Funkcialaj Ekvacioj, **48** (2005) 273–297

A q-Analog of the Garnier System

By

SAKAI Hidetaka
(University of Tokyo, Japan)

Abstract. A q-difference analog of the Garnier system is presented. It arises as the condition for preserving the connection matrix of linear q-difference equations, in close analogy with the monodromy preserving deformation of linear differential equations. The continuous limit is also studied.

Keywords and Phrases. Integrable system, q-Difference equation, Painlevé equations, Garnier system.

2000 *Mathematical Subject Classifications Numbers.* 33E17, 34M55, 39A12, 39A13.

1. Introduction

The purpose of this paper is to present a nonlinear q-difference system analogous to the Garnier system. The Garnier system is a nonlinear multi-variable system regarded as a natural extension of the sixth Painlevé equation.

Firstly we briefly look over the theory of the Garnier system. While P. Painlevé and coworkers discovered the six Painlevé equations as differential equations whose solutions do not have movable singularities other than poles ([11]), R. Fuchs derived the sixth Painlevé equation as an isomonodromic deformation equation of a linear differential equation which has 4 generic regular singular points ([2]). His result was extended by R. Garnier ([3]) for the case of $N+3$ generic singular points. Considering the equation of Fuchsian type whose Riemannian scheme is written as

$$\begin{pmatrix} x=0 & x=1 & x=t_j & x=u_k & x=\infty \\ \frac{1-\theta}{2} & \frac{1-\alpha}{2} & \frac{1-\alpha_j}{2} & -\frac{1}{2} & \frac{-1-\kappa_\infty}{2} \\ \frac{1+\theta}{2} & \frac{1+\alpha}{2} & \frac{1+\alpha_j}{2} & \frac{3}{2} & \frac{-1+\kappa_\infty}{2} \end{pmatrix}, \qquad j,k=1,\dots,N,$$

he obtained the nonlinear partial differential equations which arise as the conditions for preserving the monodromy of the Fuchsian equation. The system of equations is written as follows:

$$(1)\qquad \frac{\partial^2 u_k}{\partial t_i^2} = \frac{1}{2}\left[\frac{\mathscr{T}'(u_k)}{\mathscr{T}(u_k)} - \frac{1}{2}\frac{\mathscr{L}''(u_k)}{\mathscr{L}'(u_k)}\right]\left(\frac{\partial u_k}{\partial t_i}\right)^2 - \left[\frac{1}{2}\frac{\mathscr{T}''(t_i)}{\mathscr{T}'(t_i)} - \frac{\mathscr{L}'(t_i)}{\mathscr{L}(t_i)}\right]\frac{\partial u_k}{\partial t_i}$$

$$+\frac{1}{2}\sum_{l\neq k}\frac{\mathcal{T}(u_k)\mathcal{L}'(u_l)(u_l-t_i)^2}{\mathcal{T}(u_l)\mathcal{L}'(u_k)(u_k-t_i)^2(u_k-u_l)}\left(\frac{\partial u_l}{\partial t_i}\right)^2$$

$$-\sum_{l\neq k}\frac{u_k-t_i}{(u_l-t_i)(u_l-u_k)}\frac{\partial u_k}{\partial t_i}\frac{\partial u_l}{\partial t_i}+\frac{\mathcal{L}(t_i)^2\mathcal{T}(u_k)}{2\mathcal{T}'(t_i)^2\mathcal{L}'(u_k)(u_k-t_i)^2}I_{k,i},$$

$$\frac{\mathcal{T}'(t_i)(t_i-u_k)}{\mathcal{L}(t_i)}\frac{\partial u_k}{\partial t_i}-\frac{\mathcal{T}'(t_j)(t_j-u_k)}{\mathcal{L}(t_j)}\frac{\partial u_k}{\partial t_j}=\frac{(t_i-t_j)\mathcal{T}(u_k)}{(u_k-t_i)(u_k-t_j)\mathcal{L}'(u_k)}, \tag{2}$$

for $i,j,k=1,\dots,N$, where

$$\mathcal{T}(x)=x(x-1)\prod_{i=1}^{N}(x-t_i),$$

$$\mathcal{L}(x)=\prod_{k=1}^{N}(x-u_k),\qquad \left({}'=\frac{\mathrm{d}}{\mathrm{d}x}\right),$$

$$I_{k,i}=\kappa_\infty^2+\frac{\mathcal{T}'(0)}{\mathcal{L}(0)}\frac{\theta^2}{u_k}+\frac{\mathcal{T}'(1)}{\mathcal{L}(1)}\frac{\alpha^2}{u_k-1}+\sum_{j=1}^{N}\frac{\mathcal{T}'(t_i)}{\mathcal{L}(t_i)}\frac{\alpha_i^2-\delta_{ij}}{u_k-t_i}.$$

This system was found to be written in the form of a Hamiltonian system in the series of studies of Okamoto K. ([10]). He showed that isomonodromic deformation equation (1)–(2) is given by the system

$$\frac{\partial u_k}{\partial t_j}=\frac{\partial K_j}{\partial v_k},\qquad \frac{\partial v_k}{\partial t_j}=-\frac{\partial K_j}{\partial u_k},\qquad (j,k=1,\dots,N), \tag{3}$$

with the Hamiltonian

(4)

$$K_j=-\frac{\mathcal{L}(t_j)}{\mathcal{T}'(t_j)}\left[\sum_{l=1}^{N}\frac{\mathcal{T}(u_l)}{\mathcal{L}'(u_l)(u_l-t_j)}\left\{v_l^2-\left(\frac{\theta}{u_l}+\frac{\alpha}{u_l-1}+\sum_{i=1}^{N}\frac{\alpha_i-\delta_{ij}}{u_l-t_i}\right)v_l\right\}+\kappa\right],$$

where

$$\kappa=\frac{1}{4}\left[\left(\theta+\alpha+\sum_{j=1}^{N}\alpha_j-1\right)^2-\kappa_\infty^2\right].$$

Further progress was made by Kimura H. and Okamoto K. ([8]). Since system (3) has movable branch points, they transformed this system into a Hamiltonian system enjoying the Painlevé property, whose Hamiltonians are polynomials in canonical variables.

