Funkcialaj Ekvacioj, **48** (2005) 313–330

Existence of Local Analytic Solutions for Systems of Difference Equations with Small Step Size

By

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Abstract. We study the existence of analytic solutions of systems of difference equations where we have vector valued functions y of ε and $(\varepsilon + x)$ equals to vector valued analytic functions F of ε, x and y in a neighborhood of $(0, x^*, y^*)$ with $y^* = F(0, x^*, y^*)$. Under the assumption that the Jacobian of F with respect to y at $(0, x^*, y^*)$ minus the idendity is invertible, we first show the existence of a unique formal solution that is Gevrey-1. We also show, by applying a fixed point theorem, the existence of analytic solutions having a Gevrey-1 asymptotic expansion in small ε-sectors. This requires the construction of some bounded right inverse operators on a certain Banach space.

Keywords and Phrases. Difference equation, Difference operator, Inverse operator, Fixed point theorem, Gevrey asymptotic, Formal solution, Quasi-solution, Nagumo norm, Borel-Laplace transform.

2000 *Mathematics Subject Classification Numbers.* 39A, 47B39.

1. Introduction

We consider a system of n difference equations of the form

$$y(\varepsilon, x + \varepsilon) = F(\varepsilon, x, y(\varepsilon, x)), \tag{1}$$

where x is a complex variable, ε is a small complex parameter, and F is an analytic function of ε, x, and y in a certain domain. We look for solutions $y(\varepsilon, x)$ of equation (1) that, as $\varepsilon \to 0$, tend to some given *slow curve* $\phi_0(x)$ of equation (1) (i.e. a smooth function satisfying $F(0, x, \phi_0(x)) = \phi_0(x)$). Such solutions, if they exist, are so-called overstable solutions, in the sense that, as $\varepsilon \to 0$, they remain uniformly bounded with respect to x, in a full neighborhood of some point x_0. The existence of the solutions $y(\varepsilon, x)$ depends on the Jacobian $A_0(x)$ defined by

$$A_0(x) = \frac{\partial F}{\partial y}(0, x, \phi_0(x)) - I. \tag{2}$$

In this article, under the assumption that $A_0(x)$ as well as $A_0(x) + I$ are invertible at x_0 and some other conditions to be described later, we show the existence of overstable solutions $y(\varepsilon, x)$ of equation (1). The case where $A_0(x)$ is not invertible at x_0 will be treated in a future work where a more general version of equation (1) with an additional parameter $a(\varepsilon)$ will be considered. We also show the existence of a formal solution of equation (1) that is Gevrey-1, though the proof of the existence of overstable solutions does not rely on the existence of a formal solution. The Gevrey-1 character of the formal solution is used to prove that the overstable solution has a Gevrey-1 asymptotic expansion.

Note here that our result on the Gevrey-1 character of the formal solution (in dimension n) generalizes a result of Baesens [1] (in dimension 2). The proof of Baesens is very long and one can imagine the complexity of its generalization to dimension n. On the contrary, our proof is especially very short compared to the proof found in [1]. This is principally due to the fact that we used the Nagumo norms. We have indeed adapted a method of Canalis-Durand, Ramis, Schäfke, Sibuya [3].

We mention related works of Fruchard, Schäfke [6, 7] which treat vector difference equations of the form

$$y(x+\varepsilon) - y(x-\varepsilon) = 2\varepsilon F(x, y(x)), \tag{3}$$

and

$$y(\varepsilon, x+\varepsilon) = y(\varepsilon, x-\varepsilon) + \varepsilon F(\varepsilon, x, y(\varepsilon, x)) \tag{4}$$

respectively. We also mention another related work of El-Rabih [4], where difference equations of the form

$$y(\varepsilon, x+\varepsilon) = xy(\varepsilon, x) + \varepsilon F(\varepsilon, x, a(\varepsilon), y(\varepsilon, x)) \tag{5}$$

were considered, but in the one-dimensional case, i.e. $\varepsilon > 0$ small and $(x, a, y) \in \boldsymbol{C}^3$. More related works are those of Fruchard [5], Fruchard, Schäfke [8], Sibuya [10] and Wasow [13].

2. Assumptions

Given the system of equation (1), where ε is a small complex parameter. We assume the following:

1. The (vector valued) function $F : V \times D \to \boldsymbol{C}^n$ is an analytic function, where $V = V(\alpha, \beta, \varepsilon_0) := \{\varepsilon \mid \alpha < \arg \varepsilon < \beta, 0 < |\varepsilon| < \varepsilon_0\}$ and $D := D_{r_1}(x_0) \times D_{r_2}(y_0) \subset \boldsymbol{C} \times \boldsymbol{C}^n$; here $D_r(z)$ denotes an open disc or an open polydisc of radius $r > 0$ with centre z.
2. F is Gevrey-1 asymptotic to the formal series $\sum_{k \geq 0} f_k(x, y)\varepsilon^k$ as $V \ni \varepsilon \to 0$ uniformly for $(x, y) \in D$, where the functions f_k are holo-

morphic in D. This means that there exist positive constants A, C such that for all $(\varepsilon, x, y) \in V \times D$ and all $N \in \boldsymbol{N}^*$

$$\left| F(\varepsilon, x, y) - \sum_{k=0}^{N-1} f_k(x, y)\varepsilon^k \right| \le CA^N \Gamma(N+1)|\varepsilon|^N.$$

The infimum of all such constants A is called the *type* of the Gevrey 1 expansion. For more details about the Gevrey theory see the appendix of [2].

3. $F(0, x_0, y_0) = y_0$ for some y_0.
4. The numbers 0, 1 are not eigenvalues of $(\partial F/\partial y)(0, x_0, y_0)$.

3. Formal solutions

We consider the holomorphic function $\phi_0(x)$ satisfying $F(0, x, \phi_0(x)) = \phi_0(x)$. The existence of it is guaranteed by assumptions 3 and 4 (above) and the implicit function theorem. Replacing x by $x_0 + x$ and y by $\phi_0(x) + y$, without loss of generality, we can assume as follows:

1. F is analytic for $(x, y) \in D_{r_1}(0) \times D_{r_2}(0) \subset \boldsymbol{C} \times \boldsymbol{C}^n$.
2. F has a Gevrey-1 asymptotic expansion of type A in $\varepsilon \in V$.
3. $F(0, x, 0) = 0$ for $x \in D_{r_1}(0)$.
4. $(\partial F/\partial y)(0, 0, 0) - I$ is invertible.

Proposition 1. *Under the above assumptions* (*observe that the invertibility of* $(\partial F/\partial y)(0, 0, 0)$ *is not needed here*), *equation* (1) *has a unique formal solution* $\hat{y}(\varepsilon, x) = \sum_{j=1}^{\infty} b_j(x)\varepsilon^j$, *where* $b_j(x)$ *are analytic in* $D_r(0)$; $r < r_1$. *Moreover, for each* $\tilde{r} < r$, y *is Gevrey*-1 *on* $D_{\tilde{r}}(0)$; *i.e. there exist numbers* $M, B > 0$ *such that* $\sup_{|x| \le \tilde{r}} |b_j(x)| \le MB^j \Gamma(j+1)$ *for all* $j \in \boldsymbol{N}$.

For the proof of the theorem, we need the following two definitions:

Definition 2 ([3]). Given $\rho \in]0, r[$, we define a function d on $D_r(0)$ by

$$d(x) = \begin{cases} r - |x| & \text{if } |x| \ge \rho, \\ r - \rho & \text{if } |x| < \rho. \end{cases}$$

We consider the modified *Nagumo norms* introduced in [3]: For a nonnegative integer p and a holomorphic function f on $D_r(0)$, the Nagumo norm of f is defined by

$$\|f\|_p := \sup_{|x|<r} (|f(x)| d(x)^p).$$

We restate [3] an important property of $\| \ \|_p$:

(6) *For* $f \in \mathscr{H}(D_r)$, *we have* $\|f'\|_{p+1} \le e(p+1)\|f\|_p$.

