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Existence of Periodic Solutions for a Neutral Differential Equation with Piecewise Constant Argument

By

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Abstract. By using Mawhin's continuation theorem, the existence of periodic solutions for a neutral differential equation with piecewise constant argument is studied.

Keywords and Phrases. Neutral differential equation, Piecewise constant argument, Periodic solution.

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1. Introduction

First order delay differential equations with piecewise constant arguments were initiated by Cooke and Wiener [1] and Shah and Wiener [2]. As mentioned in [1–6], the strong interest in such equations is motivated by the fact that they represent a hybrid of continuous and discrete dynamical systems and combine the properties of both differential and difference equations.

These equations are now the subject of many investigations (see, e.g. [1–22]). In this paper, we will study one equation such that the rate of change of the unknown function also depends on its rate of change in the past or future. More specifically, we are concerned with the existence of periodic solutions of a delay neutral differential equation with piecewise constant argument of the form

$$(x(t) + cx(t+\sigma))' = f\left(t, x(t), x\left(\left[t + \frac{1}{2}\right]\right)\right), \tag{1}$$

where c is a real number different from -1 and 1, and $\sigma = l/2$ where l is an integer. Here, $[\cdot]$ is the greatest-integer function, and $f(t, x_1, x_2)$ is a real continuous vector function defined on $\boldsymbol{R}^3$ such that for some positive integer ω,

$$f(t+\omega, x_1, x_2) = f(t, x_1, x_2), \qquad (t, x_1, x_2) \in \boldsymbol{R}^3.$$

There are reasons for studying equations of the form (1). Indeed, as mentioned in [10], the term $f(t, x(t), x([t+1/2]))$ may be interpreted as the control term in the stabilization of hybrid control systems with advanced and delayed feedback, where a hybrid system is one with a continuous plant and with a discrete (sampled) controller. Furthermore, when the neutral term is missing in (1), the periodic solutions have stimulated considerable interest and have been studied in [6, 9, 11, 17].

By a solution of (1) we mean a function $x(t)$ which satisfies the conditions (i) $x(t)$ is continuous on $\boldsymbol{R}$; (ii) the function $x(t) + cx(t+\sigma)$ is differentiable at each point $t \in \boldsymbol{R}$ with the possible exception at the points $t = n - 1/2$ (n is an integer), where one-sided derivatives exist; and (iii) substitution of $x(t)$ into Eq. (1) leads to an identity on each interval $[n - 1/2, n + 1/2) \subset \boldsymbol{R}$ with integral endpoints.

In this note, existence criteria for periodic solutions of (1) will be established. For this purpose, we will make use of a continuation theorem of Mawhin. Let X and Y be two Banach spaces and $L : \operatorname{Dom} L \subset X \to Y$ is a linear mapping and $N : X \to Y$ a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim}\operatorname{Im} L < +\infty$, and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero, there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q = \operatorname{Im}(I - Q)$. It follows that $L_{|\operatorname{Dom} L \cap \operatorname{Ker} P} : (I - P)X \to \operatorname{Im} L$ has an inverse which will be denoted by K_P. If Ω is an open and bounded subset of X, the mapping N will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I - Q)N(\overline{\Omega})}$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$ there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Theorem A (Mawhin's continuation theorem [19]). *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose*

(i) *for each $\lambda \in (0,1)$ and $x \in \partial\Omega$, $Lx \neq \lambda Nx$; and*

(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$ and $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.*

Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Furthermore, let $H : X \to X$ be a linear homeomorphism such that $H|_{\operatorname{Dom} L} : \operatorname{Dom} L \to \operatorname{Dom} L$ is also a linear homeomorphism. Then we have the following result.

Theorem 1. *Let L be a Fredholm mapping of index zero, and let N be L-compact on $\overline{\Omega}$. Suppose*

(i) *for each $\lambda \in (0,1)$, $x \in \partial\Omega$, $LHx \neq \lambda Nx$; and*

(ii) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$ and $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$.*

Then the equation $LHx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$.