The canonical transformation is written as

$$s_j = \frac{t_j}{t_j - 1}, \qquad q_j = \frac{t_j \mathscr{L}(t_j)}{\mathscr{T}'(t_j)}, \qquad p_j = \sum_{k=1}^{N} \frac{(1 - t_j)\mathscr{T}(u_k)v_k}{u_k(u_k - 1)(u_k - t_j)\mathscr{L}'(u_k)},$$

and this transformation takes the system (3) to the system:

$$\frac{\partial q_k}{\partial s_j} = \frac{\partial H_j}{\partial p_k}, \qquad \frac{\partial p_k}{\partial s_j} = -\frac{\partial H_j}{\partial q_k}, \qquad (j, k = 1, \dots, N), \tag{5}$$

with the Hamiltonian

$$H_i = \frac{1}{s_i(s_i - 1)} \left[\sum_{j,k=1}^{N} E_{ijk}(s, q) p_j p_k - \sum_{j=1}^{N} F_{ij}(s, q) p_j + \kappa q_i \right]. \tag{6}$$

Here $E_{ijk}, F_{ij} \in \mathbf{C}(s)[q]$ are given by

$$E_{ijk} = E_{ikj} = \begin{cases} q_i q_j q_k, & \text{if } i, j, k \text{ are distinct}, \\ q_i q_j \left(q_j - \frac{s_j(s_i - 1)}{s_i - s_j} \right), & \text{if } j = k \neq i, \\ q_i q_j \left(q_i - \frac{s_i(s_j - 1)}{s_j - s_i} \right), & \text{if } k = i, j \neq i, \\ q_i(q_i - 1)(q_i - s_i) - \sum_{l \neq i} \frac{s_i(s_i - 1)}{s_i - s_l} q_i q_l, & \text{if } i = j = k, \end{cases}$$

$$F_{ij} = \begin{cases} (\theta + \alpha + \sum_{l=1}^{N} \alpha_l - 1) q_i q_j - \alpha_j \frac{s_j(s_i - 1)}{s_i - s_j} q_i - \alpha_i \frac{s_i(s_j - 1)}{s_j - s_i} q_j, & \text{if } i \neq j, \\ (\theta - 1) q_i(q_i - 1) + \alpha q_i(q_i - s_i) + \alpha_i(q_i - 1)(q_i - s_i) & \\ \qquad + \sum_{l \neq i} \left\{ \alpha_l q_i \left(q_i - \frac{s_i(s_l - 1)}{s_l - s_i} \right) - \alpha_i \frac{s_i(s_i - 1)}{s_i - s_l} q_l \right\}, & \text{if } i = j. \end{cases}$$

Moreover several examples of extensions of the Painlevé equations and their discretizations are known. Among them, we mention two important studies.

One is a guiding principle for integrable dynamical system, called singularity confinement. It was proposed by B. Grammaticos, A. Ramani et al. ([5, 13]) and it can be viewed as a discrete counterpart of the Painlevé property. They presented several types of discrete Painlevé equations discovered by using this method.

The second one comes from a representation theory. Noumi M. and Yamada Y. et al. constructed many types of integrable systems from representations of affine Weyl groups ([9, 7]). This is closely related to Drinfel'd-Sokolov hierarchy. However, among these integrable systems that they presented, discrete analogs of the Garnier system do not appear.

In the present article, we deal with a q-analog of the Garnier system. An origin of the Garnier system is isomonodromic deformation of linear differential equations of Fuchsian type. In a similar way we consider a deformation theory of linear q-difference equations.

As concerns linear q-difference equations, general theory was developed in classical works. In particular G. D. Birkhoff studied the generalized Riemann problem of linear q-difference equations in parallel with linear differential and difference equations ([1]). As its continuation we investigate a deformation theory of linear q-difference equations. We concern only the case that the coefficients are expressed in 2×2 matrices. Jimbo M. and the author derived q-P_{VI} system from the case of the matrix system with polynomial coefficients of degree 2 in paper [6]. (Examples of similar constructions are found in former papers [12, 4]. They are realized in 4×4-matrix system, but they fall within the frame work of [6]. See [6].) We deal with the case of degree $N+1$ in this article, as a generalization of [6].

We can take the zeros of determinat of the coefficient matrix, $\{a_i\}_{1=1}^{2N+2}$, as independent variables. We consider a simple deformation which moves (a_r, a_s) to (qa_r, qa_s) and leaves the other a_i's fixed. We denote this deformation $T_{r,s}$ $(r, s \in \{1, \dots, 2N+2\})$. As a result we find a nonlinear q-difference system associated with $T_{r,s}$, which we call q-Garnier system. It is written as

(7a)

$$\sum_{l=1}^{N} \frac{q\kappa_1 \overline{L(a_n)}\,\overline{T(\lambda_l)}}{(a_n - \overline{\lambda_l})(qa_r - \overline{\lambda_l})(qa_s - \overline{\lambda_l})\overline{L'(\lambda_l)}\overline{\mu_l}} = \sum_{l=1}^{N} \frac{\kappa_2 L(a_n)\mu_l}{(a_n - \lambda_l)(a_r - \lambda_l)(a_s - \lambda_l)L'(\lambda_l)},$$

$$\text{(7b)} \quad \frac{\overline{L(a_n)}}{L(a_n)} = -\frac{\sum_{l=1}^{N} \frac{\mu_l}{(a_n-\lambda_l)(a_r-\lambda_l)(a_s-\lambda_l)L'(\lambda_l)}}{\left(1 - \sum_{l=1}^{N} \frac{\mu_l}{(a_r-\lambda_l)(a_s-\lambda_l)L'(\lambda_l)}\right)\left(1 - \frac{\kappa_2}{q\kappa_1}\sum_{l=1}^{N} \frac{\mu_l}{(a_r-\lambda_l)(a_s-\lambda_l)L'(\lambda_l)}\right)}$$
$$\times \left(\frac{\theta_1 + \theta_2}{\kappa_1 L(0)} + a_n\left(1 + \frac{\kappa_2}{q\kappa_1}\right) + \sum_{l=1}^{N} \frac{a_n T(\lambda_l)}{(a_n - \lambda_l)\lambda_l L'(\lambda_l)\mu_l}\right.$$
$$\left. + \sum_{l=1}^{N} \frac{(\kappa_2/q\kappa_1)(qa_r a_s - a_n\lambda_l)\mu_l}{(a_r - \lambda_l)(a_s - \lambda_l)\lambda_l L'(\lambda_l)}\right),$$
$$(n \in \{1, 2, \dots, 2N+2\} \backslash \{r, s\}),$$

where $L(x) = \prod_{l=1}^{N}(x - \lambda_l)$ and $T(x) = \prod_{i=1}^{2N+2}(x - a_i)$.

