This claim can be proved by adapting the estimates in [4, p. 85–87], so we omit the details. Moreover,

$$\sigma(L) \subset \{\lambda \in \boldsymbol{C} \mid \nexists P^{-1}(\lambda) \in L(X)\}. \tag{4.7}$$

Next, by Theorem 3.2 and its corollaries, there exists a unique bounded mild solution of (4.6) on $\boldsymbol{R}$, say $x(\cdot)$, such that $\mathrm{sp}(x) \subset \mathrm{sp}(F) = \mathrm{sp}(f)$. Now we show that this yields the existence of a bounded mild solution to (4.1). In fact, by definition, for all $t \ge s$, $\int_s^t x(\xi)d\xi \in D(L)$, and

$$x(t) - x(s) = L\int_s^t x(\xi)d\xi + \int_s^t F(\xi)d\xi, \qquad \forall t \ge s. \tag{4.8}$$

Since $D(L) = D(B) \times D(B)$, if $x(t) = (u(t), v(t)) \in D(B) \times \boldsymbol{X}$, we have $\int_s^t v(\xi)d\xi \in D(B)$. Re-writing the above equation in the matrix form we have

$$\begin{pmatrix} u(t) \\ v(t) \end{pmatrix} - \begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & B \end{pmatrix} \begin{pmatrix} \int_s^t u(\xi)d\xi \\ \int_s^t v(\xi)d\xi \end{pmatrix} + \begin{pmatrix} 0 \\ \int_s^t f(\xi)d\xi \end{pmatrix}.$$

Therefore,

$$u(t) - u(s) = \int_s^t v(\xi)d\xi, \tag{4.9}$$

$$v(t) - v(s) = A\int_s^t u(\xi)d\xi + B\int_s^t v(\xi)d\xi + \int_s^t f(\xi)d\xi. \tag{4.10}$$

Now if we integrate (4.10) in t and use (4.9) (s is fixed) we come up with

$$\begin{aligned} \int_s^t v(\xi)d\xi - (t-s)v(s) &= A\int_s^t dt_1 \int_s^{t_1} u(\xi)d\xi + B\int_s^t dt_1 \int_s^{t_1} v(\xi)d\xi \\ &\quad + \int_s^t dt_1 \int_s^{t_1} f(\xi)d\xi, \\ u(t) - u(s) - (t-s)v(s) &= A\int_s^t (t-\xi)u(\xi)d\xi + B\int_s^t (u(t_1) - u(s))dt_1 \\ &\quad + \int_s^t (t-\xi)f(\xi)d\xi. \end{aligned}$$

Since $u(t) \in D(B)$ for all t, it follows from the above that

$$u(t) - u(s) = (t-s)x + A\int_s^t (t-\xi)u(\xi)d\xi + B\int_s^t u(\xi)d\xi + \int_s^t (t-\xi)f(\xi)d\xi,$$

where $x := v(s) - Bu(s)$. This shows that $u(\cdot)$ is a mild solution on $\boldsymbol{R}$. Obviously, $\mathrm{sp}(u) \subset \mathrm{sp}(f)$. □

5. Appendix

We recall now the notion of two commuting operators which will be used in the sequel.

Definition 5.1. Let A and B be operators on a Banach space G with non-empty resolvent set. We say that A and B *commute* if one of the following equivalent conditions hold:

(i) $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$ for some (all) $\lambda \in \rho(A)$, $\mu \in \rho(B)$,

(ii) $x \in D(A)$ implies $R(\mu, B)x \in D(A)$ and $AR(\mu, B)x = R(\mu, B)Ax$ for some (all) $\mu \in \rho(B)$.

Definition 5.2. Let A and B be commuting operators. Then

(i) A is said to be of class $\Sigma(\theta + \pi/2, R)$ if there are positive constants θ, R such that $0 < \theta < \pi/2$, and

$$\Sigma(\theta + \pi/2, R) \subset \rho(A) \quad \text{and} \quad \sup_{\lambda \in \Sigma(\theta+\pi/2, R)} \|\lambda R(\lambda, A)\| < \infty, \tag{5.1}$$

(ii) A and B are said to satisfy *Condition* P if there are positive constants $\theta, \theta', R, \theta' < \theta$ such that A and B are of class $\Sigma(\theta + \pi/2, R)$, $\Sigma(\pi/2 - \theta', R)$, respectively.

If A and B are commuting operators, $A + B$ is defined by $(A + B)x = Ax + Bx$ with domain $D(A + B) = D(A) \cap D(B)$.

In this paper we will use the following norm, defined by A on the space $\boldsymbol{X}$, $\|x\|_{\mathscr{T}_A} := \|R(\lambda, A)x\|$, where $\lambda \in \rho(A)$. It is seen that different $\lambda \in \rho(A)$ yields equivalent norms. We say that an operator C on $\boldsymbol{X}$ is A-closed if its graph is closed with respect to the topology induced by $\mathscr{T}_A$ on the product $\boldsymbol{X} \times \boldsymbol{X}$. It is easily seen that C is A-closable if $x_n \to 0$, $x_n \in D(C)$, $Cx_n \to y$ with respect to $\mathscr{T}_A$ in $\boldsymbol{X}$ implies $y = 0$. In this case, A-closure of C is denoted by $\bar{C}^A$.

Theorem 5.3. *Assume that A and B commute. Then the following assertions hold*:

(i) *If one of the operators is bounded, then*

$$\sigma(A + B) \subset \sigma(A) + \sigma(B). \tag{5.2}$$

(ii) *If A and B satisfy Condition* P, *then $A + B$ is A-closable, and*

$$\sigma(\overline{(A + B)}^A) \subset \sigma(A) + \sigma(B). \tag{5.3}$$

In particular, if $D(A)$ is dense in $\boldsymbol{X}$, then $\overline{(A + B)}^A = \overline{A + B}$, where $\overline{A + B}$ denotes the usual closure of $A + B$.

Proof. For the proof we refer the reader to [1, Theorems 7.2, 7.3]. □

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