# Decomposition of Phase Space for Linear Volterra Difference Equations in a Banach Space 

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#### Abstract

We study the spectrum of the solution operator for a linear Volterra difference equation in a Banach space, and obtain the decomposition of the phase space, together with that of the variation of constants formula in the phase space. As an application, we establish the existence of almost periodic solutions for forced linear Volterra difference equations with an almost periodic forcing term.


Key Words and Phrases. Volterra difference equation, Decomposition of phase space, Variation of constants formula in the phase space, Almost periodic solution, Bohr spectrum.

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## 1. Introduction

In this paper we are concerned with the Volterra difference equation

$$
\begin{equation*}
x(n+1)=\sum_{k=-\infty}^{n} Q(n-k) x(k)+p(n), \quad n \in \boldsymbol{Z}^{+}, \tag{1}
\end{equation*}
$$

where $\{Q(n)\}$ is a sequence of compact linear operators on a Banach space $X$ satisfying $\sum_{n=0}^{\infty}\|Q(n)\| e^{\gamma n}<\infty$ for some $\gamma>0$, and $p$ is an $X$-valued function on $\boldsymbol{Z}$.

In preceding papers [6, 7], under some additional conditions, we proved that the asymptotic stability of the zero solution of the equation with the forcing term free

$$
\begin{equation*}
x(n+1)=\sum_{k=-\infty}^{n} Q(n-k) x(k), \quad n \in \boldsymbol{Z}^{+} \tag{2}
\end{equation*}
$$

is equivalent to the invertibility of the characteristic operator outside the unit disk of the complex plane, and established generalizations of some stability results for finite-dimensional $X$ in $[3,4]$ to infinite-dimensional $X$.

The main purpose of this paper is to discuss asymptotic behaviors of solutions of Eq. (1) under the situation where the zero solution of Eq. (2) fails
to be asymptotically stable. In such a situation, the decomposition of the phase space for Eq. (2) associated with the spectrum of the solution operator (Theorem 1) plays quite an essential role. In fact, the decomposition of the phase space induces that of the variation of constants formula (VCF) in the phase space (Proposition 1, see also [8]) into two parts referred to as the stable part of VCF and the unstable part of VCF (Theorem 2); and by studying each part of VCF, we show that the study for Eq. (1) and (2) may be reduced in some sense to that for certain difference equations of first order in a finitedimensional space. Moreover, applying the decomposition of VCF, we will give a result on the existence of almost periodic solutions of Eq. (1), which can be viewed as a discrete analogue for Volterra difference equations of Massera's theorem [11] on the existence of periodic solutions for linear ordinary differential equations.

## 2. Preliminaries

We shall give in this section the variation of constants formula for Eq. (1) in a suitable phase space, which plays an essential role in the study on the behaviors of solutions of Eq. (1) in the subsequent sections.

Let $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{Z}^{+}, \boldsymbol{Z}^{-}$and $\boldsymbol{C}$ be the set of natural numbers, integers, nonnegative integers, nonpositive integers and complex numbers, respectively. Let $X$ be a Banach space over $C$ with norm $|\cdot|$. We denote by $\mathscr{L}(X)$ the space of all bounded linear operators on $X$ and define the norm of any $T$ belonging to $\mathscr{L}(X)$ by

$$
\|T\|=\sup \{|T x|: x \in X,|x|=1\} .
$$

Throughout the paper we assume that there exists a positive constant $\gamma$ such that

$$
K_{1}:=\sum_{n=0}^{\infty}\|Q(n)\| e^{\gamma n}<\infty
$$

Let us define the Banach space $\mathscr{B}^{\gamma}$ by

$$
\mathscr{B}^{\nu}=\left\{\phi: \boldsymbol{Z}^{-} \mapsto X\left|\sup _{\theta \in \boldsymbol{Z}^{-}}\right| \phi(\theta) \mid e^{\nu \theta}<\infty\right\}
$$

equipped with the norm

$$
\|\phi\|=\sup _{\theta \in \boldsymbol{Z}^{-}}|\phi(\theta)| e^{\nu \theta}, \quad \phi \in \mathscr{B}^{\gamma},
$$

and for any $\sigma \in \boldsymbol{Z}$ and $\phi \in \mathscr{B}^{\gamma}$ consider the initial value problem:

$$
\left\{\begin{array}{l}
x(n+1)=\sum_{j=-\infty}^{n} Q(n-j) x(j)+p(n), \quad n \geq \sigma  \tag{3}\\
x(\sigma+\theta)=\phi(\theta), \quad \theta \in \boldsymbol{Z}^{-}
\end{array}\right.
$$

The problem (3) has a unique solution, which will be denoted by $x(\cdot ; \sigma, \phi, p)$. In fact, $x(\cdot ; \sigma, \phi, p)$ is given by

$$
x(n ; \sigma, \phi, p)= \begin{cases}\phi(n), & n \leq \sigma  \tag{4}\\ R(n-\sigma) \phi(0)+\sum_{k=\sigma}^{n-1} R(n-k-1) p(k) & \\ +\sum_{k=\sigma}^{n-1} R(n-k-1)\left(\sum_{j=-\infty}^{-1} Q(k-\sigma-j) \phi(j)\right), & n \geq \sigma\end{cases}
$$

where $\{R(n)\}$ is the fundamental solution of (1) (or (2)), that is, the sequence in $\mathscr{L}(X)$ defined by

$$
\begin{equation*}
R(n+1)=\sum_{k=0}^{n} Q(n-k) R(k), \quad R(0)=I \tag{5}
\end{equation*}
$$

(see [6] and [8]). Here and hereafter we employ the convention $\sum_{\sigma}^{\sigma-1}=0$ for any $\sigma \in \boldsymbol{Z}$.

Define a family of linear operators $\{V(m)\}_{m \in \boldsymbol{Z}^{+}}$on $\mathscr{B}^{\gamma}$ by $V(m) \phi:=$ $x_{m}(0, \phi, 0)$, where the notation $x_{m}(\sigma, \phi, p)$ stands for the mapping from $\boldsymbol{Z}^{-}$into $X$ given by $\left[x_{m}(\sigma, \phi, p)\right](\theta)=x(m+\theta ; \sigma, \phi, p)$ for $\theta \in \boldsymbol{Z}^{-}$. It follows from (4) that

$$
[V(m) \phi](\theta)= \begin{cases}\phi(m+\theta), & \theta \leq-m  \tag{6}\\ R(m+\theta) \phi(0) & \\ +\sum_{k=0}^{m+\theta-1} R(m+\theta-k-1)\left(\sum_{j=-\infty}^{-1} Q(k-j) \phi(j)\right), & -m \leq \theta \leq 0\end{cases}
$$

$V(m)$ is called the solution operator of Eq. (2). One can verify that each $V(m)$ is a bounded linear operator on $\mathscr{B}^{\nu}$ with $V(0)=I$, and moreover that $\{V(m)\}_{m \in \boldsymbol{Z}^{+}}$has the semigroup property, i.e., $V(m+n)=V(m) V(n)$ for $m, n \in$ $\boldsymbol{Z}^{+}$(see [8]).

Let us define an operator $\mathscr{E}: X \mapsto \mathscr{B}^{\nu}$ by

$$
[\mathscr{E} x](\theta)= \begin{cases}x, & \theta=0 \\ 0, & \theta=-1,-2,-3 \ldots\end{cases}
$$

for any $x \in X$. Clearly, $\mathscr{E}$ is an isometry from $X$ into $\mathscr{B}^{\gamma}$ :

$$
\|\mathscr{E} x\|=|x|, \quad x \in X
$$

The following result [8] yields a representation formula for $x(\cdot ; \sigma, \phi, p)$ in $\mathscr{B}^{\gamma}$, which we call the variation of constants formula in the phase space. For the completeness we will give a proof.