This system has the Garnier system as its continuous limit (see Section 5).

We use the notations $\bar{f} = T_{r,s}(f)$, $\underline{f} = T_{r,s}^{-1}(f)$ and so forth. For detailed settings, see Section 4. These equations define the time evolution $(\lambda_k, \mu_k)_{k=1,\dots,N} \mapsto (\overline{\lambda_k}, \overline{\mu_k})_{k=1,\dots,N}$. In this process we need solve an algebraic equation of degree N and this is not a birational mapping. This is analogous to the fact that a solution of system (3), u_k, does not satisfy the Painlevé property.

By taking other variables, we can rewrite q-Garnier system to the form of a birational mapping;

$$\frac{\overline{z_{n_l}}}{z_r z_s}\left(\frac{z_{n_l}-z_r}{a_{n_l}-a_r}-\frac{z_{n_l}-z_s}{a_{n_l}-a_s}\right)=\left(1-\frac{(1-q\kappa_1/\kappa_2)(a_r-a_s)}{z_r-z_s}\right)\times\left(\frac{z_{n_l}-z_r}{a_{n_l}-a_r}\frac{1}{z_r}-\frac{z_{n_l}-z_s}{a_{n_l}-a_s}\frac{1}{z_s}\right), \tag{8a}$$

$$\overline{y_{n_l}}\left(1-\frac{(1-q\kappa_1/\kappa_2)(a_r-a_s)}{z_r-z_s}\right) = -y_{n_l}\frac{(a_{n_l}-qa_r)(a_{n_l}-qa_s)}{(z_r-z_s)^2}\left(\frac{z_{n_l}-z_r}{a_{n_l}-a_r}-\frac{z_{n_l}-z_s}{a_{n_l}-a_s}\right)\left(\frac{w_{n_l}+z_r}{a_{n_l}-qa_r}-\frac{w_{n_l}+z_s}{a_{n_l}-qa_s}\right), \tag{8b}$$

$$(l=0,1\ldots,N \text{ and } n_l\in\{1,2,\ldots,2N+2\}\backslash\{r,s\}).$$

In system (8), w_{n_l}, z_r and z_s are written by the rational function of $(y_{n_l},z_{n_l})_{l=0,1,\ldots,N}$. This system defines a time evolution $(y_{n_l},z_{n_l})_{l=0,1,\ldots,N}\mapsto(\overline{y_{n_l}},\overline{z_{n_l}})_{l=0,1,\ldots,N}$. Moreover we have two relations, that is integrals, (32) and (33). We can calculate the inverse as a rational map, so it is a birational mapping. For detailed settings see Section 6.

In Section 2 we recall known results concerning the analytic theory of linear q-difference equations. In Section 3 we treat deformations of them, and the compatibility condition between the original and the deformation equations leads to q-Garnier system in Section 4. We show in Section 5 that it reduces in the continuous limit $q\to 1$ to the Garnier system. In Section 6 we rewrite q-Garnier system to the form of a birational map. In Section 7 we study several directions of deformation and their relations.

2. Linear q-difference systems

In this section we recall classical theory of linear q-difference equations ([1]). Consider an $m\times m$ matrix system with polynomial coefficients

$$Y(qx)=A(x)Y(x),\qquad A(x)=A_0+A_1x+\cdots+A_Nx^N. \tag{9}$$

More general case of a rational $A(x)$ can be reduced to this case by solving scalar q-difference equations. Namely, if function $f(x)$ satisfies $f(qx)=(1/\prod_{i=1}^M(x-c_i))f(x)$, then the q-difference equation

$$\tilde{Y}(qx)=\frac{A(x)}{\prod_{i=1}^M(x-c_i)}\tilde{Y}(x)$$

have a solution $\tilde{Y}(x)=f(x)Y(x)$.

We assume A_0, A_N are semi-simple and invertible. Denoting by θ_h, κ_h $(1\le h\le m)$ the eigenvalues of A_0 and A_N respectively, we assume further that

$$\frac{\theta_h}{\theta_k}, \frac{\kappa_h}{\kappa_k} \notin \{q, q^2, q^3, \dots\} \qquad (1 \le h, k \le m).$$

We set $A_0 = C_0 q^{D_0} C_0^{-1}$ and $A_\infty = C_\infty q^{D_\infty} C_\infty^{-1}$, where $D_0 = \mathrm{diag}(\log \theta_r / \log q)$, $D_\infty = \mathrm{diag}(\log \kappa_r / \log q)$. Throughout this article we fix a complex number q such that $0 < |q| < 1$.

Proposition 1 ([1]). *Under the conditions above, there exist unique solutions* $Y_0(x)$, $Y_\infty(x)$ *of* (9) *with the following properties*:

(10a) $$Y_0(x) = \hat{Y}_0(x) x^{D_0},$$

(10b) $$Y_\infty(x) = q^{(N/2)\xi(\xi-1)} \hat{Y}_\infty(x) x^{D_\infty}, \qquad \xi = \frac{\log x}{\log q}.$$

Here $\hat{Y}_0(x)$ *(resp.* $\hat{Y}_\infty(x)$*) is a holomorphic and invertible matrix at* $x = 0$ *(resp. at* $x = \infty$*) such that* $\hat{Y}(0) = C_0$ *(resp.* $\hat{Y}_\infty(\infty) = C_\infty$*).*

The q-difference equation (9) entails that $\hat{Y}_\infty(x)^{\pm 1}$, $\hat{Y}_0(x)^{\pm 1}$ can be continued meromorphically in the domain $0 < |x| < \infty$. Unlike the case of Fuchsian linear differential equations on $\boldsymbol{P}^1$, the points $x = 0, \infty$ play distinguished roles in the q-difference systems. There are no branch points aside from $x = 0$ or ∞.

Furthermore we know where poles appear. Let a_i $(i = 1, \dots, mN)$ denote the zeroes of $\det A(x)$. $\hat{Y}_\infty(x)$ and $\hat{Y}_0(x)^{-1}$ have no poles, while $\hat{Y}_\infty(x)^{-1}$ and $\hat{Y}_0(x)$ are holomorphic except for possible poles at

(11a) $$\hat{Y}_\infty(x)^{-1} : q a_i, q^2 a_i, q^3 a_i, \dots,$$

(11b) $$\hat{Y}_0(x) : a_i, q^{-1} a_i, q^{-2} a_i, \dots.$$

It follows from the fact that

$$Y_0(x) = A(x)^{-1} Y_0(qx) = A(x)^{-1} A(qx)^{-1} A(q^2 x)^{-1} \dots,$$

$$Y_\infty(x) = A(q^{-1}x) Y_\infty(q^{-1}x) = A(q^{-1}x) A(q^{-2}x) A(q^{-3}x) \dots.$$

The connection matrix $P(x)$ is introduced by

(12) $$Y_\infty(x) = Y_0(x) P(x).$$

Equation (9) leads the relation $P(qx) = P(x)$. It is known to be expressible in terms of elliptic theta functions. It plays a role analogous to that of the monodromy matrices for differential equations.