Here $\mathscr{H}(D_r)$ denotes the Banach space of the holomorphic functions on D_r.

Definition 3 ([3]). We say that a series $g = \sum_{k=0}^{\infty} g_k(x)\varepsilon^k$ is majorized ("$\ll$") by a series $h(z) = \sum_{l=0}^{\infty} h_l z^l$ if

$$\|g_j\|_j \le h_j j! \qquad \text{for } j = 0, 1, 2, \dots$$

We also recall the following two properties ([3]):

If $g \ll h(z)$ *and* $\tilde{g} \ll \tilde{h}(z)$, *then*

$$g\tilde{g} \ll h(z)\tilde{h}(z), \qquad \textit{and} \tag{7}$$

$$\varepsilon \frac{d}{dx} g \ll ezh(z). \tag{8}$$

Proof of Proposition 1. For simplicity, we omit the "hats" in the following proof. We first show the existence of such a solution. We rewrite equation (1) as an equation for *formal* series (except for the term $y(\varepsilon, x+\varepsilon) - y(\varepsilon, x)$ at the moment)

$$A_0(x)y(\varepsilon, x) = y(\varepsilon, x+\varepsilon) - y(\varepsilon, x) - \varepsilon\phi_1(x) - \sum_{|\vec{p}|+k \ge 2} f_{\vec{p}k}(x) y^{\vec{p}}(\varepsilon, x)\varepsilon^k,$$

where

- $F(\varepsilon, x, y) = \sum_{|\vec{p}|+k \ge 1} f_{\vec{p}k}(x) y^{\vec{p}}(\varepsilon, x)\varepsilon^k$, and
- $\phi_1 = f_{\vec{0}1}$ and $f_{\vec{p}k}$ are n-dimensional vectors, $\vec{p} = (p_1, \dots, p_n)$, $|\vec{p}| = p_1 + \dots + p_n$ and $y^{\vec{p}} = y_1^{p_1} \dots y_n^{p_n}$ where p_i, k are nonnegative integers, and y_j is the jth entry of y.
- The entries of A_0, ϕ_1 and $f_{\vec{p}k}$ are holomorphic for $|x| < r_1$.

We proceed as in [3], but we have also to expand the term $y(\varepsilon, x+\varepsilon) - y(\varepsilon, x)$.

We have

$$y(\varepsilon, x+\varepsilon) - y(\varepsilon, x) = \sum_{r=1}^{\infty} \frac{1}{r!}\left(\varepsilon \frac{d}{dx}\right)^r y(\varepsilon, x).$$

This gives

$$A_0(x)y(\varepsilon, x) = \sum_{r=1}^{\infty} \frac{1}{r!}\left(\varepsilon \frac{d}{dx}\right)^r y(\varepsilon, x) - \varepsilon\phi_1(x) - \sum_{|\vec{p}|+k \ge 2} f_{\vec{p}k}(x) y^{\vec{p}}(\varepsilon, x)\varepsilon^k. \tag{9}$$

Putting $y(\varepsilon,x)=\sum_{j=1}^{\infty}b_j(x)\varepsilon^j$ and looking at the coefficient of ε^n, $n=0,1,\dots,$ we note that the right hand side of the above equation has coefficient 0 for ε^0 and that the coefficient of ε^1 is $-\phi_1(x)$. Furthermore, we easily see that the coefficients $\{b_j\}_{j\in\boldsymbol{N}}$ of a formal solution are uniquely determined.

Next, we show that the formal solution is Gevrey-1. To do this, we construct a majorant equation of (9) having a unique series solution that is convergent. We rewrite it as:

$$(10)\qquad y(\varepsilon,x)=A_0(x)^{-1}\sum_{r=1}^{\infty}\frac{1}{r!}\left(\varepsilon\frac{d}{dx}\right)^r y(\varepsilon,x)-A_0(x)^{-1}\phi(\varepsilon,x)$$
$$-A_0(x)^{-1}B(\varepsilon,x)y(\varepsilon,x)-A_0(x)^{-1}\sum_{|\vec{p}|\geq 2}f_{\vec{p}}(\varepsilon,x)y^{\vec{p}}(\varepsilon,x),$$

where

$$\phi(\varepsilon,x):=\sum_{k=1}^{\infty}\phi_k(x)\varepsilon^k,\quad B(\varepsilon,x):=\sum_{k=1}^{\infty}B_k(x)\varepsilon^k,\quad f_{\vec{p}}(\varepsilon,x):=\sum_{k=0}^{\infty}f_{\vec{p}k}(x)\varepsilon^k.$$

The entries of ϕ_k are those of $f_{\vec{0}k}$ and the entries of B_k are those of $f_{\vec{p}k}$ with $|\vec{p}|=1$. Note here that $B_k(x)$, $k\geq 1$ are $n\times n$ matrices. We want to find a *majorant equation* related to equation (10). For the majorization of series of vectors, we use the maximum norm, and for that of series of matrices, we use a compatible matrix norm. Then by applying the $\|\ \|_j/j!$ to the coefficients of ε^j of equation (10), we can find majorant (scalar) series $\hat{\phi}(z)$, $\hat{B}(z)$, and $\hat{f}_{\vec{p}}(z)$ of $\phi(\varepsilon,x)$, $B(\varepsilon,x)$ and $f_{\vec{p}}(\varepsilon,x)$ respectively. Note here that we use the hypothesis that F is Gevrey-1. It remains to treat the sum $\sum_{r=1}^{\infty}(1/r!)(\varepsilon d/dx)^r y(\varepsilon,x)$. Assume that $y\ll h(z)$, then by an idea of J. P. Ramis using property (8), we have:

$$\sum_{r=1}^{\infty}\frac{1}{r!}\left(\varepsilon\frac{d}{dx}\right)^r y(\varepsilon,x)\ll\sum_{r=1}^{\infty}\frac{1}{r!}(ez)^r h(z)=(e^{ez}-1)h(z).$$

Also, $\|A_0(x)^{-1}\|_0\ll C_1$, where C_1 is a constant. By these properties and those given by equations (6), (7), (8) and Cauchy's estimate for the coefficients of a convergent power series, all these series have a common radius of convergence, and $\sum_{|\vec{p}|\geq 2}\hat{f}_{\vec{p}}(z)g^{|\vec{p}|}$ is convergent if $|z|$ and $|g|$ are small enough. Then a majorant equation of (10) is given by:

$$(11)\qquad h(z)=C_1(e^{ez}-1)h(z)+C_1\hat{\phi}(z)+C_1\hat{B}(z)h(z)+C_1\sum_{n=2}^{\infty}\sum_{|\vec{p}|=n}\hat{f}_{\vec{p}}(z)h(z)^n$$

As $\hat{\phi}(0) = \hat{B}(0) = 0$, it is easy to see that this majorant equation has a unique formal solution $h(z) = \sum_{j=1}^{\infty} h_j z^j$ with nonnegative coefficients. In the same way as in [3] (i.e. essentially by induction), it follows that our formal solution y of Proposition 1 satisfies $y \ll h(z)$. By the implicit function theorem, the formal solution $h(z)$ converges for z small enough and hence $h_j \le MN^j$, $j = 1, 2, \dots$, where M, N are positive constants. Then $y \ll h(z)$ for the formal solutions y of equation (10) and h of equation (11). This means that $\|y_n(x)\|_n \le MN^n n!$. By the definition of $\| \ \|_n$, $|y_n(x)| \le MN^n d(x)^{-n} n!$ for $|x| < r$. For all δ, $0 < \delta < r - \rho$, we also have $|y_n(x)| \le MN^n \delta^{-n} n!$ for $|x| \le r - \delta$; i.e. y is Gevrey-1 uniformly on $D_{\tilde{r}}(0)$ for each $\tilde{r} \le r$. □