Proposition 1. For any $\sigma \in \boldsymbol{Z}$ and $\phi \in \mathscr{B}^{\gamma}$ we have

$$
x_{n}(\sigma, \phi, p)=V(n-\sigma) \phi+\sum_{k=\sigma}^{n-1} V(n-k-1) \mathscr{E} p(k), \quad n \geq \sigma .
$$

Proof. It is sufficient to verify the following relation:

$$
\begin{equation*}
\left[x_{n}(\sigma, \phi, p)-V(n-\sigma) \phi\right](\theta)=\sum_{k=\sigma}^{n-1}[V(n-k-1) \mathscr{E} p(k)](\theta), \quad \theta \in \boldsymbol{Z}^{-} \tag{7}
\end{equation*}
$$

(i) The case of $\theta \leq-n+\sigma$ : Since $n-\sigma+\theta \leq 0$, we get

$$
\begin{aligned}
{\left[x_{n}(\sigma, \phi, p)-V(n-\sigma) \phi\right](\theta) } & =x(n+\theta ; \sigma, \phi, p)-[V(n-\sigma) \phi](\theta) \\
& =\phi(n-\sigma+\theta)-\phi(n-\sigma+\theta)=0 .
\end{aligned}
$$

Also, for $k=\sigma, \sigma+1, \ldots, n-1$, we have $n-k-1+\theta \leq n-\sigma-1+\theta \leq-1$, and hence

$$
\sum_{k=\sigma}^{n-1}[V(n-k-1) \mathscr{E} p(k)](\theta)=\sum_{k=\sigma}^{n-1}[\mathscr{E} p(k)](n-k-1+\theta)=0
$$

Thus the relation (7) holds true.
(ii) The case of $-n+\sigma+1 \leq \theta \leq 0$ : Since $1 \leq n-\sigma+\theta$, in view of (4) and (6) we get

$$
\begin{aligned}
{[ } & \left.x_{n}(\sigma, \phi, p)-V(n-\sigma) \phi\right](\theta) \\
= & x(n+\theta ; \sigma, \phi, p)-x(n-\sigma+\theta ; 0, \phi, 0) \\
= & R(n+\theta-\sigma) \phi(0)+\sum_{k=\sigma}^{n+\theta-1} R(n+\theta-k-1)\left(\sum_{j=-\infty}^{-1} Q(k-\sigma-j) \phi(j)+p(k)\right) \\
& -\left\{R(n-\sigma+\theta) \phi(0)+\sum_{k=0}^{n-\sigma+\theta-1} R(n-\sigma+\theta-k-1)\left(\sum_{j=-\infty}^{-1} Q(k-j) \phi(j)\right)\right\} \\
= & \sum_{k=\sigma}^{n+\theta-1} R(n+\theta-k-1) p(k) .
\end{aligned}
$$

On the one hand, since $\sigma \leq n+\theta-1 \leq n-1$, it follows that

$$
\begin{aligned}
\sum_{k=\sigma}^{n-1}[V(n-k-1) \mathscr{E} p(k)](\theta)= & \sum_{k=\sigma}^{n-1} x(n-k-1+\theta ; 0, \mathscr{E} p(k), 0) \\
= & \sum_{k=\sigma}^{n+\theta-1} x(n-k-1+\theta ; 0, \mathscr{E} p(k), 0) \\
& +\sum_{k=n+\theta}^{n-1} x(n-k-1+\theta ; 0, \mathscr{E} p(k), 0) .
\end{aligned}
$$

For $\sigma \leq k \leq n+\theta-1$, we see from (4)

$$
\begin{aligned}
x(n- & k-1+\theta ; 0, \mathscr{E} p(k), 0) \\
= & R(n-k-1+\theta)[\mathscr{E} p(k)](0) \\
& +\sum_{\ell=0}^{n-k+\theta-2} R(n-k+\theta-\ell-2)\left(\sum_{j=-\infty}^{-1} Q(\ell-j)[\mathscr{E} p(k)](j)\right) \\
= & R(n-k-1+\theta) p(k),
\end{aligned}
$$

while for $n+\theta \leq k \leq n-1$,

$$
x(n-k-1+\theta ; 0, \mathscr{E} p(k), 0)=[\mathscr{E} p(k)](n-k-1+\theta)=0
$$

because of $n-k-1+\theta \leq-1$. Thus we get

$$
\sum_{k=\sigma}^{n-1}[V(n-k-1) \mathscr{E} p(k)](\theta)=\sum_{k=\sigma}^{n+\theta-1} R(n-k-1+\theta) p(k)
$$

which shows the relation (7). This completes the proof.

## 3. Decomposition of the phase space

Let $\Sigma$ be the set of characteristic roots of Eq. (2), that is, the complex numbers such that the characteristic operator $z I-\tilde{Q}(z)$ associated with Eq. (2) is not invertible in $\mathscr{L}(X)$, where $\tilde{Q}(z)$ is the $Z$-transform of $\{Q(n)\}: \tilde{Q}(z)=$ $\sum_{n=0}^{\infty} Q(n) z^{-n}$. Let $V$ be the generator of the semigroup $\{V(m)\}$, i.e., $V=$ $V(1)$. Then under the condition that $Q(n)$ are all compact operators, we can find the following relation between the set $\Sigma$ and the spectrum of $V$. In what follows, for any linear operator $T$ on a Banach space, we denote by $\sigma(T)$ the spectrum of $T$ and $P_{\sigma}(T)$ the point spectrum of $T$.

Lemma 1. If $Q(n), n \in \boldsymbol{Z}^{+}$, are compact operators, then for $|z|>e^{-\gamma}$ the following statements are equivalent:
(i) $z \in \sigma(V)$,
(ii) $z \in P_{\sigma}(V)$,
(iii) $z \in \Sigma$, that is, $z I-\tilde{Q}(z)$ is not invertible in $\mathscr{L}(X)$.

Proof. (a) (ii) $\Rightarrow$ (i). This is evident.
(b) (i) $\Rightarrow$ (iii). It is sufficient to verify that the invertibility of $z I-\tilde{Q}(z)$ in $\mathscr{L}(X)$ implies $z \in \rho(V)$, the resolvent set of $V$. To see this, given $\psi \in \mathscr{B}^{\gamma}$, consider the linear equation in $\mathscr{B}^{\gamma}$

$$
\begin{equation*}
(z I-V) \phi=\psi \tag{8}
\end{equation*}
$$

Eq. (8) is equivalent to

$$
x(1+\theta ; 0, \phi, 0)=z \phi(\theta)-\psi(\theta), \quad \theta \in \boldsymbol{Z}^{-}
$$

which, combined with the fact that $x(1 ; 0, \phi, 0)=\sum_{k=-\infty}^{0} Q(-k) \phi(k)$, becomes the system of equations in $X$

$$
\left\{\begin{array}{l}
\sum_{k=0}^{\infty} Q(k) \phi(-k)=z \phi(0)-\psi(0),  \tag{9}\\
\phi(1+\theta)=z \phi(\theta)-\psi(\theta), \quad \theta=-1,-2, \ldots
\end{array}\right.
$$

From the second equation of (9),

$$
\phi(-1)=\frac{1}{z}(\phi(0)+\psi(-1))
$$

similarly

$$
\phi(-2)=\frac{1}{z}(\phi(-1)+\psi(-2))=\frac{1}{z^{2}} \phi(0)+\frac{1}{z^{2}} \psi(-1)+\frac{1}{z} \psi(-2),
$$

and generally we have

$$
\begin{equation*}
\phi(-k)=\frac{1}{z^{k}} \phi(0)+\sum_{j=0}^{k-1} \frac{1}{z^{k-j}} \psi(-j-1), \quad k=0,1,2, \ldots . \tag{10}
\end{equation*}
$$

It follows from the first equation of (9) that

$$
\begin{aligned}
z \phi(0)-\psi(0) & =\sum_{k=0}^{\infty} Q(k) \phi(-k) \\
& =\sum_{k=0}^{\infty} Q(k) z^{-k} \phi(0)+\sum_{k=0}^{\infty} Q(k)\left(\sum_{j=0}^{k-1} \frac{1}{z^{k-j}} \psi(-j-1)\right)
\end{aligned}
$$

so that

$$
\begin{equation*}
(z I-\tilde{Q}(z)) \phi(0)=\psi(0)+\sum_{k=0}^{\infty} Q(k)\left(\sum_{j=0}^{k-1} \frac{1}{z^{k-j}} \psi(-j-1)\right) \tag{11}
\end{equation*}
$$

By the invertibility of $z I-\tilde{Q}(z), \phi(0)$ is given by

$$
\begin{equation*}
\phi(0)=(z I-\tilde{Q}(z))^{-1}\left\{\psi(0)+\sum_{k=0}^{\infty} Q(k)\left(\sum_{j=0}^{k-1} \frac{1}{z^{k-j}} \psi(-j-1)\right)\right\} . \tag{12}
\end{equation*}
$$

Hence it follows from (10) that Eq. (8) has a unique solution, denoted $R_{z} \psi$, for any $\psi \in \mathscr{B}^{\gamma}$. We must show that $R_{z} \psi$ belongs to $\mathscr{B}^{\gamma}$. Because for $|z| e^{\gamma}>1$

$$
\begin{aligned}
\left|\sum_{k=0}^{\infty} Q(k)\left(\sum_{j=0}^{k-1} \frac{1}{z^{k-j}} \psi(-j-1)\right)\right| & \leq \sum_{k=0}^{\infty}\|Q(k)\| \sum_{j=0}^{k-1} \frac{1}{|z|^{k-j}}\|\psi\| e^{\gamma(j+1)} \\
& \leq \sum_{k=0}^{\infty}\|Q(k)\| e^{\gamma k} \sum_{j=0}^{k-1} \frac{e^{\gamma}}{\left(|z| e^{\gamma}\right)^{k-j}}\|\psi\| \\
& \leq \frac{K_{1} e^{\gamma}}{|z| e^{\gamma}-1}\|\psi\|
\end{aligned}
$$

we see from (12) that

$$
\left|\left(R_{z} \psi\right)(0)\right| \leq\left\|(z I-\tilde{Q}(z))^{-1}\right\|\left(1+\frac{K_{1} e^{\gamma}}{|z| e^{\gamma}-1}\right)\|\psi\|
$$