3. Connection preserving deformation

In the theory of monodromy preserving deformation of linear differential equations, extra parameter t is introduced in the coefficient matrix and we describe the condition that the monodromy stay constant with respect to t. Analogously, in the setting of q-difference equations, one demands that the connection matrix stay pseudo-constant in t, namely that $P(x, qt) = P(x, t)$.

We restrict the subject to 2×2 matrix systems. In the differential case, this condition can be written by the existence of the deformation equations and the Garnier system is derived from the compatibility condition between the original and the deformation equations. Likewise, under appropriate conditions, it can be shown that $P(x, t)$ is pseudo-constant in t if and only if the corresponding solutions $Y(x, t) = Y_0(x, t), Y_\infty(x, t)$ satisfy

$$Y(x, qt) = B(x, t)Y(x, t), \tag{13}$$

where $B(x, t)$ is rational in x (see Proposition 2 below).

Now we have to introduce a deformation parameter t in coefficient $A(x)$. In the case of monodromy preserving deformation of Fuchsian equation, deformation parameter t_j's are configurations of regular singularities. But we have no branch points except $x = 0$ or $x = \infty$. The natural candidate for the deformation parameters are the exponents θ_j, κ_j at $x = 0, \infty$ and the zeroes of $\det A(x)$. We now take $A(x, t)$ to be of the form

$$A(x, t) = A_0 + A_1 x + \cdots + A_N x^N + A_{N+1} x^{N+1}, \tag{14}$$

$$A_{N+1} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \qquad A_0(t) \text{ has eigenvalues } \theta_1, \theta_2, \tag{15}$$

$$\det A(x, t) = \kappa_1 \kappa_2 \prod_{i=1}^{2N+2} (x - a_i). \tag{16}$$

Clearly we have

$$\kappa_1 \kappa_2 \prod_{i=1}^{2N+2} a_i = \theta_1 \theta_2. \tag{17}$$

Considering equation (13), it follows, from the configuration of possible poles, that a zero of $\det A(x)$, a_i, must move to $q^n a_i$ for some $n \in \boldsymbol{Z}$. Now we put $a_r = b_r t$, $a_s = b_s t$, $\theta_h = \psi_h t$ $(h = 1, 2)$; b_r, b_s, ψ_h $(h = 1, 2)$ and κ_h $(h = 1, 2)$, a_i $(i \in \{1, \dots, 2N+2\} \backslash \{r, s\})$ are constant in t. We denote this deformation by $T_{r,s}$, and $\bar{f} = T_{r,s}(f)$. We will consider relations among these $T_{r,s}$'s in Section 7.

In what follows we will normalize $Y_\infty(x)$ by $\hat{Y}_\infty(\infty) = I$.

Proposition 2. *We have $P(x, qt) = P(x, t)$ if and only if* (13) *holds for $Y = Y_0, Y_\infty$, where $B(x, t) = B_{r,s}(x, t)$ is a rational function of the form*

$$B_{r,s}(x, t) = \frac{x(xI + B_{r,s}^0(t))}{(x - qa_r)(x - qa_s)}. \tag{18}$$

Proof. From definition (12), the connection matrix is pseudo-constant in t if and only if

$$B_{r,s}(x,t) \stackrel{\text{def}}{=} Y_\infty(x,qt)Y_\infty(x,t)^{-1} = Y_0(x,qt)Y_0(x,t)^{-1}.$$

Using (11), we find that the only poles in $0 < |x| < \infty$ common to both sides are $x = qtb_r$ or $x = qtb_s$. Moreover (10b) along with the normalization of $Y_\infty(x)$ imply that the left hand side behaves as $I + O(x^{-1})$ at $x = \infty$. Similarly (10a) implies that the right hand side behaves like $O(x)$ at $x = 0$ (notice that D_0 is proportional to t). The proposition is an immediate consequence of these properties. □

The compatibility condition for systems (9), (13) reads

$$A(x,qt)B(x,t)_{r,s} = B(qx,t)_{r,s}A(x,t) \tag{19}$$

where $A(x,t)$ and $B(x,t)_{r,s}$ are given respectively by (14) and (18). From the next section we will parameterize $A(x)$ and derive q-Garnier system by calculating this compatibility condition as a time evolution of the space of $A(x)$, the coefficient of the linear equation (9).

4. q-Garnier system

Ignoring a gauge freedom, we have further $2N$ parameters for the space of the coefficient of linear equations (9). If we set λ_k for a zero of the $(1,2)$-element of A and set μ_k appropriately, we can consider $\mathbf{C}(a_i,\kappa_h,\theta_h)(\lambda_k,\mu_k)_{k=1,\dots N}^{i=1,\dots,2N+2;h=1,2;}$ as the space of the coefficient A of linear q-difference equation (9). Now we look at this more closely.

Define $\lambda_k = \lambda_k(t)$, $\mu_k^{(h)} = \mu_k^{(h)}(t)$ $(h = 1,2,\ k = 1,\dots,N)$ by

$$(A(\lambda_k,t))_{1,2} = 0, \qquad (A(\lambda_k,t))_{1,1} = \kappa_1\mu_k^{(1)}\prod_{l\neq k}(\lambda_k-\lambda_l), \tag{20}$$

$$(A(\lambda_k,t))_{2,2} = \kappa_2\mu_k^{(2)}\prod_{l\neq k}(\lambda_k-\lambda_l), \qquad (k=1,\dots,N)$$

so that $\mu_k^{(1)}\mu_k^{(2)} = \prod_{j=1}^{2N+2}(\lambda_k-a_j)/\prod_{l\neq k}(\lambda_k-\lambda_l)^2$. In terms of λ_k, $\mu_k^{(1)}$, $\mu_k^{(2)}$ and (14)–(16), the matrix $A(x,t)$ can be parametrized as follows.