Proof. Let $N_1 = NH^{-1}$ and $\Omega_1 = H\Omega$. Since Ω is an open and bounded subset of X and $H : X \to X$ is a linear homeomorphism, it follows that Ω_1 is an open and bounded subset of X. So from condition (i) and the fact that $\partial\Omega_1 = H\partial\Omega$, we see that for each $\lambda \in (0,1)$ and $y \in \partial\Omega_1$, $Ly \neq \lambda N_1 y$.

Next, we will prove that for each $y \in \partial\Omega_1 \cap \operatorname{Ker} L$, $QN_1 y \neq 0$ and $\deg(JQN_1, \Omega_1 \cap \operatorname{Ker} L, 0) \neq 0$. Indeed, it is easy to see that $H(\operatorname{Ker} L) = \operatorname{Ker} L$. Note that $\Omega_1 = H\Omega$, for each $y \in \partial\Omega_1 \cap \operatorname{Ker} L$, there is a $x \in \partial\Omega \cap \operatorname{Ker} L$ such that $y = Hx$. From the condition (ii), we have $QN_1 y = QNH^{-1}y = QNH^{-1}Hx = QNx \neq 0$. In view of $\deg(JQN, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, we know that $\deg(JQN_1, \Omega_1 \cap \operatorname{Ker} L, 0) \neq 0$.

It follows from N being L-compact on $\overline{\Omega}$ that $QN(\overline{\Omega})$ is bounded and $\overline{K_P(I-Q)N(\overline{\Omega})}$ is relatively compact. Thus, $QN_1(\overline{\Omega}_1) = QNH^{-1}(H\overline{\Omega}) = QN(\overline{\Omega})$ is bounded and $\overline{K_P(I-Q)N_1(\overline{\Omega}_1)} = \overline{K_P(I-Q)NH^{-1}(H\overline{\Omega})} = \overline{K_P(I-Q)N(\overline{\Omega})}$ is relatively compact. Since N_1 is L-compact on $\overline{\Omega}$ and L is a Fredholm mapping of index zero, by Theorem A, the equation $Ly = N_1 y$ has at least one solution in $\overline{\Omega}_1 \cap \operatorname{Dom} L$. That is, the equation $LHx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$. The proof is complete.

2. Existence criteria

We will establish existence criteria based on combinations of the following conditions, where D and M are positive constants:

(a_1) $f(t, x_1, x_2) > 0$ for $t \in \boldsymbol{R}$ and $x_1 \geq D$, $x_2 \geq D$,
(a_2) $f(t, x_1, x_2) < 0$ for $t \in \boldsymbol{R}$ and $x_1 \geq D$, $x_2 \geq D$,
(b_1) $f(t, x_1, x_2) < 0$ for $t \in \boldsymbol{R}$ and $x_1 \leq -D$, $x_2 \leq -D$,
(b_2) $f(t, x_1, x_2) > 0$ for $t \in \boldsymbol{R}$ and $x_1 \leq -D$, $x_2 \leq -D$,
(c_1) $f(t, x_1, x_2) \geq -M$ for $(t, x_1, x_2) \in \boldsymbol{R}^3$,
(c_2) $f(t, x_1, x_2) \leq M$ for $(t, x_1, x_2) \in \boldsymbol{R}^3$.

Theorem 2. *Suppose either one of the following set of conditions hold*:
(i) (a_1), (b_1) *and* (c_1), *or,*
(ii) (a_2), (b_2) *and* (c_1), *or,*
(iii) (a_1), (b_1) *and* (c_2), *or,*
(iv) (a_2), (b_2) *and* (c_2).
Then (1) *has an* ω*-periodic solution.*

We only give the proof in case (a_1), (b_1) and (c_1) hold, since the other cases can be treated in similar manners.