4. Quasi-solutions

So far, we have shown that equation (1) has a unique fomal solution $\hat{y}(\varepsilon, x) = \sum_{j=1}^{\infty} b_j(x) \varepsilon^j$ with

$$\sup_{|x| \le \tilde{r}} |b_j(x)| \le MB^j \Gamma(j+1),$$

for some numbers $M, B > 0$ i.e. it is Gevrey-1. Recall that F is also Gevrey-1 asymptotic to $\sum_{j \ge 0} f_j(x, y) \varepsilon^j$. Next, we take the Borel transform $\tilde{b}(t, x)$ of $\hat{y}(\varepsilon, x)$, i.e. we define:

$$\tilde{b}(t, x) := \sum_{j=1}^{\infty} b_j(x) \frac{t^{j-1}}{(j-1)!}, \qquad x \in D_{\tilde{r}}(0),\ |t| < \frac{1}{B}.$$

Let $T = 1/B$. We define $\tilde{y}(\varepsilon, x)$ to be the Laplace transform (with some corrective term) of $\tilde{b}(t, x)$. We put

$$\tilde{y}(\varepsilon, x) := \int_0^T \tilde{b}(t, x) e^{-t/\varepsilon}\, dt + \varepsilon \tilde{b}(T, x) e^{-T/\varepsilon}.$$

Then, we know that $\tilde{y}(\varepsilon, x)$ is Gevrey-1 asymptotic to $\hat{y}(\varepsilon, x)$.

Theorem 4. *As given above, $\tilde{y}(\varepsilon, x)$ is a quasi-solution of equation* (1), *i.e. there exists $K > 0$ such that*

$$\tilde{R}(\varepsilon, x) := \tilde{y}(\varepsilon, x + \varepsilon) - F(\varepsilon, x, \tilde{y}(\varepsilon, x))$$

satisfies

$$|\tilde{R}(\varepsilon, x)| \le K e^{-T/|\varepsilon|}, \qquad \varepsilon \in V,\ \varepsilon \to 0,\ x \in D_{\tilde{r}}(0).$$

Proof. As $\tilde{y}(\varepsilon, x)$ and $F(\varepsilon, x, y)$ are Gevrey-1 in ε, there exist a good covering [9, 11, 12] $S_j = S(\alpha_j, \beta_j, \varepsilon_0)$, $1 \le j \le m$ of the punctured disk $D(0, \varepsilon_0)$ and functions $y_j(\varepsilon, x)$ and $F_j(\varepsilon, x, y)$ defined for $\varepsilon \in S_j$ and satisfying

$$y_1 = \tilde{y}, \qquad F_1 = \tilde{F}, \qquad |y_{j+1}(\varepsilon, x) - y_j(\varepsilon, x)| = \mathcal{O}(e^{-\gamma/|\varepsilon|}), \qquad \text{and}$$
$$|F_{j+1}(\varepsilon, x, y) - F_j(\varepsilon, x, y)| = \mathcal{O}(e^{-\gamma/|\varepsilon|}),$$

for $\varepsilon \in S_j \cap S_{j+1}$, $x \in D_{\tilde{r}}(0)$ and γ some positive constant. The functions $y_j(\varepsilon, x)$ and $F_j(\varepsilon, x, y)$ can be chosen, similarly to $\tilde{y}(\varepsilon, x)$ given in the previous section, as truncated Borel-Laplace transforms of $\hat{y}(\varepsilon, x)$ and $F(\varepsilon, x, y)$ on $[0, Te^{2j\pi i}]$. Trivially, the same estimates still hold for $y_j(\varepsilon, x + \varepsilon)$ if $x \in D_{\tilde{r}-\varepsilon}(0)$, i.e.

$$|y_{j+1}(\varepsilon, x + \varepsilon) - y_j(\varepsilon, x + \varepsilon)| = \mathcal{O}(e^{-\gamma/|\varepsilon|}), \qquad \varepsilon \in S_j \cap S_{j+1}.$$

Also, $\tilde{y}(\varepsilon, x + \varepsilon) = y_1(\varepsilon, x + \varepsilon)$. So, we have to reduce the x-domain $D_{\tilde{r}}(0)$ into $D_{\tilde{r}-\varepsilon}(0)$. Let

$$R_j(\varepsilon, x) := y_j(\varepsilon, x + \varepsilon) - F_j(\varepsilon, x, y_j(\varepsilon, x)).$$

Now, by using standard estimates, it follows that the differences $R_{j+1} - R_j$ are also exponentially small:

$$|R_{j+1}(\varepsilon, x) - R_j(\varepsilon, x)| = \mathcal{O}(e^{-\gamma/|\varepsilon|})$$

for $\varepsilon \in S_j \cap S_{j+1}$, $j = 1, \ldots, n$ and $x \in D_{\tilde{r}-\varepsilon}(0)$. The theorem of Ramis-Sibuya implies that R_j, $j = 1, \ldots, m$, have asymptotic expansions of Gevrey order 1 with a common right hand side. Since one of these R_j is $\tilde{R}$ in question, this right hand side is $\hat{y}(\varepsilon, x + \varepsilon) - \hat{F}(\varepsilon, x, \hat{y}(\varepsilon, x)) = 0$, because $\hat{y}$ is a formal solution of equation (1). It immediately follows that all R_j are exponentially small. □

In the next sections, we show directly the existence of an analytic solution of equation (1) by applying the fixed point theorem on some suitable Banach space. We also show that this solution is exponentially close to the quasi-solution $\tilde{y}(\varepsilon, x)$ and is Gevrey-1 asymptotic to $\hat{y}(\varepsilon, x)$.

5. Analytic solutions

Theorem 5. *Under the assumptions of Section* 2, *for each subsector* $\tilde{V} = V(\tilde{\alpha}, \tilde{\beta}, \varepsilon_1)$ *of* V *having sufficiently small angular opening and with sufficiently small* $\varepsilon_1 > 0$, *there exists an analytic solution* $y : \tilde{V} \times D_r(0) \to \boldsymbol{C}^n$ *of equation* (1) *which is Gevrey*-1 *asymptotic to* $\hat{y}(\varepsilon, x)$, *provided that* $r > 0$ *is sufficiently small.*

The proof needs the following theorem:

Theorem 6. *Given* $\lambda_1, \ldots, \lambda_n \in \boldsymbol{C} \backslash \{0, 1\}$, *there exist a neighborhood* Ω *of* 0 *and bounded linear operators* $T_{\varepsilon j} : \mathscr{H}_b(\Omega_\varepsilon) \to \mathscr{H}_b(\Omega_\varepsilon)$ *such that* $y_j := T_{\varepsilon j}(g)$, $g \in \mathscr{H}_b(\Omega_\varepsilon)$ *implies that* $y_j(\varepsilon, x + \varepsilon) = \lambda_j y_j(\varepsilon, x) + g(x)$ *whenever* $x \in \Omega_\varepsilon$ *and* $x + \varepsilon \in \Omega_\varepsilon$. *Here* $\Omega_\varepsilon = \Omega + [-\varepsilon/2, \varepsilon/2]$ *and* $\mathscr{H}_b(\Omega_\varepsilon)$ *denotes the Banach space* *of the holomorphic bounded functions on* Ω_ε *equipped with the supremum norm* $\|y\| = \sup_{\varepsilon \in V, x \in \Omega_\varepsilon} |y(\varepsilon, x)|$, *with* $y = (y_j)_{j=1}^n$. *Thus* $T_\varepsilon : \mathscr{H}_b^n(\Omega_\varepsilon) \to \mathscr{H}_b^n(\Omega_\varepsilon)$ *defined by* $T_\varepsilon(g_j)_{j=1}^n = (T_{\varepsilon j}(g_j))_{j=1}^n$ *is a bounded right inverse of the operator*

$$\gamma_\varepsilon((y_j)_{j=1}^n)(\varepsilon, x) = (y_j(\varepsilon, x+\varepsilon) - \lambda_j y_j(\varepsilon, x))_{j=1}^n. \tag{12}$$

Observe that the images of γ_ε *are not necessarily defined on all of* Ω_ε.