$$A(x,t) = \begin{pmatrix} \kappa_1 W(x) & \kappa_2 wL(x) \\ -\kappa_1\frac{T(x)-Z(x)W(x)}{wL(x)} & \kappa_2 Z(x) \end{pmatrix}.$$

Here

$$L(x) = \prod_{l=1}^{N}(x-\lambda_l), \qquad T(x) = \frac{\det A(x)}{\kappa_1\kappa_2} = \prod_{i=1}^{2N+2}(x-a_i),$$

$$W(x) = L(x)\left(x - \alpha + \sum_{l=1}^{N} \frac{\mu_l^{(1)}}{x - \lambda_l}\right), \qquad Z(x) = L(x)\left(x - \beta + \sum_{l=1}^{N} \frac{\mu_l^{(2)}}{x - \lambda_l}\right),$$

$$\alpha = \frac{1}{\kappa_1 - \kappa_2}\left((-1)^{N+1}\frac{\theta_1 + \theta_2}{\prod_{l=1}^{N}\lambda_l} - \kappa_1\sum_{l=1}^{N}\frac{\mu_l^{(1)}}{\lambda_l} - \kappa_2\sum_{l=1}^{N}\frac{\mu_l^{(2)}}{\lambda_l} - \kappa_2\left(\sigma_1 - 2\sum_{l=1}^{N}\lambda_l\right)\right),$$

$$\beta = \frac{1}{\kappa_1 - \kappa_2}\left(-(-1)^{N+1}\frac{\theta_1 + \theta_2}{\prod_{l=1}^{N}\lambda_l} + \kappa_1\sum_{l=1}^{N}\frac{\mu_l^{(1)}}{\lambda_l} + \kappa_2\sum_{l=1}^{N}\frac{\mu_l^{(2)}}{\lambda_l} + \kappa_1\left(\sigma_1 - 2\sum_{l=1}^{N}\lambda_l\right)\right),$$

$$\sigma_1 = a_1 + a_2 + \cdots + a_{2N+2}.$$

The quantity $w = w(t)$ is related to the 'gauge' freedom, and does not enter the final result for the q-Garnier system. Introducing μ_k by

$$\mu_k = \frac{T(\lambda_k)}{L'(\lambda_k)\mu_k^{(1)}} = L'(\lambda_k)\mu_k^{(2)},$$

we can identify $\mathbf{C}(a_i, \kappa_h, \theta_h)(\lambda_k, \mu_k)_{k=1,\ldots N}^{i=1,\ldots,2N+2;\, h=1,2;}$ with the space of A.

Compatibility (19) is written by the form of a set of q-difference equations among the quantities λ_k, $\mu_k^{(1)}$, etc.

Theorem 3. *The equation of compatibility* (19) *is equivalent to the q-Garnier system* (7):

$$\sum_{l=1}^{N}\frac{q\kappa_1\overline{L(a_n)}\,\overline{T(\lambda_l)}}{(a_n - \overline{\lambda_l})(qa_r - \overline{\lambda_l})(qa_s - \overline{\lambda_l})\overline{L'(\lambda_l)}\overline{\mu_l}} = \sum_{l=1}^{N}\frac{\kappa_2 L(a_n)\mu_l}{(a_n - \lambda_l)(a_r - \lambda_l)(a_s - \lambda_l)L'(\lambda_l)},$$

$$\frac{\overline{L(a_n)}}{L(a_n)} = -\frac{\sum_{l=1}^{N}\frac{\mu_l}{(a_n-\lambda_l)(a_r-\lambda_l)(a_s-\lambda_l)L'(\lambda_l)}}{\left(1 - \sum_{l=1}^{N}\frac{\mu_l}{(a_r-\lambda_l)(a_s-\lambda_l)L'(\lambda_l)}\right)\left(1 - \frac{\kappa_2}{q\kappa_1}\sum_{l=1}^{N}\frac{\mu_l}{(a_r-\lambda_l)(a_s-\lambda_l)L'(\lambda_l)}\right)}$$

$$\times\left(\frac{\theta_1 + \theta_2}{\kappa_1 L(0)} + a_n\left(1 + \frac{\kappa_2}{q\kappa_1}\right) + \sum_{l=1}^{N}\frac{a_n T(\lambda_l)}{(a_n - \lambda_l)\lambda_l L'(\lambda_l)\mu_l}\right.$$

$$\left. + \sum_{l=1}^{N}\frac{(\kappa_2/q\kappa_1)(qa_r a_s - a_n\lambda_l)\mu_l}{(a_r - \lambda_l)(a_s - \lambda_l)\lambda_l L'(\lambda_l)}\right), \qquad (n \in \{1, 2, \ldots, 2N+2\}\backslash\{r, s\})$$

with the condition about the gauge freedom:

$$\frac{\overline{w}}{w} = \frac{q\kappa_1}{\kappa_2}\,\frac{1 - \sum_{l=1}^{N}\frac{\overline{T(\lambda_l)}}{(qa_r-\overline{\lambda_l})(qa_s-\overline{\lambda_l})\overline{L'(\lambda_l)}\overline{\mu_l}}}{1 - \frac{q\kappa_1}{\kappa_2}\sum_{l=1}^{N}\frac{\overline{T(\lambda_l)}}{(qa_r-\overline{\lambda_l})(qa_s-\overline{\lambda_l})\overline{L'(\lambda_l)}\overline{\mu_l}}}. \tag{21}$$

We will omit the proof and show it for the birational version in Section 6, because the calculation is essentially same as that of the birational form.

Now we study the q-Garnier system itself, in detail. In system (7) the suffix n is running through $\{1, 2, \dots, 2N+2\} \backslash \{r, s\}$, but what we need is only N equations. System (7) with respect to n_i ($n_i \in \{1, \dots, 2N+2\} \backslash \{r, s\}, i = 1, \dots, N$) assure the rest of them (see the proof of Theorem 4 in Section 6).

The time evolution $(\lambda_k, \mu_k)_{k=1,\dots,N} \mapsto (\overline{\lambda_k}, \overline{\mu_k})_{k=1,\dots,N}$ is determined by these $2N$ equations. But this process is a little bit complicated.

Firstly $\overline{L(a_n)}$ can be written by $(\lambda_k, \mu_k)_{k=1,\dots,N}$. Since monic polynomial of degree N is determined by values at distinct N points, we can write down $\overline{L(x)}$ by using Lagrange's interpolations. Namely,

$$\overline{L(x)} = \prod_{l=1}^{N} (x - \overline{\lambda_l}) = \Psi(x) \left(1 + \sum_{i=1}^{N} \frac{\overline{L(a_{n_i})}}{\Psi'(a_{n_i})} \frac{1}{x - a_{n_i}} \right), \tag{22}$$

where $\Psi(x) = \prod_{i=1}^{N} (x - a_{n_i})$. Then $\overline{\lambda_k}$ is a zero of the polynomial, $\overline{L(x)}$.