First we make the simple observation that $x(t)$ is an ω-periodic solution of the following equation

$$(2) \quad x(t) + cx(t+\sigma) = x(0) + cx(0+\sigma) + \int_0^t f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds, \quad t \in \boldsymbol{R},$$

if and only if $x(t)$ is an ω-periodic solution of (1).

Let X_ω be the Banach space of all real ω-periodic continuous functions of the form $x = x(t)$ which is defined on $\boldsymbol{R}$ and endowed with the usual linear structure as well as the norm $\|x\|_1 = \max_{0 \le t \le \omega}|x(t)|$. Let Y_ω be the Banach space of all real continuous functions of the form $y = \alpha t + h(t)$ such that $y(0) = 0$, where $\alpha \in \boldsymbol{R}$ and $h(t) \in X_\omega$, endowed with the usual linear structure as well as the norm $\|y\|_2 = |\alpha| + \|h\|_1$. Let the zero element of X_ω and Y_ω be denoted by θ_1 and θ_2 respectively.

Define the mappings $L : X_\omega \to Y_\omega$ and $N : X_\omega \to Y_\omega$ respectively by

$$Lx(t) = x(t) - x(0), \qquad t \in \boldsymbol{R}. \tag{3}$$

and

$$Nx(t) = \int_0^t f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds, \qquad t \in \boldsymbol{R}. \tag{4}$$

Let

(5)

$$\bar{h}(t) = \int_0^t f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds - \frac{t}{\omega}\int_0^\omega f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds, \qquad t \in \boldsymbol{R}.$$

Since $\bar{h} \in X_\omega$ and $\bar{h}(0) = 0$, N is a well-defined operator from X_ω into Y_ω. Let us define $P : X_\omega \to X_\omega$ and $Q : Y_\omega \to Y_\omega$ respectively by

$$Px(t) = x(0), \qquad t \in \boldsymbol{R}, \tag{6}$$

for $x = x(t) \in X_\omega$ and

$$Qy(t) = \alpha t, \qquad t \in \boldsymbol{R}, \tag{7}$$

for $y(t) = \alpha t + h(t) \in Y_\omega$. Furthermore, let us define $H : X_\omega \to X_\omega$ by

$$Hx(t) = x(t) + cx(t + \sigma), \qquad t \in \boldsymbol{R}. \tag{8}$$

Lemma 1. *Let the mapping H be defined by* (8). *Then $H : X_\omega \to X_\omega$ is a linear homeomorphism.*

Proof. First of all, H is clearly linear. We will prove that $H : X_\omega \to X_\omega$ is an one to one and onto mapping. To see this, it suffices to show that for each $u = u(t) \in X_\omega$, there is a unique $x = x(t) \in X_\omega$ such that

$$u(t) = x(t) + cx(t + \sigma), \qquad t \in \boldsymbol{R}. \tag{9}$$

In case $|c| < 1$, it is easy to see that $\sum_{i=0}^{\infty}(-1)^i c^i u(t + i\sigma)$ is uniformly convergent on compact intervals of R. If we define $x(t)$ by

$$x(t) = \sum_{i=0}^{\infty}(-1)^i c^i u(t + i\sigma), \qquad t \in \boldsymbol{R}, \tag{10}$$

then $x = x(t) \in X_\omega$. Furthermore, it is not difficult to check that $x(t)$ satisfies (9). Similarly, in case $|c| > 1$, if we define $x(t)$ as

$$x(t) = \sum_{i=0}^{\infty}(-1)^i \left(\frac{1}{c}\right)^{i+1} u(t-(i+1)\sigma), \qquad t \in \boldsymbol{R}, \tag{11}$$

then $x = x(t) \in X_\omega$ and (9) holds.