The construction of this right inverse and the x-domain along with the proof will be given in the next section.

As an application of Theorem 5 we have the following:

Theorem 7. *Consider an analytic (matrix valued) function* $A : V \times D_r(x_0) \to M_{nn}(\boldsymbol{C})$, *where* $r > 0$ *and* $V = V(\alpha, \beta, \varepsilon_0)$. *We suppose that* A *has a Gevrey-*1 *asymptotic expansion* $\sum_{n \geq 0} L_n(x)\varepsilon^n$ *as* $V \ni \varepsilon \to 0$ *uniformly for* $x \in D_r(x_0)$. *We write the matrices* $A(\varepsilon, x)$ *in blocks*

$$A(\varepsilon, x) = \begin{pmatrix} A_1(\varepsilon, x) & A_2(\varepsilon, x) \\ A_3(\varepsilon, x) & A_4(\varepsilon, x) \end{pmatrix},$$

where A_1, A_2, A_3 *and* A_4 *are, respectively,* m *by* m, m *by* $n-m$, $n-m$ *by* m *and* $n-m$ *by* $n-m$ *matrices. We assume, furthermore, that* $A_2(0, x_0) = A_3(0, x_0) = 0$ *and that* $A_1(0, x_0)$ *and* $A_4(0, x_0)$ *are invertible and have no common eigenvalues. Then for all subsectors* $\tilde{V} = V(\tilde{\alpha}, \tilde{\beta}, \varepsilon_1)$ *of* V *having sufficiently small angular opening and with sufficiently small* $0 < \varepsilon_1 < \varepsilon_0$, *there exist* $r_1 > 0$ *and an analytic matrix function* $P : \tilde{V} \times D_{r_1}(x_0) \to M_{n,n}(\boldsymbol{C})$ *having a Gevrey-*1 *asymptotic expansion as* $\tilde{V} \ni \varepsilon \to 0$ *and an analogous block decomposition*

$$P(\varepsilon, x) = \begin{pmatrix} I & P_2(\varepsilon, x) \\ P_3(\varepsilon, x) & I \end{pmatrix},$$

such that $P(\varepsilon, x+\varepsilon)^{-1} A(\varepsilon, x) P(\varepsilon, x) = B(\varepsilon, x)$ *is block diagonal with*

$$B(\varepsilon, x) = \begin{pmatrix} B_1(\varepsilon, x) & 0 \\ 0 & B_4(\varepsilon, x) \end{pmatrix},$$

and $B_1(0, x_0) = A_1(0, x_0)$, $B_4(0, x_0) = A_4(0, x_0)$.

Proof. The proof is analogous to the well known one for singularly perturbed differential equations. We need to find a non singular matrix P such that

$$A(\varepsilon, x)P(\varepsilon, x) = P(\varepsilon, x+\varepsilon)B(\varepsilon, x).$$

This gives:

$$B_1(\varepsilon,x) = A_1(\varepsilon,x) + A_2(\varepsilon,x)P_3(\varepsilon,x),$$
$$P_3(\varepsilon,x+\varepsilon)B_1(\varepsilon,x) = A_3(\varepsilon,x) + A_4(\varepsilon,x)P_3(\varepsilon,x),$$
$$P_2(\varepsilon,x+\varepsilon)B_4(\varepsilon,x) = A_2(\varepsilon,x) + A_1(\varepsilon,x)P_2(\varepsilon,x),$$
$$B_4(\varepsilon,x) = A_4(\varepsilon,x) + A_3(\varepsilon,x)P_2(\varepsilon,x).$$

Thus $P_3(\varepsilon,x)$ has to satisfy:

$$P_3(\varepsilon,x+\varepsilon) = (A_3(\varepsilon,x) + A_4(\varepsilon,x)P_3(\varepsilon,x))(A_1(\varepsilon,x) + A_2(\varepsilon,x)P_3(\varepsilon,x))^{-1}, \tag{13}$$

which implies,

$$\begin{aligned} P_3(\varepsilon,x+\varepsilon) &= A_3(\varepsilon,x)A_1^{-1}(\varepsilon,x) + A_4(\varepsilon,x)P_3(\varepsilon,x)A_1^{-1}(\varepsilon,x) \\ &\quad - A_3(\varepsilon,x)A_1^{-1}(\varepsilon,x)A_2(\varepsilon,x)P_3(\varepsilon,x)A_1^{-1}(\varepsilon,x) + \mathcal{O}(P_3^2). \end{aligned}$$

This equation satisfies the hypotheses of Theorem 5 because at $\varepsilon = 0$, $x = x_0$ its linear part reduces to $P_3 \mapsto A_4(0,x_0)P_3A_1^{-1}(0,x_0)$. Neither 0 nor 1 are eigenvalues of this mapping by the assumption of our theorem. So, by Theorem 5, for subsectors $\tilde{V}$ of V as mentioned there, there exists an analytic solution $P_3 : \tilde{V} \times D_{r_1}(x_0) \to M_{n-m,m}$ of (13) having a Gevrey-1 asymptotic expansion. Similarly, we can solve the respective equation for P_2. The invertibility of the matrix P with blocks I, P_2, P_3, I for sufficiently small ε follows from $P_2(0,x_0) = 0$, $P_3(0,x_0) = 0$. Put $B_1(\varepsilon,x) = A_1(\varepsilon,x) + A_2(\varepsilon,x)P_3(\varepsilon,x)$ and $B_4(\varepsilon,x) = A_4(\varepsilon,x) + A_3(\varepsilon,x)P_2(\varepsilon,x)$. Then the four equations equivalent to $A(\varepsilon,x)P(\varepsilon,x) = P(\varepsilon,x+\varepsilon)B(\varepsilon,x)$ are satisfied and hence P has the properties required in our theorem. □

6. Proof of Theorem 6

6.1. Construction of the operators $T_{\varepsilon j}$

For simplicity, we prove the theorem only for $\varepsilon > 0$, $\varepsilon \in]0,\varepsilon_0]$.