The rest of equations, (7a), determines $\overline{\mu_k}$ from $\overline{\lambda_l}$, λ_l and μ_l ($l = 1, \dots, N$), because these equations are linear about $1/\overline{\mu_k}$.

Remark 1. In order to calculate $\overline{\mu_k}$ we need the inverse of the matrix $P_{i,l} = (1/(a_{n_i} - \lambda_l))_{l=1,\dots,N}^{i=1,\dots,N}$. It is written as $Q_{k,i} = \left(-\frac{\Psi(\lambda_k)}{L'(\lambda_k)} \frac{L(a_{n_i})}{(a_{n_i} - \lambda_k)\Psi'(a_{n_i})} \right)_{i=1,\dots,N}^{k=1,\dots,N}$.

Proof. We consider the partial fraction of the rational function

$$\frac{L(x)}{(x - a_{n_i})\Psi(x)} = \sum_{i=1}^{N} \frac{1}{x - a_{n_i}} \frac{L(a_{n_i})}{(a_{n_i} - \lambda_l)\Psi'(a_{n_i})}.$$

Substituting $x = \lambda_k$ to this rational function, we get equation

$$\sum_{i=1}^{N} \frac{\Psi(\lambda_k)}{(\lambda_k - a_{n_i})L'(\lambda_k)} \frac{L(a_{n_i})}{(a_{n_i} - \lambda_l)\Psi'(a_{n_i})} = \sum_{i=1}^{N} Q_{k,i} P_{i,l} = \delta_{k,l}. \quad \square$$

In the process of determination of $(\overline{\lambda_k}, \overline{\mu_k})_{k=1,\dots,N}$ we have to solve an algebraic equation of degree N. It means that the time evolution is not uniquely determined. In the case of the Garnier differential system, this corresponds to the fact that u_k of (3) does not satisfy the Painlevé property, while q_k of (5), a symmetric function of u_k, satisfies the Painlevé property.

The inverse of this transformation, $(\lambda_k, \mu_k) = T_{r,s}^{-1}(\overline{\lambda_k}, \overline{\mu_k})$, can be calculated by (7a) and the equations

$$\frac{L(a_n)}{\overline{L(a_n)}} = -\frac{\sum_{l=1}^{N} \frac{\overline{T(\lambda_l)}}{(a_n-\overline{\lambda_l})(qa_r-\overline{\lambda_l})(qa_s-\overline{\lambda_l})\overline{L'(\lambda_l)}\overline{\mu_l}}}{\left(1-\sum_{l=1}^{N} \frac{\overline{T(\lambda_l)}}{(qa_r-\overline{\lambda_l})(qa_s-\overline{\lambda_l})\overline{L'(\lambda_l)}\overline{\mu_l}}\right)\left(1-\frac{q\kappa_1}{\kappa_2}\sum_{l=1}^{N} \frac{\overline{T(\lambda_l)}}{(qa_r-\overline{\lambda_l})(qa_s-\overline{\lambda_l})\overline{L'(\lambda_l)}\overline{\mu_l}}\right)}$$
$$\times\left(q\frac{\theta_1+\theta_2}{\kappa_1\overline{L(0)}} + a_n\left(1+\frac{q\kappa_1}{\kappa_2}\right) + \sum_{l=1}^{N}\frac{a_n\overline{\mu_l}}{(a_n-\overline{\lambda_l})\overline{\lambda_l}\overline{L'(\lambda_l)}}\right.$$
$$\left. + \frac{q\kappa_1}{\kappa_2}\sum_{l=1}^{N}\frac{(qa_ra_s-a_n\overline{\lambda_l})\overline{T(\lambda_l)}}{(qa_r-\overline{\lambda_l})(qa_s-\overline{\lambda_l})\overline{\lambda_l}\overline{L'(\lambda_l)}\overline{\mu_l}}\right).$$

Remark 2. In the case $N = 1$ $(r = 1,\ s = 2)$, system (7) is written as

(23a)
$$\overline{\lambda_1}\lambda_1 = a_3a_4\frac{1-\frac{\theta_1}{\kappa_1a_3a_4}\frac{\mu_1}{(a_1-\lambda_1)(a_2-\lambda_1)}}{1-\frac{\mu_1}{(a_1-\lambda_1)(a_2-\lambda_1)}}\,\frac{1-\frac{\theta_2}{\kappa_1a_3a_4}\frac{\mu_1}{(a_1-\lambda_1)(a_2-\lambda_1)}}{1-\frac{\kappa_2}{q\kappa_1}\frac{\mu_1}{(a_1-\lambda_1)(a_2-\lambda_1)}}$$

(23b)
$$\overline{\mu_1}\mu_1 = \frac{q\kappa_1}{\kappa_2}(\overline{\lambda_1}-a_3)(\overline{\lambda_1}-a_4)(\lambda_1-a_1)(\lambda_1-a_2).$$

By the change of variables

$$(\lambda, v) = \left(\lambda_1, \frac{(a_1-\lambda_1)(a_2-\lambda_1)}{\mu_1}\right),$$

the time evolution about (λ, v) coincides with the q-P_{VI} system ([6]). □

5. Continuous limit

From the construction one expects that in the continuous limit the q-Garnier system reduces to the Garnier system. By continuous limit we mean that the limit $\varepsilon = 1 - q \to 0$.

We go into the details of a calculation of continuous limit although it is rather complicated. We consider a limit of $T_{2j+1,2j+2}$. In equations (7), we look at only the case of even n sufficiently. In what follows in this section, the number n is replaced by $2n + 2$. Formulas used in detailed calculations are listed in Remark 3 in the last part of this section.