To show uniqueness, let $y = y(t) \in X_\omega$ which also satisfies

$$u(t) = y(t) + cy(t+\sigma), \qquad t \in \boldsymbol{R}. \tag{12}$$

From (9), (12) and the fact that $x(t)$ and $y(t)$ are ω-periodic, we see that for any $t \in \boldsymbol{R}$,

$$|x(t) - y(t)| = |c|\,|x(t+\sigma) - y(t+\sigma)|. \tag{13}$$

By (13) and the fact that $x(t)$ and $y(t)$ are ω-periodic, we have

$$\begin{aligned} \max_{0 \le t \le \omega} |x(t) - y(t)| &= \sup_{t \in \boldsymbol{R}} |x(t) - y(t)| \\ &= |c| \sup_{t \in \boldsymbol{R}} |x(t+\sigma) - y(t+\sigma)| \\ &= |c| \, \max_{0 \le t \le \omega} |x(t) - y(t)|. \end{aligned} \tag{14}$$

Since $|c| \neq 1$, (14) implies $x(t) = y(t)$ for $t \in \boldsymbol{R}$.

Next, we will prove that H and H^{-1} are continuous mappings. By the definition of H, for any $x = x(t) \in X_\omega$, we have

$$|1 - |c|\,|\,\|x\|_1 \le \|Hx\|_1 \le (1 + |c|)\|x\|_1. \tag{15}$$

From the second inequality of (15), we see that H is a continuous mapping. On the other hand, from the first inequality of (15) and the fact that $|c| \neq 1$, for any $x = x(t) \in X_\omega$, we have

$$\|H^{-1}x\|_1 \le \frac{1}{|1 - |c|\,|}\|x\|_1. \tag{16}$$

Therefore H^{-1} is also a continuous mapping. The proof is complete.

Lemma 2. *Let the mapping L be defined by* (3). *Then*

$$\operatorname{Dom} L = X_\omega \qquad \textit{and} \qquad \operatorname{Ker} L = \boldsymbol{R}. \tag{17}$$

The proof is omitted since it can be verified in a straightforward manner.

Lemma 3. *Let the mapping* L *be defined by* (3). *Then*

$$\operatorname{Im} L = \{ y \in X_\omega \mid y(0) = 0 \} \subset Y_\omega. \tag{18}$$

It suffices to show that for each $y = y(t) \in X_\omega$ that satisfies $y(0) = 0$, there is a $x = x(t) \in X_\omega$ such that

$$y(t) = x(t) - x(0), \qquad t \in \boldsymbol{R}. \tag{19}$$

Indeed, let $x = x(t) = y(t) \in X_\omega$. Then it may easily be checked that (19) holds.

Lemma 4. *The mapping* L *defined by* (3) *is a Fredholm mapping of index zero.*

Indeed, from Lemma 2, Lemma 3 and the definition of Y_ω, $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = 1 < +\infty$. From (18), we see that $\operatorname{Im} L$ is closed in Y_ω. Hence L is a Fredholm mapping of index zero.

Lemma 5. *Let the mapping* L, P *and* Q *be defined by* (3), (6) *and* (7) *respectively. Then* $\operatorname{Im} P = \operatorname{Ker} L$ *and* $\operatorname{Im} L = \operatorname{Ker} Q$.

Indeed, from Lemma 2, Lemma 3 and the defining conditions (6) and (7), it is easy to see that $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Im} L = \operatorname{Ker} Q$.

Lemma 6. *Let* L *and* N *be defined by* (3) *and* (4) *respectively. Suppose* Ω *is an open and bounded subset of* X_ω. *Then* N *is* L*-compact on* $\overline{\Omega}$.

Proof. It is easy to see that for any $x \in \overline{\Omega}$,

$$QNx(t) = \frac{t}{\omega}\int_0^\omega f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds, \tag{20}$$

so,

$$\|QNx\|_2 = \left|\frac{1}{\omega}\int_0^\omega f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds\right|, \tag{21}$$

and

$$\begin{aligned}(I - Q)Nx(t) = &\int_0^t f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds \\ &- \frac{t}{\omega}\int_0^\omega f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds, \qquad t \in \boldsymbol{R}.\end{aligned} \tag{22}$$

These lead us to

$$K_P(I-Q)Nx(t) = \int_0^t f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)ds - \frac{t}{\omega}\int_0^\omega f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)ds, \qquad t \in \boldsymbol{R}. \tag{23}$$

By (21), we see that $QN(\overline{\Omega})$ is bounded. Noting that (7) holds and N is a completely continuous mapping, by means of the Arzela-Ascoli theorem we know that $\overline{K_P(I-Q)N(\overline{\Omega})}$ is relatively compact. Thus N is L-compact on $\overline{\Omega}$. The proof is complete.