Therefore (10), together with (12), yields that for $k \in N$

$$
\begin{aligned}
\left|\left(R_{z} \psi\right)(-k)\right| e^{-\gamma k} & \leq \frac{1}{\left(|z| e^{\gamma}\right)^{k}}\left|\left(R_{z} \psi\right)(0)\right|+\sum_{j=0}^{k-1} \frac{e^{\gamma}}{\left(|z| e^{\gamma}\right)^{k-j}}\|\psi\| \\
& \leq\left\|(z I-\tilde{Q}(z))^{-1}\right\|\left(1+\frac{K_{1} e^{\gamma}}{|z| e^{\gamma}-1}\right)\|\psi\|+\frac{e^{\gamma}}{|z| e^{\gamma}-1}\|\psi\|
\end{aligned}
$$

consequently $R_{z} \psi \in \mathscr{B}^{\gamma}$, as required. In particular, we have

$$
\left\|R_{z} \psi\right\| \leq\left\|(z I-\tilde{Q}(z))^{-1}\right\|\left(1+\frac{K_{1} e^{\gamma}}{|z| e^{\gamma}-1}\right)\|\psi\|+\frac{e^{\gamma}}{|z| e^{\gamma}-1}\|\psi\|,
$$

and so the mapping $R_{z}: B^{\gamma} \mapsto B^{\gamma}$ is bounded.
By the definition of $R_{z}$ it is obvious that $(z I-V) R_{z}=R_{z}(z I-V)=I$. Thus, we conclude $z \in \rho(V)$.
(c) (iii) $\Rightarrow$ (ii). Suppose that $z \in \sigma(\tilde{Q}(z))$. Observe that $\tilde{Q}(z)$ is a compact operator. Then $z \in P_{\sigma}(\tilde{Q}(z))$ by the Riesz-Schauder theorem. So there exists
a nonzero $\phi(0) \in X$ with $\tilde{Q}(z) \phi(0)=0$. It is easy to see that $\phi \in \mathscr{B}^{\gamma}$, given by $\phi(-k):=z^{-k} \phi(0)$ for $k \in N$, satisfies $(z I-V) \phi=0$ and hence $z \in P_{\sigma}(V)$.

Now we recall that the essential spectrum $\sigma_{e}(V)$ of $V$ is defined as the set of $z \in \sigma(V)$, for which at least one of the following holds (see [9, §4.3], [1, § 1.4]):
(i) $\mathscr{R}(z I-V)$, the range of the operator $z I-V$, is not closed,
(ii) the point $z$ is the limit point of $\sigma(V)$,
(iii) the generalized eigenspace $\bigcup_{k \geq 1} \mathscr{N}\left((z I-V)^{k}\right)$ for $z$ is of infinite dimension,
where $\mathscr{N}\left((z I-V)^{k}\right)$ denotes the null space of the operator $(z I-V)^{k}$. Also the essential spectral radius $r_{e}(V)$ of $V$ is defined by

$$
r_{e}(V):=\sup \left\{|z|: z \in \sigma_{e}(V)\right\} .
$$

Then, we have the following estimate:
Lemma 2. The essential spectral radius $r_{e}(V)$ of $V$ satisfies $r_{e}(V) \leq e^{-\gamma}$.
Proof. Given $n \in \boldsymbol{N}$, we claim that $\alpha(V(n)) \leq e^{-\gamma n}$, where $\alpha(V(n))$ denotes the $\alpha$-measure (or Kuratowski measure of noncompactness) of the bounded linear operator $V(n)$ (see $[9, \S 4.3],[1, \S 1.2])$. Let $\Omega$ be a bounded set in $\mathscr{B}^{\nu}$. Then for arbitrary $\eta>0$ there exists a finite cover $\left\{O_{i}\right\}_{i=1}^{\ell}$ of $\Omega$ such that $d\left(O_{i}\right)<\alpha(\Omega)+\eta$ for $i=1, \ldots, \ell$, where $d\left(O_{i}\right)$ denotes the diameter of the set $O_{i}$. Since the coefficients $Q(k)$ are compact, the operator $R(s)$ in (5) is compact for $s=1,2, \ldots$. Therefore it follows from (6) that $\{[V(s) \phi](0): 1 \leq s \leq n$, $\phi \in \Omega\}$ is a relative compact set of $X$, and that $W:=\{(x(1 ; 0, \phi, 0), \ldots$, $x(n ; 0, \phi, 0)): \phi \in \Omega\}$ is relatively compact in $X^{n}$, the $n$-copies of $X$, endowed with the norm $|x|=\max _{1 \leq s \leq n}|x(s)|$ for $x=(x(1), \ldots, x(n)) \in X^{n}$. Hence there corresponds a finite cover $\left\{W_{j}\right\}_{j=1}^{m}$ of $W$ satisfying $d\left(W_{j}\right)<\eta, j=1, \ldots, m$. Then $\left\{O_{i, j}\right\}_{i, j}$, defined by $O_{i, j}:=\left\{\phi \in O_{i}:(x(1 ; 0, \phi, 0), \ldots, x(n ; 0, \phi, 0)) \in W_{j}\right\}$, gives a finite refinement of $\left\{O_{i}\right\}_{i}$, and for $\phi, \psi \in O_{i, j}$ we have

$$
\begin{aligned}
\|V(n) \phi-V(n) \psi\| & \leq \max _{1 \leq s \leq n}|x(s ; 0, \phi, 0)-x(s ; 0, \psi, 0)|+e^{-\gamma n}\|\phi-\psi\| \\
& \leq \eta+e^{-\gamma n}(\alpha(\Omega)+\eta) .
\end{aligned}
$$

Since $\eta$ is arbitrary, it follows that $\alpha(V(n) \Omega) \leq e^{-\gamma n} \alpha(\Omega)$, and therefore that $\alpha(V(n)) \leq e^{-\gamma n}$.

By Nussbaum's formula [1, § 1.4], we get

$$
r_{e}(V)=\lim _{n \rightarrow \infty} \sqrt[n]{\alpha\left(V^{n}\right)}=\lim _{n \rightarrow \infty} \sqrt[n]{\alpha(V(n))} \leq e^{-\gamma}
$$

as desired.

Let $\Lambda \subset \sigma(V)$ be a spectral set. Then, there exists a closed Jordan curve $C$ in $\rho(V)$ which contains $\Lambda$ in its interior and no points of $\Lambda^{\prime}:=\sigma(V) \backslash \Lambda$. Then, it is well known that the space $\mathscr{B}^{\gamma}$ is expressed as the direct sum of $V$ invariant closed subspaces: $\mathscr{B}^{\nu}=E^{4} \oplus E^{\Lambda^{\prime}}$ with $E^{\Lambda}=\Pi^{4}\left(\mathscr{B}^{\nu}\right)$ and $E^{\Lambda^{\prime}}=$ $\left(I-\Pi^{4}\right)\left(\mathscr{B}^{\nu}\right)$, where

$$
\Pi^{4}:=\frac{1}{2 \pi i} \int_{C}(z I-V)^{-1} d z
$$

is the projection from $\mathscr{B}^{\gamma}$ onto $E^{\Lambda}$, and that the relations $\sigma\left(\left.V\right|_{E^{4}}\right)=\Lambda$ and $\sigma\left(\left.V\right|_{E^{1^{\prime}}}\right)=\Lambda^{\prime}$ hold. Now consider subsets of $\Sigma$ as follows: $\quad \Sigma^{c}:=\Sigma \cap\{|z|=1\}$ and $\quad \Sigma^{u}:=\Sigma \cap\{|z|>1\}$. Then by Lemma $1 \quad \Sigma^{u}=\sigma(V) \cap\{|z|>1\} \quad$ and $\Sigma^{c}=\sigma(V) \cap\{|z|=1\}$. We also see from Lemma 2 that each element in $\Sigma^{u} \cup \Sigma^{c}$ does not belong to the essential spectrum $\sigma_{e}(V)$ of $V$; hence $\Sigma^{u} \cup \Sigma^{c}$ is finite, so that both $E^{\Sigma^{u}}$ and $E^{\Sigma^{c}}$ are of finite dimension. In particular, if we set $E^{u}=E^{\Sigma^{u}}$, $E^{c}=E^{\Sigma^{c}}$ and $E^{s}=E^{\Sigma^{\prime}}, \Sigma^{\prime}$ being $\sigma(V) \backslash\left(E^{c} \cup E^{u}\right), \mathscr{B}^{\gamma}$ can be written as the direct sum: $\quad \mathscr{B}^{\gamma}=E^{u} \oplus E^{c} \oplus E^{s}$. Put $V^{u}(n)=\left.V(n)\right|_{E^{u}}, V^{c}(n)=\left.V(n)\right|_{E^{c}}$ and $V^{s}(n)=\left.V(n)\right|_{E^{s}}$. Then it follows that $\sigma\left(V^{s}\right)=\Sigma^{u}, \sigma\left(V^{c}\right)=\Sigma^{c}$ and $\sigma\left(V^{s}\right)=\Sigma^{\prime}$, where we denote $V^{u}(1)$ by $V^{u}$, and similarly for $V^{c}$ and $V^{s}$.