We set $T_{\mathrm{odd}}(x) = \prod_{i=0}^{N}(1 - x/a_{2i+1})$ and $T_{\mathrm{even}}(x) = \prod_{i=0}^{N}(1 - x/a_{2i+2})$ (so we have $T(x) = (\prod_{i=1}^{2N+2} a_i)T_{\mathrm{odd}}(x)T_{\mathrm{even}}(x)$). Besides we introduce new variables (u_k, v_k) which are defined as $q^{-u_kv_k} = \kappa_2\mu_k/(\theta_2T_{\mathrm{odd}}(\lambda_k))$ and $u_k = \lambda_k$, then equation (7a) is written as

$$\sum_{l=1}^{N}\frac{q^2\theta_1\overline{L(a_{2n+2})}\,\overline{T_{\mathrm{even}}(u_l)}q^{\overline{u_l}\,\overline{v_l}}}{(a_{2n+2}-\overline{u_l})(qa_{2j+1}-\overline{u_l})(qa_{2j+2}-\overline{u_l})\overline{L'(u_l)}}$$
$$= \sum_{l=1}^{N}\frac{\theta_2L(a_{2n+2})T_{\mathrm{odd}}(u_l)q^{-u_lv_l}}{(a_{2n+2}-u_l)(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}.$$

Here we have

$$\sum_{l=1}^{N}\frac{T_{\mathrm{odd}}(u_l)q^{-u_lv_l}}{(a_{2n+2}-u_l)(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}$$

$$=\sum_{l=1}^{N}\frac{T_{\mathrm{odd}}(u_l)}{(a_{2n+2}-u_l)(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}$$

$$-\sum_{l=1}^{N}\frac{T_{\mathrm{odd}}(u_l)(1-q^{-u_lv_l})}{(a_{2n+2}-u_l)(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}$$

$$=\frac{T_{\mathrm{odd}}(a_{2j+2})}{(a_{2n+2}-a_{2j+2})(a_{2j+1}-a_{2j+2})L(a_{2j+2})}$$

$$-\frac{T_{\mathrm{odd}}(a_{2n+2})}{(a_{2n+2}-a_{2j+2})(a_{2j+1}-a_{2n+2})L(a_{2n+2})}$$

$$-\sum_{l=1}^{N}\frac{T_{\mathrm{odd}}(u_l)(1-q^{-u_lv_l})}{(a_{2n+2}-u_l)(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}.$$

Calculation about the left hand side is in the same way. Now we introduce notation of q-number $[\alpha]_q=(1-q^{\alpha})/(1-q)$. This tends to α as $q\to 1$. By using this notation, we can write equation (7a) as

$$\frac{q^2\theta_1\overline{L(a_{2n+2})}\,\overline{T_{\mathrm{even}}(a_{2j+1})}}{q(a_{2n+2}-qa_{2j+1})(a_{2j+2}-a_{2j+1})\overline{\overline{L(a_{2j+1})}}}$$

$$-\frac{\theta_2L(a_{2n+2})T_{\mathrm{odd}}(a_{2j+2})}{(a_{2n+2}-a_{2j+2})(a_{2j+1}-a_{2j+2})L(a_{2j+2})}+\frac{\theta_2T_{\mathrm{odd}}(a_{2n+2})}{(a_{2n+2}-a_{2j+2})(a_{2j+1}-a_{2n+2})}$$

$$-\varepsilon\sum_{l=1}^{N}\frac{q^2\theta_1\overline{L(a_{2n+2})}\,\overline{T_{\mathrm{even}}(u_l)}[\overline{u_l}\overline{v_l}]_q}{(a_{2n+2}-\overline{u_l})(qa_{2j+1}-\overline{u_l})(qa_{2j+2}-\overline{u_l})\overline{\overline{L'(u_l)}}}$$

$$+\varepsilon\sum_{l=1}^{N}\frac{\theta_2L(a_{2n+2})T_{\mathrm{odd}}(u_l)[-u_lv_l]_q}{(a_{2n+2}-u_l)(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}=0.$$

Putting

(24) $$a_{2n+1}=q^{-\alpha_n/2}t_n,\qquad a_{2n+2}=q^{\alpha_n/2}t_n\quad(n=0,\dots,N),\qquad \kappa_1=q^{-K/2},$$

$$\kappa_2=q^{K/2},\qquad \theta_1=q^{-\theta/2}\prod_{i=0}^{N}(-t_i),\qquad \theta_2=q^{\theta/2}\prod_{i=0}^{N}(-t_i),$$

(notice relation (17)), we get

$$\frac{\mathscr{L}(t_n)\mathscr{T}'(t_j)}{\mathscr{L}(t_j)(t_j-t_n)}\left[\frac{\theta+\alpha_0-1}{t_j}+\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{\alpha_j-1-\alpha_i}{t_j-t_i}-\sum_{l=1}^{N}\left(\frac{\alpha_j-1+\frac{\mathrm{d}u_l}{\mathrm{d}t_j}}{t_j-u_l}-\frac{\frac{\mathrm{d}u_l}{\mathrm{d}t_j}}{t_n-u_l}\right)-\frac{\alpha_j-1}{t_j-t_n}\right]$$
$$-\frac{\alpha_n\mathscr{T}'(t_n)}{(t_n-t_j)^2}-\sum_{l=1}^{N}\frac{2\mathscr{L}(t_n)\mathscr{T}(u_l)v_l}{(t_j-u_l)^2(t_n-u_l)\mathscr{L}'(u_l)}=0,$$

where $\mathscr{T}(x)=x\prod_{i=0}^{N}(x-t_i)$ and $\mathscr{L}(x)=\prod_{i=1}^{N}(x-u_i)$. In this calculation we used the Taylor expansion

$$u_l(qt_j)=u_l((1-\varepsilon)t_j)=u_l(t_j)-\varepsilon t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}+\varepsilon^2\frac{t_j^2}{2}\frac{\mathrm{d}^2u_l}{\mathrm{d}t_j^2}+\cdots.$$

The equation obtained is linear with respect to $\mathrm{d}u_l/\mathrm{d}t_j$'s. Considering the matrix $N_{k,n}=\frac{\mathscr{T}(u_k)t_n(t_n-t_j)\mathscr{L}(t_n)}{\mathscr{T}'(t_n)u_k(u_k-t_j)(u_k-t_n)\mathscr{L}'(u_k)}$, which is the inverse of the matrix $M_{n,l}=1/(t_n-u_l)$ $(n\in\{0,\dots,N\}\backslash\{j\}$ and $k,l=1,\dots,N)$, we can calculate $\mathrm{d}u_k/\mathrm{d}t_j$. By using the formulas (28b) and (28f) given below in Remark 3, we obtain

$$\frac{\mathrm{d}u_k}{\mathrm{d}t_j}=\frac{\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\frac{\mathscr{T}(u_k)}{\mathscr{L}'(u_k)(u_k-t_j)}\left(\frac{\theta}{u_k}+\sum_{i=0}^{N}\frac{\alpha_i-\delta_{lj}}{u_k-t_i}-2v_k\right). \tag{25}$$