Lemma 7. *Suppose $g(t)$ is a real, bounded and continuous function on $[a,b)$ and $\lim_{t\to b^-} g(t)$ exists. Then there is a point $\xi \in (a,b)$ such that*

$$\int_a^b g(s)ds = g(\xi)(b-a). \tag{24}$$

The above result is only a slight extension of the integral mean value theorem and is easily proved.

Lemma 8. *Suppose conditions (a_1) and (b_1) hold. Suppose further that $x(t) \in X_\omega$ satisfies*

$$\int_0^\omega f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)ds = 0. \tag{25}$$

Then there is $t_1 \in [0,\omega]$ such that $|x(t_1)| < D$.

Proof. From (25) and Lemma 7, we have $\xi_i \in (i-1/2, i+1/2) \cap (0,\omega)$ for $i = 0, 1, \dots, \omega$ such that

$$\begin{aligned}\sum_{i=0}^{\omega} f(\xi_i, x(\xi_i), x(i)) &= \int_0^{1/2} f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)ds \\ &\quad + \sum_{i=1}^{\omega-1}\int_{i-1/2}^{i+1/2} f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)ds \\ &\quad + \int_{\omega-1/2}^{\omega} f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)ds \\ &= \int_0^\omega f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)ds = 0.\end{aligned} \tag{26}$$

Our assertion is true if one of $x(\xi_i)$, $x(i)$, $i = 0, 1, \dots, \omega$, has absolute value less than D. Otherwise, by (a_1), (b_1) and (26), there should be $x(\eta_1)$ and $x(\eta_2)$ among $x(\xi_i)$, $x(i)$, $i = 0, 1, \dots, \omega$, such that $x(\eta_1) \geq D$ and $x(\eta_2) \leq -D$. Since

$x(t)$ is continuous, in view of the intermediate value theorem, there is $x(\eta_3)$ such that $-D < x(\eta_3) < D$, (here $\eta_1 > \eta_3 > \eta_2$ or $\eta_2 > \eta_3 > \eta_1$). Since $x(t)$ is periodic, there is $t_1 \in [0, \omega]$ such that $|x(t_1)| = |x(\eta_3)| < D$. The proof is complete.

Now, we consider the following auxiliary equation

(27)
$$x(t) + cx(t+\sigma) - x(0) - cx(0+\sigma) = \lambda \int_0^t f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds, \qquad t \in \boldsymbol{R},$$

where $\lambda \in (0,1)$.

Lemma 9. *Suppose* (a_1), (b_1) *and* (c_1) *are satisfied. Then there are positive constants* D_1 *such that for any* ω*-periodic solution* $x(t)$ *of* (27),

(28) $$|x(t)| \le D_1, \qquad t \in [0, \omega].$$

Proof. Let $x(t)$ be a ω-periodic solution of (27). By (27), we have

(29) $$\int_0^\omega f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds = 0.$$

By Lemma 8, there is $t_1 \in [0, \omega]$ such that

(30) $$|x(t_1)| < D.$$

If we write

(31) $$G_+(t) = \max\left\{ f\left(t, x(t), x\left(\left[t + \frac{1}{2}\right]\right)\right), 0\right\}, \qquad t \in \boldsymbol{R},$$

and

(32) $$G_-(t) = \max\left\{ -f\left(t, x(t), x\left(\left[t + \frac{1}{2}\right]\right)\right), 0\right\}, \qquad t \in \boldsymbol{R},$$

then $G_+(t)$ and $G_-(t)$ are piecewise continuous functions on $\boldsymbol{R}$, and that