We consider a system of difference equations of the form

$$y_j(\varepsilon,x+\varepsilon) = \lambda_j y_j(\varepsilon,x) + g_j(x), \tag{14}$$

where $\lambda_j \in \boldsymbol{C}$, $\lambda_j \neq 0,1$ for $j = 1,\dots,n$ and where $g_j(x)$ are some analytic functions on certain horizontally convex x-domains Ω to be described later. On the Banach space $\mathscr{H}_b(\Omega_\varepsilon)$ of the holomorphic bounded functions on Ω_ε, we construct bounded operators $T_{\varepsilon j}$ such that $y_j := T_{\varepsilon j}(g_j)$ implies $\gamma_{\varepsilon j}(y_j) = g_j(x)$, where $\gamma_{\varepsilon j}$ is the operator defined by

$$\gamma_{\varepsilon j} y_j(\varepsilon,x) = y_j(\varepsilon,x+\varepsilon) - \lambda_j y_j(\varepsilon,x). \tag{15}$$

$T_{\varepsilon j}$ is related to a right inverse of the operator Δ_ε given by

$$\Delta_\varepsilon y_j(\varepsilon, x) = \frac{1}{\varepsilon}(y_j(\varepsilon, x+\varepsilon) - y_j(\varepsilon, x)). \tag{16}$$

To see this, we define

$$Z_j(\varepsilon, x) := \lambda_j^{-x/\varepsilon} := \exp\left(-\frac{\log \lambda_j}{\varepsilon} x\right), \tag{17}$$

where the branch of log will be chosen later. Note that $1/Z_j(\varepsilon, x)$ is the homogeneous solution of $\gamma_{\varepsilon j} y_j(\varepsilon, x) = 0$. We rewrite equation (14) as

$$\Delta_\varepsilon \tilde{y}_j(\varepsilon, x) = \tilde{g}_j(\varepsilon, x), \tag{18}$$

where

$$\tilde{y}_j(\varepsilon, x) = Z_j(\varepsilon, x) y_j(\varepsilon, x), \qquad \tilde{g}_j(\varepsilon, x) = \frac{Z_j(\varepsilon, x)}{\varepsilon\lambda_j} g_j(\varepsilon, x).$$

We know that, on suitable spaces of analytic functions, equation (18) admits an analytic solution $\tilde{y}_j(\varepsilon, x)$ of the form $\tilde{y}_j(\varepsilon, x) = V_\varepsilon(\tilde{g}_j(\varepsilon, x))$, where $V_\varepsilon = S - \varepsilon U_\varepsilon$ is a right inverse operator of Δ_ε [7]. Then equation (14) admits an analytic solution $y_j(\varepsilon, x)$ given by

$$\begin{aligned} y_j(\varepsilon, x) &= \frac{\tilde{y}_j(\varepsilon, x)}{Z_j(\varepsilon, x)} = \frac{(S - \varepsilon U_\varepsilon)(\tilde{g}_j(\varepsilon, x))}{Z_j(\varepsilon, x)} \\ &= \frac{1}{Z_j(\varepsilon, x)}(S - \varepsilon U_\varepsilon)\left(\frac{Z_j(\varepsilon, x) g_j(\varepsilon, x)}{\varepsilon\lambda_j}\right), \end{aligned} \tag{19}$$

that is,

$$y_j(\varepsilon, x) := T_{\varepsilon j}(g_j(\varepsilon, x)) = (I_{1j} + I_{2j} + I_{3j}) g_j(\varepsilon, x), \tag{20}$$

where

$$I_{1j} g_j(\varepsilon, x) := \frac{4}{\varepsilon Z_j(\varepsilon, x)} \int_{-1/8}^{1/8} \int_{\gamma_{x,t}^-} \frac{Z_j(\varepsilon, \xi) g_j(\varepsilon, \xi)}{\lambda_j} \, d\xi dt, \tag{21}$$

$$I_{2j} g_j(\varepsilon, x) := \frac{-4}{\varepsilon Z_j(\varepsilon, x)} \int_{-1/8}^{1/8} \int_{\gamma_{x,t}^-} \frac{Z_j(\varepsilon, \xi) g_j(\varepsilon, \xi)}{\lambda_j(1 - e^{(2i\pi/\varepsilon)(\xi - x)})} \, d\xi dt, \tag{22}$$

$$I_{3j} g_j(\varepsilon, x) := \frac{4}{\varepsilon Z_j(\varepsilon, x)} \int_{-1/8}^{1/8} \int_{\gamma_{x,t}^+} \frac{Z_j(\varepsilon, \xi) g_j(\varepsilon, \xi)}{\lambda_j(1 - e^{-(2i\pi/\varepsilon)(\xi - x)})} \, d\xi dt, \tag{23}$$

with

- $\gamma_{x,t}^-$ is an ascending path joining $x^- + \varepsilon t$ to $x - \varepsilon/2$ (avoiding $x - \varepsilon$ and x) such that $\operatorname{Im} \xi$ is increasing as ξ varies on it, and
- $\gamma_{x,t}^+$ is an ascending path joining $x - \varepsilon/2$ to $x^+ + \varepsilon t$ (avoiding $x - \varepsilon$ and x) such that $\operatorname{Im} \xi$ is increasing as ξ varies on it, and where
- x^- and x^+ are points with minimal and maximal imaginary parts, respectively, in the horizontally convex x-domain Ω.

6.2. Domain description

In order that all integrals I_{ij} for all $i = 1, 2, 3$ and $j = 1, \ldots, n$ define bounded operators, the x-domain for which they are all bounded, must satisfy restrictive conditions. As in [7, 6], the x-domain Ω is some horizontally convex domain, c-ascending (c to be determined) and with extreme points x^- and x^+ of minimal and maximal imaginary parts respectively. However, Ω will be here some rhombus to be described in more details below.

In order to see what further conditions we need to impose on our choice of Ω, we first discuss heuristically the boundedness of the integrals I_{1j}, I_{2j} and I_{3j}. Then once that is done, we show the boundedness of those integrals in details.

6.2.1. Heuristic estimation of I_{1j}, I_{2j} and I_{3j}

- We first consider I_{1j} (see equation (21)). Note that

$$\frac{Z_j(\varepsilon, \xi)}{Z_j(\varepsilon, x)} = \exp\left(\frac{1}{\varepsilon}(\log \lambda_j)(x - \xi)\right).$$

Here the rhombus is very close to a segment. Its interior angles at the points $x^\pm$ are assumed to be small. We put $x^+ = Ae^{i\Psi}$ and $x^- = -Ae^{i\Psi}$, $A > 0$. Observing that ξ varies from $x^- + \varepsilon t$ to $x - \varepsilon/2$, we may assume for the moment that $\arg(x - \xi) \approx \arg(x - x^-) \approx \arg(x^+ - x^-) = \Psi$. Here, we obtain: $(x - \xi) \log \lambda_j \approx |x - \xi| \, |\log \lambda_j|^{i(\Psi + \Phi_j)}$, where $\Phi_j = \arg \log \lambda_j$. I_{1j} becomes bounded, as $\varepsilon \to 0^+$, if $Z_j(\varepsilon, \xi)/Z_j(\varepsilon, x)$ is. Observe that the imaginary part of $\log \lambda_j$ and hence its argument Φ_j are not yet completely determined. For the moment, we want that $\cos(\Psi + \Phi_j) < 0$ for all j, so that $\operatorname{Re}((x - \xi) \log \lambda_j)$ is negative.

- Next, we consider I_{2j} (see equation (22)). Since

$$\frac{1}{1 - e^{(2i\pi/\varepsilon)(\xi - x)}} = \mathcal{O}\left(\exp\left(\frac{2\pi}{\varepsilon} \operatorname{Im}(\xi - x)\right)\right),$$

we have

$$\frac{Z_j(\varepsilon,\xi)}{Z_j(\varepsilon,x)(1-e^{(2i\pi/\varepsilon)(\xi-x)})} = \mathcal{O}\left(\exp\left(\frac{1}{\varepsilon}\ \mathrm{Re}((x-\xi)(\log\lambda_j+2i\pi))\right)\right).$$

Here, we obtain: $(x-\xi)(\log\lambda_j+2i\pi)) \approx |x-\xi|(|\log\lambda_j|e^{i(\Psi+\Phi_j)}+2\pi e^{i(\Psi+\pi/2)})$. So, we want $|\log\lambda_j|\cos(\Psi+\Phi_j)-2\pi\sin\Psi<0$. Since $\cos(\Psi+\Phi_j)<0$, then we restrict for the moment $\Psi\in\,]0,\pi[$.