Consequently, we obtain the following result.
Theorem 1. Let $V(n)$ be the solution operator of Eq. (2). Then there corresponds a decomposition of the phase space

$$
\mathscr{B}^{\nu}=E^{u} \oplus E^{c} \oplus E^{s}
$$

with the following properties:
(i) $\operatorname{dim}\left(E^{u} \oplus E^{c}\right)<\infty$,
(ii) $V(n) E^{u} \subset E^{u}, V(n) E^{c} \subset E^{c}$, and $V(n) E^{s} \subset E^{s}$,
(iii) $V^{s}(n)$ has the semigroup property in $\boldsymbol{Z}^{+}$, while $V^{u}(n)$ and $V^{c}(n)$ have in $\boldsymbol{Z}$,
(iv) there exist constants $K \geq 1$ and $\alpha>\varepsilon>0$ such that

$$
\begin{aligned}
\left\|V^{s}(n)\right\| \leq K e^{-\alpha n}, & n \in \boldsymbol{Z}^{+}, \\
\left\|V^{u}(-n)\right\| \leq K e^{-\alpha n}, & n \in \boldsymbol{Z}^{+} \\
\left\|V^{c}(n)\right\| \leq K e^{\varepsilon|n|}, & n \in \boldsymbol{Z}
\end{aligned}
$$

Proof. It remains to show (iii) and (iv). Because of $\sigma\left(V^{u}\right)=\Sigma^{u}$, we see $\left(V^{u}\right)^{-1} \in \mathscr{L}\left(E^{u}\right)$. If we define $V^{u}(-n)=\left(V^{u}\right)^{-n}$ for $n \in \boldsymbol{N}, V^{u}(n)$ satisfies the semigroup property on $n \in \boldsymbol{Z}$, and likewise for $V^{c}(n)$. This shows (iii).

Since $\sigma\left(V^{s}\right)=\Sigma^{\prime}$, it follows from Lemma 2 that the spectral radius of $V^{s}$ is less than one, so that there exist $K \geq 1$ and $\alpha>0$ such that $\left\|V^{s}(n)\right\| \leq$ $K e^{-\alpha n}, n \in \boldsymbol{Z}^{+}$, which proves the first inequality of (iv). Also, noting that
$r_{\sigma}\left(V^{u}(-1)\right)=r_{\sigma}\left(\left(V^{u}\right)^{-1}\right)<1$, we obtain the second inequality of (iv). The third one follows from the fact that $r_{\sigma}\left(V^{c}(-1)\right)=r_{\sigma}\left(\left(V^{c}\right)^{-1}\right)=1=r_{\sigma}\left(V^{c}(1)\right)$.

## 4. Decomposition of variation of constants formula

Corresponding to the decomposition of the phase space $\mathscr{B}^{\gamma}$ in the previous section, one can naturally decompose the variation of constants formula in Proposition 1 into two parts. Consider the equations

$$
\begin{gather*}
\xi(n)=V^{s}(n-\sigma) \xi(\sigma)+\sum_{k=\sigma}^{n-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} P(k)), \quad n \geq \sigma,  \tag{13}\\
\eta(n)=V^{u c}(n-\sigma) \eta(\sigma)+\sum_{k=\sigma}^{n-1} V^{u c}(n-k-1) \Pi^{u c}(\mathscr{E} p(k)), \quad n \geq \sigma, \tag{14}
\end{gather*}
$$

where $V^{u c}(n)$ denotes the restriction of $V(n)$ to the space $E^{u c}:=E^{u} \oplus E^{c}$. The equations (13) and (14) are called the stable part of VCF and the unstable part of VCF, respectively. Then we get:

Theorem 2. For the solution $x(n ; \sigma, \phi, p)$ of Eq. (1) the $E^{s}$-component $\Pi^{s} x_{n}(\sigma, \phi, p)$ and the $E^{u c}$-component $\Pi^{u c} x_{n}(\sigma, \phi, p)$ of $x_{n}(\sigma, \phi, p)$ satisfy the stable part and the unstable part of VCF, respectively.

Conversely, suppose that functions $\xi(n)$ and $\eta(n)$ on $\boldsymbol{Z}$ with $\xi(n) \in E^{s}$ and $\eta(n) \in E^{u c}$ satisfy the stable part and the unstable part of VCF, respectively, for $n \geq \sigma$. Then the function $x(n)$ defined by $x(n)=[\xi(n)+\eta(n)](0), n \in \boldsymbol{Z}$, is a solution of Eq. (1).

Proof. The former part is obvious from Proposition 1 and Theorem 1 (iii). We will verify the latter part. Let $\xi(n)$ and $\eta(n), n \in \boldsymbol{Z}$, be solutions of Eq. (13), respectively, and define $\tilde{x}(n)=\xi(n)+\eta(n)$ for $n \in \boldsymbol{Z}$. It then follows that

$$
\begin{aligned}
V(n & -\sigma) \tilde{x}(\sigma)+\sum_{k=\sigma}^{n-1} V(n-k-1)(\mathscr{E}(p(k)) \\
& =V^{s}(n-\sigma) \xi(\sigma)+V^{u c}(n-\sigma) \eta(\sigma)+\sum_{k=\sigma}^{n-1} V(n-k-1)(\mathscr{E}(p(k)) \\
& =\left\{\xi(n)-\sum_{k=\sigma}^{n-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} p(k))\right\}
\end{aligned}
$$

$$
\begin{aligned}
& +\left\{\eta(n)-\sum_{k=\sigma}^{n-1} V^{u c}(n-k-1) \Pi^{u c}(\mathscr{E} p(k))\right\}+\sum_{k=\sigma}^{n-1} V(n-k-1)(\mathscr{E}(p(k)) \\
= & \xi(n)+\eta(n)=\tilde{x}(n),
\end{aligned}
$$

which, together with Proposition 1, implies that $\tilde{x}(n)=x_{n}(\sigma, \tilde{x}(\sigma), p)$. Hence, setting $x(n):=[\xi(n)+\eta(n)](0)$, we have

$$
x(n)=[\tilde{x}(n)](0)=x(n ; \sigma, \tilde{x}(\sigma), p)
$$

for any $n$ and $\sigma$ with $n \geq \sigma$. We claim that $x_{n}=\tilde{x}(n)$ for every $n \in \boldsymbol{Z}$. Indeed, given $m \in \boldsymbol{N}$, we see that for $-m \leq \theta \leq 0$,

$$
\begin{align*}
x(n+\theta) & =x(n+\theta ; n-m, \tilde{x}(n-m), p)  \tag{15}\\
& =x_{n}(\theta ; n-m, \tilde{x}(n-m), p) \\
& =\left[x_{n}(n-m, \tilde{x}(n-m), p)\right](\theta) \\
& =[\tilde{x}(n)](\theta)
\end{align*}
$$

Since $m$ is arbitrary, (15) holds for all $\theta \in \boldsymbol{Z}^{-}$, and so $x_{n}=\tilde{x}(n)$ for every $n \in \boldsymbol{Z}$. This yields $x(n)=x\left(n ; \sigma, x_{\sigma}, p\right)$, which implies that $x(n)$ is a solution of Eq. (1).

In what follows we shall show that the unstable part of VCF is reduced to a certain type of first order difference equation in a finite dimensional space.