(33) $$f\left(t, x(t), x\left(\left[t + \frac{1}{2}\right]\right)\right) = G_+(t) - G_-(t), \qquad t \in \boldsymbol{R},$$

as well as

(34) $$\left| f\left(t, x(t), x\left(\left[t + \frac{1}{2}\right]\right)\right)\right| = G_+(t) + G_-(t), \qquad t \in \boldsymbol{R}.$$

Thus, in view of (c_1) and (27), we have

(35) $$|G_-(t)| = G_-(t) \le M, \qquad t \in \boldsymbol{R},$$

and

$$\int_0^{\omega} G_-(s)ds \le \omega M. \tag{36}$$

From (29), (33) and (35), we have

$$\int_0^{\omega} G_+(s)ds = \int_0^{\omega} G_-(s)ds \le \omega M. \tag{37}$$

By (34) and (37), we further have

$$\int_0^{\omega} \left| f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)\right| ds \le 2\omega M. \tag{38}$$

From (27), for any t' and $t'' \in [0, \omega]$, we have

$$x(t') + cx(t'+\sigma) - x(t'') - cx(t''+\sigma) = \lambda \int_{t'}^{t''} f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right) ds. \tag{39}$$

Let $d = \sup_{u,v\in[0,\omega]} |x(u) - x(v)|$, $\alpha = \max_{0\le t\le\omega} x(t)$ and $\beta = \min_{0\le t\le\omega} x(t)$. It is easy to see that $d = \alpha - \beta$ and

$$\beta \le x(t_1) \le \alpha. \tag{40}$$

In case $|c| < 1$, by (38) and (39) and the fact that $x(t)$ is ω-periodic, for any $t', t'' \in [0, \omega]$, we have

$$\begin{aligned} |x(t') - x(t'')| &\le |c|\,|x(t'+\sigma) - x(t''+\sigma)| + \left| \int_{t'}^{t''} f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right) ds \right| \\ &\le |c|d + \int_0^{\omega} \left| f\left(s, x(s), x\left(\left[s+\frac{1}{2}\right]\right)\right)\right| ds \\ &\le |c|d + 2\omega M, \end{aligned}$$

thus,

$$d \le d_1, \tag{41}$$

where $d_1 = 2\omega M/(1 - |c|)$. In view of (30), (40) and (41), we have for any $t \in [0, \omega]$

$$\begin{aligned} -(D + d_1) &\le -|x(t_1)| - (\alpha - \beta) \le x(t_1) - (\alpha - \beta) \le \beta \le x(t) \le \alpha \\ &\le x(t_1) + (\alpha - \beta) \le |x(t_1)| + (\alpha - \beta) \le D + d_1, \end{aligned} \tag{42}$$

that is

$$\max_{0 \le t \le \omega} |x(t)| \le D_{11}, \tag{43}$$

where $D_{11} = D + d_1$.

In case $|c| > 1$, by (38), (39) and the fact that $x(t)$ is ω-periodic, for any $t', t'' \in [0, \omega]$, we have

$$\begin{aligned} |c|\,|x(t'+\sigma) - x(t''+\sigma)| & \\ &\le |x(t') - x(t'')| + \left| \int_{t'}^{t''} f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds \right| \\ &\le d + \int_0^{\omega} \left| f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) \right| ds \\ &\le d + 2\omega M. \end{aligned} \tag{44}$$

Thus,

$$d \le d_2, \tag{45}$$

where $d_2 = 2\omega M/(|c| - 1)$. Similarly, from (30), (40) and (45), we have

$$\max_{0 \le t \le \omega} |x(t)| \le D_{12} \tag{46}$$

where $D_{12} = D + d_2$. The proof is complete.