- Finally, we consider I_{3j} (see equation (23)), where

$$\frac{Z_j(\varepsilon,\xi)}{Z_j(\varepsilon,x)(1-e^{(-2i\pi/\varepsilon)(\xi-x)})} = \mathcal{O}\left(\exp\left(\frac{1}{\varepsilon}\ \mathrm{Re}((x-\xi)(\log\lambda_j-2i\pi))\right)\right).$$

Putting $\tilde{\Phi}_j=\arg(\log\lambda_j-2i\pi)$, then, heuristically speaking, as ξ varies from x to x^+, we have $\xi-x\approx|\xi-x|e^{i\Psi}$. We want then $\cos(\Psi+\tilde{\Phi}_j)>0$ for all j, so that $\mathrm{Re}((x-\xi)(\log\lambda_j-2i\pi))$ is negative.

To sum up, heuristically, we need to choose Ψ and thus the x-domain Ω such that the following are satisfied:

1. $\Psi\in\,]0,\pi[$,
2. $\cos(\Psi+\Phi_j)<0$ for all j, with $\Phi_j=\arg\log\lambda_j$, and
3. $\cos(\Psi+\tilde{\Phi}_j)>0$ for all j, with $\tilde{\Phi}_j=\arg(\log\lambda_j-2i\pi)$.

We choose Ψ to be slightly bigger than $\pi/2$, say, $\Psi=\pi/2+\delta_1$, where $\delta_1>0$ is very small. We now choose $\log\lambda_j$, for $j=1,\dots,n$, such that:

1. $\mathrm{Im}\log\lambda_j\in[0,2\pi]$,
2. $\Phi_j=\arg\log\lambda_j\in[0,\pi[$, (then $\mathrm{Im}\log\lambda_j>0$ if $\mathrm{Re}\log\lambda_j<0$),
3. if $\mathrm{Re}\log\lambda_j\geq 0$, then $\mathrm{Im}\log\lambda_j<2\pi$ (recall that $\lambda_j\neq 0$ for all j).

Then, $\mathrm{Im}(\log\lambda_j-2i\pi)\in[-2\pi,0]$, and $\tilde{\Phi}_j=\arg(\log\lambda_j-2i\pi)\in[-\pi,0[$. With such a choice of $\log\lambda_j$, the above conditions are satisfied if δ_1 is chosen also such that:

- For $\Phi_j=\pi-\delta_2$ with small $\delta_2>0$, $\delta_1-\delta_2<0$, or $\max_j\ \Phi_j+\delta_1<\pi$.
- For $\tilde{\Phi}_j=-\delta_3$ with small $\delta_3>0$, $\delta_1-\delta_3<0$, or $\max_j\ \tilde{\Phi}_j+\delta_1<0$.

6.2.2. Choice of Ω

For arbitrary $A>0$, choose $x^+=Ae^{i\Psi}$ and $x^-=-Ae^{i\Psi}$, and let the rhombus Ω (a c-ascending domain) be determined by x^+, x^- and its interior angles 2δ at these points where $\max_j\ \Phi_j+\delta_1+\delta<\pi$, $\max_j\ \tilde{\Phi}_j+\delta_1+\delta<0$, and $0<\delta<\delta_1$.

6.3. Completion of the proof of Theorem 6

We prove Theorem 6 with Ω as described above and $0<c<1/2$. We show that all I_{ij} are bounded for $i=1,2,3$ and $j=1,\dots,n$. Without loss of generality, we may assume that $x \in \Omega + [0, \varepsilon]$. We write $x - \varepsilon/2 = \tilde{x} + \mu$ where $\tilde{x} \in \Omega$ and $\mu \in [-\varepsilon/2, \varepsilon/2]$. We consider the following three cases:

1. $\operatorname{Im} x^- + \varepsilon c/8 < \operatorname{Im} x < \operatorname{Im} x^+ - \varepsilon c/8$.
2. $\operatorname{Im} x \le \operatorname{Im} x^- + \varepsilon c/8$.
3. $\operatorname{Im} x \ge \operatorname{Im} x^+ - \varepsilon c/8$.

Case 1: We describe the paths $\gamma^-_{x,t}$ and $\gamma^+_{x,t}$. Such a path $\gamma^-_{x,t}$ (respectively $\gamma^+_{x,t}$) is chosen very close to γ^-_x (respectively γ^+_x) joining x^- to x (respectively x to x^+) that we used in our heuristic estimation of the integrals I_{1j}, I_{2j}, I_{3j} and that lead to the construction of the x-domain Ω. Indeed, we select a c-ascending path from x^- to $\tilde{x}$ and modify it to a c-ascending path from $x^- + \mu$ to $\tilde{x} + \mu = x - \varepsilon/2$ and then complete with a horizontal segment from $x^- + \varepsilon t$ to $x^- + \mu$. Then the c-ascending path from $x^- + \mu$ to $x - \varepsilon/2$ differs from that of x^- to $\tilde{x}$ at most by the order of ε.

• Consider I_{1j} (see equation (21)). On the segment from $x^- + \mu$ to $x - \varepsilon/2$, we have

$$\frac{Z_j(\varepsilon, \xi)}{Z_j(\varepsilon, x)} = \mathcal{O}\left(\exp\left(\operatorname{Re}\left(\frac{\log \lambda_j}{\varepsilon}(\tilde{x} - \tilde{\xi})\right)\right)\right)$$
$$= \mathcal{O}\left(\exp\left(\frac{1}{\varepsilon}|\tilde{x} - \tilde{\xi}|\,|\log \lambda_j| \cos(\Psi + \Phi_j + \tilde{\mu})\right)\right),$$

where $\tilde{\xi} = \xi - \mu$, $\Psi = \arg(x^+ - x^-)$ and $\arg(\tilde{x} - \tilde{\xi}) = \Psi + \tilde{\mu}$, $\tilde{\mu} \in [-\delta, \delta]$. Then,

$$\frac{Z_j(\varepsilon, \xi)}{Z_j(\varepsilon, x)} = \mathcal{O}\left(\exp\left(-\frac{\alpha}{\varepsilon}(|\tilde{x}| - |\tilde{\xi}|)\right)\right), \tag{24}$$

where

$$\alpha = \min_{j, |\tilde{\mu}| \le \delta} (|\log \lambda_j|\, |\cos(\Psi + \Phi_j + \tilde{\mu})|) > 0. \tag{25}$$

Hence, the integral part of I_{1j} from $x^- + \mu$ to $x - \varepsilon/2$ denoted by $\widetilde{I_{1j}}$ satisfies

$$|\widetilde{I_{1j}}| \le \frac{\|g_j\|}{\varepsilon \min\limits_j |\lambda_j|} \int_{x^-}^{\tilde{x}} \exp\left(-\frac{\alpha}{\varepsilon}(|\tilde{x}| - |\tilde{\xi}|)\right) d(|\xi|) \tag{26}$$
$$\le \frac{K\|g_j\|}{\varepsilon} \int_0^\infty e^{-(\alpha/\varepsilon)t}\, dt \le \frac{K\|g_j\|}{\alpha} < \infty,$$

where K is a constant. On the horizontal part from $x^- + \varepsilon t$ to $x^- + \mu$, equations (24), (25) still hold and hence the integral part of I_{1j} which corresponds to this horizontal part is also bounded.