Let $d$ denote the dimension of the subspace $E^{u c}$, which is known to be finite (Theorem 1), and $\left\{\phi_{1}, \ldots, \phi_{d}\right\}$ be a basis of $E^{u c}$. Also denote its dual basis by $\left\{\psi_{1}, \ldots, \psi_{d}\right\}$. Each $\psi_{i}$ may be considered as an element in $\left(\mathscr{B}^{\gamma}\right)^{*}$, the dual space of $\mathscr{B}^{\gamma}$, by the zero extension, i.e., $\left.\psi_{i}\right|_{E^{s}}=0$ for $i=1, \ldots, d$. Now set

$$
\Phi=\left(\phi_{1}, \ldots, \phi_{d}\right), \quad \Psi=\left(\begin{array}{c}
\psi_{1} \\
\vdots \\
\psi_{d}
\end{array}\right) .
$$

$\Phi$ and $\Psi$ are called a basis vector of $E^{u c}$ and the dual vector associated with $\Phi$, respectively. We use the notation $\langle$,$\rangle for the pairing between the dual space$ and the original one, and also write by $\langle\Psi, \Phi\rangle$ the matrix $\left(\left\langle\psi_{i}, \phi_{j}\right\rangle\right)_{i, j}$, which is the identity matrix of degree $d$. In this setting the projection $\Pi^{u c}:=\Pi^{u}+\Pi^{c}$ onto $E^{u c}$ is expressed in terms of $\Phi$ and $\Psi$ as

$$
\Pi^{u s} \phi=\Phi\langle\Psi, \phi\rangle, \quad \phi \in \mathscr{B}^{\gamma} .
$$

Recall that $V$ induces a linear transform of $E^{u c}$, denoted $V^{u c}$, and let $B$ be its representation matrix with respect to the basis $\Phi: V^{u c} \Phi=\Phi B$.

Let $x(n)$ be a solution of Eq. (1) and $z(n) \in \boldsymbol{C}^{d}$ the coordinate of $\Pi^{u c} x_{n}$ with respect to the basis vector $\Phi$, i.e., $\Pi^{u c} x_{n}=\Phi z(n)$. Then:

Theorem 3. If $x(n)$ is a solution of Eq. (1), then its coordinate $z(n)$ satisfies the difference equation of first order in $\boldsymbol{C}^{d}$ :

$$
\begin{equation*}
z(n+1)=B z(n)+\langle\Psi, \mathscr{E} p(k)\rangle \tag{16}
\end{equation*}
$$

Conversely, given any solution $z(n)$ of $(16)$ the sequence $x_{*}(n)$ defined by

$$
x_{*}(n)=\left[\Phi z(n)+\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} p(k))\right](0)
$$

is a solution of Eq. (1). Furthermore, given any solution $x(n)$ of Eq. (1), the solution $x_{*}(n)$ with $z(n)$ satisfying $\Phi z(n)=\Pi^{u c} x_{n}$ satsifies

$$
\left|x(n)-x_{*}(n)\right| \leq K\left(x_{\sigma}, p\right) e^{-\alpha(n-\sigma)}, \quad n \geq \sigma
$$

where $K\left(x_{\sigma}, p\right)$ is a constant depending on $\left\|x_{\sigma}\right\|$ and $\sup _{n \in \boldsymbol{Z}}|p(n)|$.
Proof. Let $x(n)$ be a solution of Eq. (1) and $z(n)$ the coordinate of its unstable component. Then we see

$$
\begin{aligned}
z(n) & =\langle\Psi, \Phi\rangle z(n)=\langle\Psi, \Phi z(n)\rangle \\
& =\left\langle\Psi, \Pi^{u c} x_{n}\right\rangle \\
& =\left\langle\Psi, \Pi^{u c}\left(V(n-\sigma) x_{\sigma}+\sum_{k=\sigma}^{n-1} V(n-k-1)(\mathscr{E} p(k))\right)\right\rangle \\
& =\left\langle\Psi, V(n-\sigma) \Pi^{u c} x_{\sigma}+\sum_{k=\sigma}^{n-1} V(n-k-1) \Pi^{u c}(\mathscr{E} p(k))\right\rangle \\
& =\langle\Psi, V(n-\sigma) \Phi z(\sigma)\rangle+\sum_{k=\sigma}^{n-1}\langle\Psi, V(n-k-1) \Phi\langle\Psi, \mathscr{E} p(k)\rangle\rangle \\
& =\left\langle\Psi, \Phi B^{n-\sigma} z(\sigma)\right\rangle+\sum_{k=\sigma}^{n-1}\left\langle\Psi, \Phi B^{n-k-1}\right\rangle\langle\Psi, \mathscr{E} p(k)\rangle \\
& =B^{n-\sigma} z(\sigma)+\sum_{k=\sigma}^{n-1} B^{n-k-1}\langle\Psi, \mathscr{E} p(k)\rangle
\end{aligned}
$$

Thus, we find that $z(n)$ satisfies Eq. (16).

Conversely, let $z(n)$ be a solution of Eq. (16). Then it follows that

$$
z(n)=B^{n-\sigma} z(\sigma)+\sum_{k=\sigma}^{n-1} B^{n-k-1}\langle\Psi, \mathscr{E} p(k)\rangle,
$$

so that

$$
\Phi z(n)=V^{u c}(n-\sigma) \Phi z(\sigma)+\sum_{k=\sigma}^{n-1} V^{u c}(n-k-1) \Pi^{u c}(\mathscr{E} p(k)) .
$$

Hence $\Phi z(n)$ is a solution of the unstable part of VCF. On the other hand, since $p(n)$ is bounded, we get

$$
\begin{align*}
\sum_{k=-\infty}^{n-1}\left\|V^{s}(n-k-1)(\mathscr{E} p(k))\right\| & \leq \sum_{k=-\infty}^{n-1} K e^{-\alpha(n-k-1)}\left\|\Pi^{s} \mathscr{E}\right\|\left(\sup _{\tau \in \boldsymbol{Z}}\|p(\tau)\|\right)  \tag{17}\\
& \leq \frac{K}{1-e^{-\alpha}}\left\|\Pi^{s}\right\|\left(\sup _{\tau \in \boldsymbol{Z}}\|p(\tau)\|\right)<\infty
\end{align*}
$$

Furthermore, $\xi(n):=\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} p(k))$ satisfies the stable part of VCF. Therefore by Theorem 2 we deduce that $x_{*}(n)=[\Phi z(n)+\xi(n)](0)$ is a solution of Eq. (1).

Finally, let $x(n)$ be a solution of Eq. (1) and $x_{*}(n)$ the one given in the theorem. Then by Proposition 1 we see

$$
\begin{aligned}
x_{n}-\left(x_{*}\right)_{n}= & x_{n}-\Phi z(n)-\xi(n) \\
= & V(n-\sigma)\left(x_{\sigma}-\Phi z(\sigma)\right)+\sum_{k=\sigma}^{n-1} V(n-k-1)(\mathscr{E} p(k)) \\
& -\sum_{k=\sigma}^{n-1} V^{u c}(n-k-1) \Pi^{u c}(\mathscr{E} p(k))-\sum_{k=-\infty}^{n-1} V^{s}(n-k-1)(\mathscr{E} p(k)) \\
= & V^{s}(n-\sigma) \Pi^{s} x_{\sigma}-\sum_{k=-\infty}^{\sigma-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} p(k)) .
\end{aligned}
$$

Since $\left\|V^{s}(n-\sigma)\left(\Pi^{s} x_{\sigma}\right)\right\| \leq K e^{-\alpha(n-\sigma)}\left\|\Pi^{s}\right\|\left\|x_{\sigma}\right\|$ and

$$
\sum_{k=-\infty}^{\sigma-1}\left\|V^{s}(n-k-1)(\mathscr{E} p(k))\right\| \leq \frac{K\left\|\Pi^{s}\right\|\|p\|_{Z}}{1-e^{-\alpha}} e^{-\alpha(n-\sigma)},
$$

we get
$\left|x(n)-x_{*}(n)\right| \leq\left\|x_{n}-x_{* n}\right\| \leq \frac{K\left\|\Pi^{s}\right\|}{1-e^{-\alpha}}\left(\left\|x_{\sigma}\right\|+\sup _{n \in \boldsymbol{Z}}|p(n)|\right) e^{-\alpha(n-\sigma)}, \quad$ for $n \geq \sigma$,
and the proof is completed.
Theorem 3 asserts that the study of asymptotic behaviors of solutions for Eq. (1) may be reduced to that for a finite difference equation.

## 5. Boundedness and almost periodicity of solutions

Based on the decomposition of VCF, in this section, we will establish some results on the existence of almost periodic solutions for Eq. (1) in connection with the existence of bounded solutions. In particular, in the case where the zero solution of Eq. (2) is hyperbolic, we prove unique existence of almost periodic solutions and give its explicit representation.