We now turn to the proof of Theorem 2. Let L, N, P and Q be defined by (3), (4), (6) and (7) respectively. By Lemma 9, there is a positive constant D_1 such that any ω-periodic solution $x(t)$ of (27) satisfies (28). Set

$$\Omega = \{x \in X_\omega \mid \|x\|_1 < \bar{D}\},$$

where $\bar{D}$ is a fixed number which satisfies $\bar{D} > D + D_1$. Ω is an open and bounded subset of X_ω. Furthermore, in view of Lemma 4 and Lemma 6, L is a Fredholm mapping of index zero and N is L-compact on $\bar{\Omega}$. Noting that $\bar{D} > D_1$, by Lemma 9, for each $\lambda \in (0,1)$ and $x \in \partial\Omega$, $LHx \ne \lambda Nx$. Next note that a function $x \in \partial\Omega \cap \operatorname{Ker} L$ must be constant: $x(t) \equiv \bar{D}$ or $x(t) \equiv -\bar{D}$. Hence by (a_1), (b_1) and (20),

$$QNx(t) = \frac{t}{\omega} \int_0^{\omega} f\left(s, x(s), x\left(\left[s + \frac{1}{2}\right]\right)\right) ds = \frac{t}{\omega} \int_0^{\omega} f(s, x, x) ds,$$

so

$$QNx \ne \theta_2.$$

The isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$ is defined by $J(t\alpha) = \alpha$ for $\alpha \in \boldsymbol{R}$ and $t \ge 0$. Then

$$JQNx = \frac{1}{\omega}\int_0^{\omega} f(s, x, x)ds \neq 0. \tag{47}$$

In particular, we see that if $x = \overline{D}$, then

$$JQNx = \frac{1}{\omega}\int_0^{\omega} f(s, \overline{D}, \overline{D})ds > 0, \tag{48}$$

and if $x = -\overline{D}$, then

$$JQNx = \frac{1}{\omega}\int_0^{\omega} f(s, -\overline{D}, -\overline{D})ds < 0. \tag{49}$$

Consider the mapping

$$h(x, s) = sx + (1 - s)JQNx, \qquad 0 \le s \le 1. \tag{50}$$

From (15) and (50), for each $s \in [0, 1]$ and $x = \overline{D}$, we have

$$h(x, s) = s\overline{D} + (1 - s)\frac{1}{\omega}\int_0^{\omega} f(s, \overline{D}, \overline{D})ds > 0, \tag{51}$$

Similarly, from (49) and (50), for each $s \in [0, 1]$ and $x = -\overline{D}$, we have

$$h(x, s) = -s\overline{D} + (1 - s)\frac{1}{\omega}\int_0^{\omega} f(s, -\overline{D}, -\overline{D})ds < 0. \tag{52}$$

By (51) and (52), $h(x, s)$ is a homotopy. This shows that

$$\deg(JQNx, \Omega \cap \operatorname{Ker} L, \theta_1) = \deg(x, \Omega \cap \operatorname{Ker} L, \theta_1) \neq 0.$$

By Theorem 1, we see that equation $LHx = Nx$ has at least one solution in $\overline{\Omega} \cap \operatorname{Dom} L$. In other words, (1) has an ω-periodic solution $x(t)$. The proof is complete.

3. Example

Consider the equation

$$\left(x(t) + cx\left(t + \frac{1}{2}\right)\right)' = \left(-3 - \cos \pi t + x(t) + x\left(\left[t + \frac{1}{2}\right]\right)\right) \times \exp\left(-3 - \cos \pi t + x(t) + x\left(\left[t + \frac{1}{2}\right]\right)\right), \tag{53}$$

where c is constant and $|c| \neq 1$. We can show that it has a nontrivial 2-periodic solution. Indeed, let $D > 2$ and $M = 1$, then the conditions in (i) of

Theorem 1 are satisfied. Therefore (53) has a 2-periodic solution. Furthermore, this solution is nontrivial since $f(t,0)$ is not identically zero.

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