• Next, we consider I_{2j} (see equation (22)). As above, we divide the integration path of I_{2j} into two parts:

$$I_{2j} := I_{2j\text{-}horizontal} + I_{2j\text{-}ascending},$$

where

$$I_{2j\text{-}horizontal} g_j(\varepsilon, x) := \frac{-4}{\varepsilon Z_j(\varepsilon, x)} \int_{-1/8}^{1/8} \int_{x^- + \varepsilon t}^{x^- + \mu} \frac{Z_j(\varepsilon, \xi) g_j(\varepsilon, \xi)}{\lambda_j (1 - e^{(2i\pi/\varepsilon)(\xi - x)})} \, d\xi dt$$

and

$$I_{2j\text{-}ascending} g_j(\varepsilon, x) := \frac{-4}{\varepsilon Z_j(\varepsilon, x)} \int_{-1/8}^{1/8} \int_{x^- + \mu}^{x - \varepsilon/2} \frac{Z_j(\varepsilon, \xi) g_j(\varepsilon, \xi)}{\lambda_j (1 - e^{(2i\pi/\varepsilon)(\xi - x)})} \, d\xi dt.$$

i – First, we consider $I_{2j\text{-}horizontal}$. We have: $\operatorname{Im} \xi = \operatorname{Im} x^- + \varepsilon t = \operatorname{Im} x^-$ and $\operatorname{Im}(x - \xi) = \operatorname{Im}(x - x^-) > \varepsilon c/8$. Then

$$\left| \frac{1}{1 - e^{(2i\pi/\varepsilon)(\xi - x)}} \right| \le \frac{e^{\pi c/4}}{e^{\pi c/4} - 1} \le \frac{1}{\frac{\pi c}{8}} < \frac{4}{c}.$$

Since equations (24), (25) still hold in this case, $I_{2j\text{-}horizontal}$ is bounded.

ii – Next, we consider $I_{2j\text{-}ascending}$. Since $(\operatorname{Im} \tilde{x} - \operatorname{Im} \tilde{\xi}) \ge c|\tilde{x} - \tilde{\xi}|$ for $\tilde{x}$, $\tilde{\xi}$ defined above, we have:

$$\begin{aligned} \left| \frac{1}{1 - e^{(2i\pi/\varepsilon)(\xi - x)}} \right| &= \mathcal{O}\left(\exp\left(\frac{2\pi}{\varepsilon} \operatorname{Im}(\xi - x) \right) \right) \\ &= \mathcal{O}\left(\exp\left(\frac{2\pi}{\varepsilon} \operatorname{Im}(\tilde{\xi} - \tilde{x}) \right) \right) \\ &\le \exp\left(-\frac{2\pi c}{\varepsilon} (|\tilde{x}| - |\tilde{\xi}|) \right). \end{aligned}$$

Here also equations (24), (25) remain valid. Then,

$$\begin{aligned} I_{2j\text{-}ascending} &\le \frac{\|g_j\|}{\varepsilon \min_j |\lambda_j|} \int_{x^-}^{\tilde{x}} \exp\left(-\left(\frac{2\pi c + \alpha}{\varepsilon} \right) (|\tilde{x}| - |\tilde{\xi}|) \right) d(|\tilde{\xi}|) \\ &\le \frac{\|g_j\|}{(2\pi c + \alpha) \min_j |\lambda_j|} < \infty. \end{aligned}$$

This shows that I_{2j} is bounded.

• Similarly, we can show that I_{3j} (see equation (23)) is also bounded (Indeed the same estimations hold except that α has to be replaced by $\tilde{\alpha}$ depending on $\tilde{\Phi}_j$ instead of Φ_j).

Case 2: The above estimations still hold for I_{3j}. However those of I_{1j} and I_{2j} fail and the above path $\gamma_{x,t}^{-}$ is no longer sufficient for the estimates. Some *polygonal path* must be instead considered. We choose this paths exactly as in [7], and we can show the boundedness of I_{1j} and I_{2j} in the same way. We refer the reader there for a detailed discussion.

Case 3: Here the estimations of case 1 still hold except that of I_{3j}. Also, some *polygonal path* must be considered exactly as in [7] and we refer the reader there for a detailed discussion. This shows that $T_{\varepsilon j}$ (see equation (20)) is bounded for all $j = 1, \dots, n$ on $\mathscr{H}_b(\Omega_\varepsilon)$. Then T_ε is also bounded on $\mathscr{H}_b^n(\Omega_\varepsilon)$ and $\|T_\varepsilon g\| \le L\|g\|$, with $\|g\| = \max_j |g_j|$.

Corollary 8. *Equation* (14) *has an analytic solution* $(y_j(\varepsilon, x))_{j=1}^n$ *bounded on* $\mathscr{H}_b^n(\Omega_\varepsilon)$ *given by* $(y_j(\varepsilon, x))_{j=1}^n = (T_{\varepsilon j}(g_j))_{j=1}^n$.

7. Proof of Theorem 5

For simplicity, we consider $\varepsilon > 0$. In equation (1), we write $F(\varepsilon, x, y(\varepsilon, x)) = A(\varepsilon, x)y(\varepsilon, x) + f(\varepsilon, x) + H(\varepsilon, x, y(\varepsilon, x))$, where $A(\varepsilon, x) = (\partial F/\partial y)(\varepsilon, x, 0)$, $f(\varepsilon, x) = F(\varepsilon, x, 0)$, and H is the remaining part of F. To solve equation (1), we rewrite it as a fixed point equation on a certain closed subset of a Banach space to be constructed later. This invokes the use of the right inverse T_ε that we constructed earlier. Let δ be some positive number $< 1/(6L)$ (recall that $L = \|T_\varepsilon\|$). There exists a *constant* matrix P such that $B(\varepsilon, x) = P^{-1}A(\varepsilon, x)P$, and that $B(0,0) = bldiag(B_1(0,0), \dots, B_\nu(0,0))$, $\nu \le n$ has block diagonal form with $B_j(0,0) = \lambda_j I_j + N_j$,

$$N_j = \begin{pmatrix} 0 & \delta & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ & & & & 0 \\ & & & \ddots & \delta \\ 0 & \cdots & & & 0 \end{pmatrix}.$$

By the transformation $y(\varepsilon, x) = PY(\varepsilon, x)$, equation (1) becomes:

$$\begin{aligned} Y(\varepsilon, x+\varepsilon) &= B(\varepsilon, x)Y(\varepsilon, x) + P^{-1}f(\varepsilon, x) + P^{-1}H(\varepsilon, x, PY(\varepsilon, x)) \\ &= DY(\varepsilon, x) + (B(\varepsilon, x) - D)Y(\varepsilon, x) + \tilde{f}(\varepsilon, x) + \tilde{H}(\varepsilon, x, Y(\varepsilon, x)), \end{aligned}$$

where $D = diag(\lambda_1, \dots, \lambda_n)$. This gives a system:

$$\gamma_{\varepsilon j} Y_j(\varepsilon, x) = [B(\varepsilon, x) - DY(\varepsilon, x)]_j + \tilde{f}_j(\varepsilon, x) + \tilde{H}_j(\varepsilon, x, Y(\varepsilon, x)), \tag{27}$$

where $j = 1, \dots, n$, and $\gamma_{\varepsilon j} Y_j(\varepsilon, x) = Y_j(\varepsilon, x+\varepsilon) - \lambda_j Y_j(\varepsilon, x)$. Note here that $\tilde{H}_j$ is a function of ε, x and $Y = (Y_1, \dots, Y_n)$. We denote the right hand side of equation (27) by Q_j and our system is now

$$\gamma_{\varepsilon j} Y_j(\varepsilon, x) = Q_j(\varepsilon, x, Y_1(\varepsilon, x), \dots, Y_n(\varepsilon, x)) =: Q_j(\varepsilon, x, Y).$$