We first recall that a function $p$ from $\boldsymbol{Z}$ into a Banach space $Y$ with norm $\|\cdot\|$ is called almost periodic, if for arbitrary $\varepsilon>0$ there exists an $\ell=\ell(\varepsilon) \in N$ with the property that for any $a \in \boldsymbol{Z}$ there is a $\tau \in \boldsymbol{Z}$ satisfying $a \leq \tau<a+\ell$ and

$$
\|p(n+\tau)-p(n)\|<\varepsilon \quad \text { for } n \in \boldsymbol{Z}
$$

If $p: \boldsymbol{Z} \mapsto Y$ is almost periodic, then the limit

$$
a(\lambda ; p)=\lim _{N \rightarrow \infty} \frac{1}{2 N} \sum_{k=-N}^{N} p(k) \lambda^{-k}
$$

exists whenever $\lambda$ belongs to the unit circle $\{|z|=1\}$. The limit $a(\lambda ; p)$ is called the Bohr transform of $p$, and the set $\sigma_{b}(p)$, which consists of all numbers $\lambda \in\{|z|=1\}$ with $a(\lambda ; p) \neq 0$, is called the Bohr spectrum of $p$.

Theorem 4. Suppose that $p(n)$ is a bounded function. Then
(i) the function $\xi(n)=\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} p(k))$ satisfies the stable part of VCF for all $n$ and $\sigma$ with $n \geq \sigma$. Moreover, if $\xi^{\prime}(n)$ is bounded on $\boldsymbol{Z}$ and satisfies the stable part of VCF for all $n$ and $\sigma$ with $n \geq \sigma$, then $\xi(n) \equiv \xi^{\prime}(n)$ for $n \in \boldsymbol{Z}$.
(ii) In addition, if $p(n)$ is almost periodic, then $\xi(n)$ is an almost periodic solution of the stable part of VCF such that $\sigma_{b}(\xi) \subset \sigma_{b}(p)$.

Proof. (i) The former part has already been shown in the proof of Theorem 3. We now prove the latter part. Let $\xi^{\prime}(n)$ be another bounded solution of (13) on $\boldsymbol{Z}$. Since $\xi(n)-\xi^{\prime}(n)=V^{s}(n-\sigma)\left(\xi(\sigma)-\xi^{\prime}(\sigma)\right)$, we see

$$
\left\|\xi(n)-\xi^{\prime}(n)\right\| \leq K e^{-\alpha(n-\sigma)}\left\{\sup _{\tau \in \mathbb{Z}}\|\xi(n)\|+\sup _{\tau \in Z}\left\|\xi^{\prime}(n)\right\|\right\} \rightarrow 0
$$

as $\sigma \rightarrow-\infty$, and hence $\xi(n) \equiv \xi^{\prime}(n)$.
(ii) Let $p(n)$ be almost periodic. We first claim that $\xi(n)$ is almost periodic. Since $p(n)$ is an almost periodic function, given $\varepsilon>0$, there exists a trigonometric polynomial $p^{\varepsilon}(n)$ such that

$$
\sup _{n \in \boldsymbol{Z}}\left|p(n)-p^{\varepsilon}(n)\right|<\frac{1-e^{-\alpha}}{K\left\|\Pi^{s} \mathscr{E}\right\|} \varepsilon
$$

where $p^{\varepsilon}(n)$ is of the form

$$
p^{\varepsilon}(n)=\sum_{j=1}^{m} x_{j} \mu_{j}^{n}, \quad x_{j} \in X, \mu_{j} \in \sigma_{b}(p) .
$$

Put

$$
\xi^{\varepsilon}(n):=\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}\left(\mathscr{E} \mathscr{E}^{\varepsilon}(k)\right)=\sum_{j=1}^{m} \sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}\left(\mathscr{E} x_{j}\right) \mu_{j}^{k} .
$$

Then by a similar estimate to (17) we have

$$
\begin{equation*}
\left\|\xi(n)-\xi^{\varepsilon}(n)\right\| \leq \frac{K}{1-e^{-\alpha}}\left\|\Pi^{s} \mathscr{E}\right\|\left(\sup _{\tau \in \boldsymbol{Z}}\left|p(\tau)-p^{\varepsilon}(\tau)\right|\right)<\varepsilon \tag{18}
\end{equation*}
$$

Now set $\Theta(n):=\left(\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}\left(\mathscr{E} x_{j}\right) \mu_{j}^{k}\right) \mu_{j}^{-n}$. Then it follows that

$$
\begin{aligned}
\Theta(n+1) & =\left(\sum_{k=-\infty}^{n} V^{s}(n-k) \Pi^{s}\left(\mathscr{E} x_{j}\right) \mu_{j}^{k}\right) \mu_{j}^{-n-1} \\
& =\left(\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}\left(\mathscr{E} x_{j}\right) \mu_{j}^{k}\right) \mu_{j}^{-n}=\Theta(n) \quad \text { for } n \in \boldsymbol{Z}
\end{aligned}
$$

so that $\Theta(n)$ is independent of $n$; in particular

$$
\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}\left(\mathscr{E} x_{j}\right) \mu_{j}^{k}=\Theta(0) \mu_{j}^{n}
$$

which is almost periodic. This implies that $\xi^{\varepsilon}(n)$ is almost periodic, and hence so is $\xi(n)$ by (18).

We next verify the relation $\sigma_{b}(\xi) \subset \sigma_{b}(p)$. Since

$$
\sigma_{b}\left(\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}\left(\mathscr{E} x_{j}\right) \mu_{j}^{k}\right)=\sigma_{b}\left(\left\{\Theta(0) \mu_{j}^{n}\right\}_{n}\right)=\left\{\mu_{j}\right\} \subset \sigma_{b}(p),
$$

we see that $\sigma_{b}\left(\xi^{\varepsilon}\right) \subset \sigma_{b}(p)$. Assume that $\sigma_{b}(\xi) \subset \sigma_{b}(p)$ does not hold. Then there exist $\lambda \in \sigma_{b}(\xi) \backslash \sigma_{b}(p)$ and an $N_{0} \in \boldsymbol{N}$ such that

$$
\left\|\frac{1}{2 N} \sum_{k=-N}^{N} \xi(k) \lambda^{-k}\right\| \geq 2 \varepsilon_{0} \quad \text { for } N \geq N_{0}
$$

where $\varepsilon_{0}$ is some positive number. Then, by (18) we get

$$
\begin{aligned}
\left\|\frac{1}{2 N} \sum_{k=-N}^{N} \xi^{\varepsilon_{0}}(k) \lambda^{-k}\right\| & =\left\|\frac{1}{2 N} \sum_{k=-N}^{N}\left(\xi^{\varepsilon_{0}}(k)-\xi(k)\right) \lambda^{-k}+\frac{1}{2 N} \sum_{k=-N}^{N} \xi(k) \lambda^{-k}\right\| \\
& \geq\left\|\frac{1}{2 N} \sum_{k=-N}^{N} \xi(k) \lambda^{-k}\right\|-\frac{2 N+1}{2 N} \varepsilon_{0} \\
& \geq \frac{\varepsilon_{0}}{2} \quad \text { for } N \geq N_{0}
\end{aligned}
$$

which implies that $\lambda \in \sigma_{b}\left(\xi^{\varepsilon_{0}}\right) \subset \sigma_{b}(p)$. This is a contradiction.
Theorem 5. Suppose that $p(n)$ is an almost periodic function, and that Eq. (16) has a bounded solution on $\boldsymbol{Z}^{+}$. Then there exists an almost periodic solution $z(n)$ of $E q$. (16) such that $\sigma_{b}(z) \subset \sigma_{b}(p)$.

Proof. The proof will be devided into three steps:
Step 1. We will first show that Eq. (16) has a bounded solution on the whole $\boldsymbol{Z}$. Let $\left\{n_{k}\right\} \subset \boldsymbol{N}$ be any sequence with $n_{k} \rightarrow \infty$ as $k \rightarrow \infty$. Since $p(n)$ is almost periodic, taking a subsequence if necessary, we may assume that $p\left(n+n_{k}\right)$ converges to some almost periodic function $q(n)$ uniformly on $\boldsymbol{Z}$ as $k \rightarrow \infty$. Let us denote by $u(n)$ a bounded solution of Eq. (16) on $\boldsymbol{Z}^{+}$and set $u^{k}(n):=u\left(n+n_{k}\right)$ for $n \geq-n_{k}$. Then we have

$$
\begin{equation*}
u^{k}(n+1)=B u^{k}(n)+\left\langle\Phi, \mathscr{E} p\left(n+n_{k}\right)\right\rangle \quad \text { for } n \geq-n_{k} . \tag{19}
\end{equation*}
$$

Because of the boundedness of $u(n)$, by the diagonalization we may assume that $u^{k}(n)$ converges to some bounded function $u^{*}(n)$ uniformly on any finite subset of $\boldsymbol{Z}$. Letting $k \rightarrow \infty$ in (19), we get

$$
\begin{equation*}
u^{*}(n+1)=B u^{*}(n)+\langle\Phi, \mathscr{E} q(n)\rangle \quad \text { for } n \in \boldsymbol{Z} \tag{20}
\end{equation*}
$$

Applying the same argument this time to the sequence $v^{k}(n):=u^{*}\left(n-n_{k}\right)$, we may assume that $v^{k}(n)$ converges to some bounded function $v(n)$ uniformly
on any finite subset of $\boldsymbol{Z}$. Noting that $q\left(n-n_{k}\right)$ converges to $p(n)$ uniformly on $\boldsymbol{Z}$ as $k \rightarrow \infty$, we find that $v(n)$ is a bounded solution of Eq. (16) on $\boldsymbol{Z}$.

Step 2. We claim that the existence of bounded solutions of Eq. (16) on $\boldsymbol{Z}$ implies that of almost periodic solutions. Let $\mathscr{S}$ be the set of all bounded solutions of Eq. (16) on $\boldsymbol{Z}$. $\mathscr{S}$ is not empty by Step 1. Now put

$$
\rho_{0}:=\inf _{z \in \mathscr{S}}\|z\|,
$$

where $\|z\|=\sup _{n \in \boldsymbol{Z}}|z(n)|$ and $|\cdot|$ denotes the Euclidean norm in $\boldsymbol{C}^{d}$. Then we will verify that $\rho_{0}=\|z\|$ holds for some $z \in \mathscr{S}$. Indeed, we can choose a sequence $\left\{z^{k}\right\}_{k \in N}$ in $\mathscr{S}$ such that $\rho_{0} \leq\left\|z^{k}\right\| \leq \rho_{0}+1 / k$ for $k \in N$. Since $\left\{z^{k}(n)\right\}_{k}$ is uniformly bounded on $\boldsymbol{Z}$, we may assume that $\left\{z^{k}(n)\right\}_{k}$ converges to some bounded function $\hat{z}(n)$ uniformly on any finite set of $\boldsymbol{Z}$ as $k \rightarrow \infty$. In the same way as in Step 1, we find that $\hat{z} \in \mathscr{S}$. Letting $k \rightarrow \infty$ in the inequality $\left|z^{k}(n)\right| \leq\left\|z^{k}\right\| \leq \rho_{0}+1 / k$, we have $|\hat{z}(n)| \leq \rho_{0}$ for $n \in \boldsymbol{Z}$, so that $\|\hat{z}\|=\rho_{0}$.

Our claim is proved if we show that $\hat{z}(n)$ is almost periodic. For this it is sufficient to verify that given any sequence $\left\{n_{k}\right\} \subset \boldsymbol{Z},\left\{\hat{z}\left(n+n_{k}\right)\right\}$ contains a subsequence which converges uniformly on $\boldsymbol{Z}$ (see [5]). We will show this by a contradiction. So assume that $\left\{\hat{z}\left(n+n_{k}\right)\right\}$ contains no subsequences which converge uniformly on $\boldsymbol{Z}$. Then there exist a $\delta>0$ and sequences $\left\{n_{k}^{(1)}\right\},\left\{n_{k}^{(2)}\right\} \subset$ $\left\{n_{k}\right\}$ and $\left\{m_{k}\right\} \subset \boldsymbol{Z}$ such that

$$
\begin{equation*}
\left|\hat{z}\left(m_{k}+n_{k}^{(1)}\right)-\hat{z}\left(m_{k}+n_{k}^{(2)}\right)\right| \geq \delta, \quad \text { for } k \in \boldsymbol{N} . \tag{21}
\end{equation*}
$$

Since $\left\{\hat{z}\left(n+m_{k}+n_{k}^{(j)}\right)\right\}$ is uniformly bounded in $\boldsymbol{Z}$, one can assume that $\lim _{k \rightarrow \infty} \hat{z}\left(n+m_{k}+n_{k}^{(j)}\right)=\hat{z}^{(j)}(n)$ uniformly on any finite subset in $\boldsymbol{Z}$ for some bounded functions $\hat{\boldsymbol{z}}^{(j)}(n)$ on $\boldsymbol{Z}$. Also since we may assume that $\lim _{k \rightarrow \infty} p\left(n+n_{k}\right)=q(n)$ and $\lim _{k \rightarrow \infty} q\left(n+m_{k}\right)=r(n)$ uniformly on $\boldsymbol{Z}$ for some almost periodic functions $q$ and $r$, we deduce that $\lim _{k \rightarrow \infty} p\left(n+m_{k}+n_{k}\right)=r(n)$ uniformly on $\boldsymbol{Z}$. By the same reasoning as in Step 1, we see that both $\hat{\boldsymbol{z}}^{(1)}(n)$ and $\hat{z}^{(2)}(n)$ satisfy

$$
\begin{equation*}
z(n+1)=B z(n)+\langle\Phi, \mathscr{E} r(n)\rangle \tag{22}
\end{equation*}
$$

for $n \in \boldsymbol{Z}$. Now set

$$
z^{+}(n):=\frac{\hat{z}^{(1)}(n)+\hat{z}^{(2)}(n)}{2}, \quad z^{-}(n):=\frac{\hat{z}^{(1)}(n)-\hat{z}^{(2)}(n)}{2} .
$$

Then, $z^{+}(n)$ satisfies Eq. (22) on $\boldsymbol{Z}$, while $z^{-}(n)$ is a bounded solution of the equation $z(n+1)=B z(n)$ on $\boldsymbol{Z}$. In particular, $z^{-}(n)$ must be almost periodic. Therefore we have $\inf _{n \in \boldsymbol{Z}}\left|z^{-}(n)\right|>0$ since $\left|z^{-}(0)\right|>0$ by (21). Noting $\left\|z^{(1)}\right\|=$ $\left\|z^{(2)}\right\|=\rho_{0}$, we get

$$
\left|z^{+}(n)\right|^{2}+\left|z^{-}(n)\right|^{2}=\frac{1}{2}\left(\left|z^{(1)}(n)\right|^{2}+\left|z^{(2)}(n)\right|^{2}\right) \leq \rho_{0}^{2}
$$

so that $\left\|z^{+}\right\|<\rho_{0}$. By the same argument as the last paragraph in Step 1, we may assume that $\lim _{k \rightarrow \infty} z^{+}\left(n-m_{k}-n_{k}\right)=z^{*}(n)$ for some $z^{*} \in \mathscr{S}$. Thus we arrive at $\left\|z^{*}\right\|=\left\|z^{+}\right\|<\rho_{0}=\|\hat{z}\|$, which contradicts to the definition of $\rho_{0}$.

Step 3. Once the existence of an almost periodic solution of Eq. (16) is guaranteed, one can obtain a solution with the desired property as follows: Let $\hat{z}(n)$ be the one in Step 2. We claim that

$$
\begin{equation*}
(B-\lambda) a(\lambda ; \hat{z})=0, \quad \text { for } \lambda \notin \sigma_{b}(p) . \tag{23}
\end{equation*}
$$

Indeed,

$$
\begin{aligned}
(B-\lambda) a(\lambda ; \hat{z})= & \lim _{N \rightarrow \infty} \frac{1}{2 N} \sum_{k=-N}^{N}(B-\lambda) \hat{z}(k) \lambda^{-k} \\
= & \lim _{N \rightarrow \infty} \frac{1}{2 N} \sum_{k=-N}^{N}\{\hat{z}(k+1)-\langle\Psi, \mathscr{E} p(k)\rangle-\lambda \hat{z}(k)\} \lambda^{-k} \\
= & \lim _{N \rightarrow \infty} \frac{1}{2 N}\left(\hat{z}(N+1) \lambda^{-N}-\hat{z}(-N) \lambda^{N+1}\right) \\
& -\lim _{N \rightarrow \infty}\left\langle\Psi, \mathscr{E}\left(\frac{1}{2 N} \sum_{k=-N}^{N} p(k) \lambda^{-k}\right)\right\rangle \\
= & -\langle\Psi, \mathscr{E} a(\lambda ; p)\rangle,
\end{aligned}
$$

so that we have $(B-\lambda) a(\lambda ; \hat{z})=0$ for $\lambda \notin \sigma_{b}(p)$, as required. By virtue of (23) we see that $a(\lambda ; \hat{z})=0$ for $\lambda \notin \sigma(B) \cup \sigma_{b}(p), \sigma(B)$ being the set of eigenvalues of the matrix $B$. Now let $w(n):=\sum_{\lambda \in \sigma(B) \backslash \sigma_{b}(p)} a(\lambda ; \hat{z}) \lambda^{n}$ and consider the function $z(n)=\hat{z}(n)-w(n)$. We notice from (23) that

$$
w(n+1)=\sum_{\lambda \in \sigma(B) \backslash \sigma_{b}(p)} \lambda a(\lambda ; \hat{z}) \lambda^{n}=\sum_{\lambda \in \sigma(B) \backslash \sigma_{b}(p)} B a(\lambda ; \hat{z}) \lambda^{n}=B w(n),
$$

and hence $z(n)$ is a solution of (16). Moreover, by its definition $z(n)$ satisfies $\sigma_{b}(z) \subset \sigma_{b}(p)$. Consequently, the solution $z(n)$, so obtained, is a desired one.