Earlier, we constructed a right inverse $T_{\varepsilon j}$ of $\gamma_{\varepsilon j}$ on the Banach space $\mathscr{H}_b(\Omega_\varepsilon)$. So, it is sufficient to solve

$$Y(\varepsilon, x) = T_\varepsilon Q(\varepsilon, x, Y),$$

where $Q = (Q_1, \dots, Q_n)$, $T_\varepsilon h = (T_{\varepsilon 1} h_1, \dots, T_{\varepsilon n} h_n)$ and $\|T_\varepsilon h\| \le L\|h\|$, with $\|h\| = \max_j |h_j|$. We want to show that $T_\varepsilon Q$ is a contraction on a certain closed subset of a Banach space $\mathscr{B}$. Let $\mathscr{B}$ be the set of all functions $Y(\varepsilon, x)$ (with values in $\boldsymbol{C}^n$) analytic and bounded for $x \in \Omega$ and $\varepsilon \in]0, \varepsilon_1]$, where $0 < \varepsilon_1 < \varepsilon_0$ is to be determined. Considering the maximum norm of Y and with the usual addition and scalar multiplication, $\mathscr{B}$ becomes a Banach space. Next, let M be the closed subset of $\mathscr{B}$ of all $Y(\varepsilon, x)$ satisfying $\|Y\| \le \rho$ (to be determined) and such that $\tilde{H}(\varepsilon, x, Y(\varepsilon, x))$ is defined for $x \in \Omega$, $\varepsilon \in]0, \varepsilon_1]$, $\|Y\| \le \rho$. Note that M is non empty. Now, we show that $T_\varepsilon Q$ has a fixed point on M if ρ, Ω are sufficiently small. We have

$$Q(\varepsilon, x, W) - Q(\varepsilon, x, V) = \left(\int_0^1 \frac{\partial Q}{\partial Y}(\varepsilon, x, V + t(W - V))dt\right)(W - V).$$

Recall that

$$Q_j(\varepsilon, x, Y) = [B(\varepsilon, x) - D]_j Y(\varepsilon, x) + \tilde{f}_j(\varepsilon, x) + \tilde{H}_j(\varepsilon, x, Y(\varepsilon, x)).$$

We can choose our domain Ω such that the Lipschitz constant of $\tilde{H}(\varepsilon, x, Y(\varepsilon, x))$ is not too large and $\max_{\varepsilon, x}\|B(\varepsilon, x) - B(0,0)\| \le \delta$. Then, we have:

$$\left|\frac{\partial Q}{\partial Y}(\varepsilon, x, V + t(W - V))dt\right| \le \max_{\varepsilon, x} \|B(\varepsilon, x) - D\| + k(\zeta),$$

with $\zeta = \sup\|\tilde{y}\|$ and some function $k(\zeta)$ that tends to 0 as $\zeta \to 0$ (Indeed $k(\zeta) \le C\rho$ for some $C > 0$). Choosing $\rho < \delta/C$, it follows then by our choice of $\delta < 1/(6L)$ that

$$\|Q(\varepsilon, x, W) - Q(\varepsilon, x, V)\| < \frac{1}{2L}\|W - V\| \qquad \text{and hence} \qquad \|TQ(W - V)\| \le \frac{1}{2}.$$

Next, as $Q(0, x, 0) = 0$ for all $x \in \Omega$, we can also choose ε_1 such that for $0 < \varepsilon < \varepsilon_1$, $\|Q(\varepsilon, x, 0)\| \le \rho/(2L)$. This implies that $T_\varepsilon Q : M \to M$ is a contraction with a contraction factor at most $1/2$. Thus there exists $Y \in M$

satisfying $T_\varepsilon QY = Y$, namely there exists y an analytic solution of equation (1). It remains to show that this analytic solution y differs from the quasi-solution, constructed by taking the Borel Laplace transform of $\hat{y}$, by an exponentially small function, and that it is asymptotic to $\hat{y}$. To see this, note that $d(\varepsilon, x) = y(\varepsilon, x) - \tilde{y}(\varepsilon, x)$ satisfies

$$d(\varepsilon, x + \varepsilon) = G(\varepsilon, x, d(\varepsilon, x)), \tag{28}$$

with

$$G(\varepsilon, x, d(\varepsilon, x)) = F(\varepsilon, x, \tilde{y}(\varepsilon, x) + d(\varepsilon, x)) - F(\varepsilon, x, \tilde{y}(\varepsilon, x)) - \tilde{R}(\varepsilon, x),$$

where $\tilde{R}(\varepsilon, x)$ is the exponential function in Theorem 4. Moreover, $d(\varepsilon, x) = 0$ is a quasi-solution of equation (28), i.e. $G(\varepsilon, x, 0)$ is exponentially small. In the same way as we did for equation (1) of y, we can show that equation (28) has an analytic solution d. Since $G(\varepsilon, x, 0)$ is also exponentially small, we can show in addition that equation (28) has indeed an analytic solution d that is *exponentially small.* In fact, it suffices to apply the fixed point theorem on a Banach space as we did above for the y equation but where the supremum norm is replaced by the weighted norm $\|y\| = \sup_{\varepsilon \in V, x \in \Omega_\varepsilon} |y(\varepsilon, x)| e^{\alpha/|\varepsilon|}$. Finally, since exponentially small functions are Gevrey-1, it follows that the analytic solution $y(\varepsilon, x)$ of equation (1) is also Gevrey-1 and that its asymptotic expansion is the formal solution $\hat{y}(\varepsilon, x)$.

Acknowledgements. The author would like to thank her advisor Reinhard Schäfke for all his continuous support and help. She also would like to thank Boele Braaksma and Augustin Fruchard for their remarks. Finally, she wishes to thank the National Council of Scientific Research in Lebanon because most of this work was done when the author was a scholarship holder of the latter organism.

References

[1] Baesens, C., Gevrey series and dynamic bifurcations for analytic slow-fast mappings, Nonlinearity, **8** (1995), 179–201.

[2] Canalis-Durand, M., Solutions Gevrey d'équations différentielles singulièrement perturbées, preprint Université d'Aix-Marseille III, 1999.

[3] Canalis-Durand, M., Ramis, J.-P., Schäfke, R. and Sibuya, Y., Gevrey solutions of singularly perturbed differential equations, J. Reine Angew. Math., **518** (2000), 95–129.

[4] El-Rabih, A., Canards solutions of difference equations with small step size, J. of Diff. Eq. Appl., **9** (2003), 911–931.

[5] Fruchard, A., The sum of a function, Analysis, **16** (1996), 65–88.

[6] Fruchard, A. and Schäfke, R., Exponentially small splitting of separatrices for difference equations with small step size, J. Dynam. Control Systems, **2** (1996), 193–238.

[7] Fruchard, A. and Schäfke, R., Analytic solutions of difference equations with small step size, J. of Difference Equations and Applications, **7** (2001), 651–684.

[8] Fruchard, A. and Schäfke, R., Bifurcation delay and difference equations, Nonlinearity, **16** (2003), 2199–2220.

[9] Ramis, J.-P., Dévissage Gevrey, Astérisque (Soc. Math. France), **59–60** (1978), 173–204.

[10] Sibuya, Y., Sur réduction analytique d'un système d'équations différentielles ordinaires linéaires contenant un paramètre, J. Fac. Sci., Univ. Tokyo, I, **7** (1958), 527–540.

[11] Sibuya, Y., A theorem concerning uniform simplification at a transition point and the problem of resonance, SIAM J. Math. Anal., **2** (1981), 653–668.

[12] Sibuya, Y., Gevrey property of formal solutions in a parameter, Preprint School of Mathemetics, University of Minnesota, Minneapolis, 1989.

[13] Wasow, W., *Asymptotic expansions for ordinary differential equations*, Interscience, New York, 1965.

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(Ricevita la 11-an de majo, 2004)
(Reviziita la 21-an de februaro, 2005)