In view of Theorem 3 through Theorem 5, we get the following theorem.
Theorem 6. Let $p(n)$ be an almost periodic function. If Eq. (1) has a bounded solution on $\boldsymbol{Z}^{+}$, it has an almost periodic solution $x(n)$ such that $\sigma_{b}(x) \subset \sigma_{b}(p)$.

Proof. Let $y(n)$ be a bounded solution of Eq. (1) on $\boldsymbol{Z}^{+}$. Since $y_{n}$ is bounded in $\mathscr{B}^{\gamma}$, it follows from Theorems 3 and 5 that there exists an almost periodic solution $z(n)$ of $(16)$ with $\sigma_{b}(z) \subset \sigma_{b}(p)$. Notice that $\Phi_{z}(n)$ is a $\mathscr{B}^{\gamma}$ valued almost periodic function such that $\sigma_{b}(\Phi z)=\sigma_{b}(z) \subset \sigma_{b}(p)$. Let $\xi(n)$ be the one defined in the proof of Theorem 4. Then, it follows from Theorems 3 and 4 that $x(n):=[\Phi z(n)+\xi(n)](0)$ is a desired solution.

When it is the case where the zero soluiton of Eq. (2) is hyperbolic, that is, no characteristic roots of Eq. (2) belong to the unit circle $\{|z|=1\}$, the situation is quite simple. In fact, we can ensure the unique existence of bounded solutions of Eq. (1).

Theorem 7. Suppose that the characteritic operator $z I-\tilde{Q}(z)$ of $E q$. (2) is invertible in $\mathscr{L}(X)$ for $|z|=1$. Then we have:
(i) for any bounded function $p: \boldsymbol{Z} \mapsto X$, Eq. (1) has a unique bounded solution on $\boldsymbol{Z}$, which is represented by the formula

$$
\begin{equation*}
x(n)=\left[\sum_{k=-\infty}^{n-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} p(k))-\sum_{k=n}^{\infty} V^{u}(n-k-1) \Pi^{s}(\mathscr{E} p(k))\right](0), \tag{24}
\end{equation*}
$$

(ii) if $p(n)$ is periodic, then the solution $x(n)$ is also periodic.

Proof. (i) Uniqueness. We first show the uniqueness of bounded solutions on $\boldsymbol{Z}$. Let $x(n)$ be a bounded solution of Eq. (1). Then $\xi(n)=\Pi^{s} x_{n}$ is a solution of the stable part of VCF:

$$
\begin{equation*}
\xi(n)=V^{s}(n-\sigma) \xi(\sigma)+\sum_{k=\sigma}^{n-1} V^{s}(n-k-1)(\mathscr{E} p(k)), \quad n \geq \sigma . \tag{25}
\end{equation*}
$$

Since $x(n)$ is bounded, $\xi(n)$ is also bounded on $\boldsymbol{Z}$; and therefore by Theorem 4(i) we get

$$
\xi(n)=\sum_{k=-\infty}^{n-1} V^{s}(n-k-1)(\mathscr{E} p(k)), \quad n \in \boldsymbol{Z}
$$

By our assumption it follows that $E^{c}=\{0\}$, so that $\eta(n)=\Pi^{u} x(n)$ satisfies the unstable part of VCF:

$$
\eta(n)=V^{u}(n-\sigma) \eta(\sigma)+\sum_{k=\sigma}^{n-1} V^{u}(n-k-1) \Pi^{u}(\mathscr{E} p(k)), \quad n \geq \sigma .
$$

In view of Theorem 1 (iii), we see

$$
\begin{align*}
\eta(\sigma) & =V^{u}(\sigma-n) V^{u}(n-\sigma) \eta(\sigma)  \tag{26}\\
& =V^{u}(\sigma-n)\left\{\eta(n)-\sum_{k=\sigma}^{n-1} V^{u}(n-k-1) \Pi^{u}(\mathscr{E} p(k))\right\} \\
& =V^{u}(\sigma-n) \eta(n)-\sum_{k=\sigma}^{n-1} V^{u}(\sigma-k-1) \Pi^{u}(\mathscr{E} p(k))
\end{align*}
$$

By the same reasoning as the argument for $\xi(n)$, we deduce that

$$
\left\|V^{u}(\sigma-n) \eta(n)\right\| \leq K e^{-\alpha(n-\sigma)}\left\|\Pi^{u}\right\|\left(\sup _{\sigma \in \boldsymbol{Z}}\left\|x_{\sigma}\right\|\right) \rightarrow 0, \quad \text { as } n \rightarrow \infty
$$

and

$$
\sum_{k=\sigma}^{\infty}\left\|V^{u}(\sigma-k-1) \Pi^{u}(\mathscr{E} p(k))\right\|<\infty
$$

Hence, letting $n \rightarrow \infty$ in (26), we get

$$
\eta(\sigma)=-\sum_{k=\sigma}^{\infty} V^{u}(\sigma-k-1)(\mathscr{E} p(k)), \quad n \geq \sigma
$$

in particular

$$
\eta(n)=-\sum_{k=n}^{\infty} V^{u}(n-k-1)(\mathscr{E} p(k)), \quad n \in \boldsymbol{Z}
$$

We thus have

$$
\begin{aligned}
x_{n} & =\xi(n)+\eta(n) \\
& =\sum_{k=-\infty}^{n-1} V^{s}(n-k-1)(\mathscr{E} p(k))-\sum_{k=n}^{\infty} V^{s}(n-k-1)(\mathscr{E} p(k)), \quad n \in \boldsymbol{Z},
\end{aligned}
$$

consequently
$x(n)=x_{n}(0)=\left[\sum_{k=0}^{n-1} V^{s}(n-k-1) \Pi^{s}(\mathscr{E} p(k))-\sum_{k=n}^{\infty} V^{u}(n-k-1) \Pi^{s}(\mathscr{E} p(k))\right](0)$.
Existence. Next let us show the existence of a bounded solution of Eq.
(1). To do this, consider a function $\zeta: \boldsymbol{Z} \rightarrow \mathscr{B}^{\gamma}$ defined by

$$
\zeta(n):=\sum_{k=-\infty}^{n-1} V^{s}(n-k-1)(\mathscr{E} p(k))-\sum_{k=n}^{\infty} V^{u}(n-k-1)(\mathscr{E} p(k)), \quad n \in \boldsymbol{Z}
$$

We notice that $\sup _{n \in \boldsymbol{Z}}\|\zeta(n)\|<\infty$ since

$$
\begin{aligned}
\|\zeta(n)\| \leq & \sum_{k=-\infty}^{n-1} K e^{-\alpha(n-k-1)}\left\|\Pi^{s} \mathscr{E}\right\|\left(\sup _{n \in \boldsymbol{Z}}\|p(\tau)\|\right) \\
& +\sum_{k=n}^{\infty} K e^{\alpha(n-k-1)}\left\|\Pi^{u} \mathscr{E}\right\|\left(\sup _{n \in \boldsymbol{Z}}\|p(\tau)\|\right) \\
\leq & \frac{K\left(1+e^{-\alpha}\right)}{1-e^{\alpha}}\left(\left\|\Pi^{s}\right\|+\left\|\Pi^{u}\right\|\right)\left(\sup _{n \in \boldsymbol{Z}}\|p(\tau)\|\right), \quad n \in \boldsymbol{Z} .
\end{aligned}
$$

Since the first and the second terms of $\zeta(n)$ satisfy the stable and the unstable parts of VCF, respectively, we conclude from Theorem 2 that $z(n):=[\zeta(n)](0)$ gives a bounded solution of Eq. (1).
(ii) If $p(n)$ is a periodic function, $\zeta(n)$ is also periodic, and hence so is $z(n)$.

Corollary 1. Suppose that $z I-\tilde{Q}(z)$ is invertible in $\mathscr{L}(X)$ for $|z|=1$. Then, for any almost periodic function $p: \boldsymbol{Z} \mapsto X$, Eq. (1) has a unique almost periodic solution $x(n)$, which is given by the formula in Theorem 7. Furthermore, the relation $\sigma_{b}(x) \subset \sigma_{b}(p)$ holds.

Proof. By Theorem 7 Eq. (1) has a unique bounded solution $x(n)$ on $\boldsymbol{Z}$, which is given by (24); in particular Theorem 6 implies that Eq. (1) has an almost periodic solution $x^{\prime}(n)$ with $\sigma_{b}\left(x^{\prime}\right) \subset \sigma_{b}(p)$ as well. Since an almost periodic solution is necessarily bounded on $\boldsymbol{Z}, x^{\prime}(n)$ must coincide with $x(n)$. This completes the proof.

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