Next we rewrite equation (7b) using the notation of q-numbers. It can be written as

$$\left(\frac{\prod_{i=0}^{N}(-a_{2i+1})T_{\mathrm{odd}}(a_{2j+2})}{(a_{2j+2}-a_{2j+1})L(a_{2j+2})}+\varepsilon\sum_{l=1}^{N}\frac{\prod_{i=0}^{N}(-a_{2i+1})T_{\mathrm{odd}}(u_l)\left[-u_lv_l+\frac{\theta+\sum_{i=0}^{N}\alpha_i-K}{2}\right]_q}{(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}\right)$$
$$\times\left(\frac{\prod_{i=0}^{N}(-a_{2i+1})T_{\mathrm{odd}}(a_{2j+2})}{(a_{2j+2}-a_{2j+1})L(a_{2j+2})}+\varepsilon\sum_{l=1}^{N}\frac{\prod_{i=0}^{N}(-a_{2i+1})T_{\mathrm{odd}}(u_l)\left[-u_lv_l+\frac{\theta+\sum_{i=0}^{N}\alpha_i+K}{2}-1\right]_q}{(a_{2j+1}-u_l)(a_{2j+2}-u_l)L'(u_l)}\right)$$
$$=\left(\frac{q\theta_1\overline{T_{\mathrm{even}}(a_{2j+1})}}{(a_{2n+2}-qa_{2j+1})(a_{2j+1}-a_{2j+2})\overline{L(a_{2j+1})}}\right.$$
$$\left.+\varepsilon\sum_{l=1}^{N}\frac{q^2\theta_1\overline{T_{\mathrm{even}}(u_l)}[\overline{u_l}\overline{v_l}]_q}{(a_{2n+2}-\overline{u_l})(qa_{2j+1}-\overline{u_l})(qa_{2j+2}-\overline{u_l})\overline{L'(u_l)}}\right)$$

$$\times\left(\frac{\theta_2(a_{2n+2}-qa_{2j+1})T_{\mathrm{odd}}(a_{2j+2})}{q(a_{2j+2}-a_{2j+1})L(a_{2j+2})}-\varepsilon^2 a_{2n+2}q^{-(\theta+\sum_{i=0}^{N}\alpha_i)/2}[k_1]_q[k_2]_q\right.$$

$$\left.-\varepsilon\sum_{l=1}^{N}\left(\frac{a_{2n+2}\theta_1 T_{\mathrm{even}}(u_l)[u_l v_l]_q}{(a_{2n+2}-u_l)u_l L'(u_l)}+\frac{(qa_{2j+1}a_{2j+2}-a_{2n+2}u_l)\theta_2 T_{\mathrm{odd}}(u_l)[-u_l v_l]_q}{q(a_{2j+1}-u_l)(a_{2j+2}-u_l)u_l L'(u_l)}\right)\right),$$

where $k_1=(\theta+\sum_{i=0}^{N}\alpha_i-K)/2,\ k_2=(\theta+\sum_{i=0}^{N}\alpha_i+K)/2-1$.

This equation leads to $0=0$ as the limit $\varepsilon\to 0$. Furthermore, as the coefficient of ε^1's term of this equation, the following equation appears:

$$\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{(\alpha_j-1)t_j+\alpha_i t_i}{t_j-t_i}+\sum_{l=1}^{N}\frac{\left(\alpha_j-1+\frac{\mathrm{d}u_l}{\mathrm{d}t_j}\right)t_j}{u_l-t_j}$$
$$=\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\sum_{l=0}^{N}\frac{\mathscr{T}(u_l)[\theta+\sum_{i=0}^{N}\alpha_i-1-2u_l v_l]}{\mathscr{L}'(u_l)u_l(u_l-t_j)^2}. \tag{26}$$

But this can be derived from equation (25). We have to look at the coefficients of ε^2 term in equation (7b). Notice that

$$[\alpha]_q=\frac{1-q^\alpha}{1-q}=\alpha+\varepsilon\frac{\alpha(1-\alpha)}{2}+\cdots,$$

and we obtain

$$0=\frac{1}{2}\frac{\mathscr{T}'(t_j)}{t_j\mathscr{L}(t_j)}\left[\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{(\alpha_j-1)t_j+\alpha_i t_i}{t_j-t_i}\sum_{l=1}^{N}+\frac{\left(\alpha_j-1+\frac{\mathrm{d}u_l}{\mathrm{d}t_j}\right)t_j}{u_l-t_j}\right]$$
$$\times\left[\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{(\alpha_j+1)t_j+\alpha_i t_i}{t_j-t_i}+\sum_{l=1}^{N}\frac{(\alpha_j+1)t-t\frac{\mathrm{d}u_l}{\mathrm{d}t_j}}{u_l-t_j}\right.$$
$$\left.-2\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\sum_{l=1}^{N}\frac{(t_n-t_j)\mathscr{T}(u_l)v_l}{(t_n-u_l)(t-u_l)^2\mathscr{L}'(u_l)}\right]$$
$$+\frac{1}{2}\frac{\mathscr{T}'(t_j)}{t_j\mathscr{L}(t_j)}\left[\sum_{l=1}^{N}\left(\frac{t_j^2\frac{\mathrm{d}^2u_l}{\mathrm{d}t_j^2}}{u_l-t_j}-t_j^2\frac{\left(\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-1\right)\left(\frac{\mathrm{d}u_l}{\mathrm{d}t_j}+\alpha_j-1\right)}{(u_l-t_j)^2}\right)\right.$$
$$\left.-\sum_{\substack{i=0\\ i\neq j}}^{N}\left(t_j\frac{(\alpha_j-1)t_j+\alpha_i t_i}{(t_j-t_i)^2}+\frac{\alpha_i t_i}{t_j-t_i}\right)\right]$$

$$
\begin{aligned}
&-\sum_{l=1}^{N}\frac{(t_n-t_j)\mathscr{T}(u_l)v_l}{(t_n-u_l)\mathscr{L}'(u_l)(u_l-t_j)^2}\left[\sum_{\substack{m=1\\ m\neq l}}^{N}\frac{t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-t_j\frac{\mathrm{d}u_m}{\mathrm{d}t_j}}{u_l-u_m}-\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}}{u_l-t_i}\right.\\
&\left.+\frac{t_j\left(\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-1\right)}{u_l-t_j}-\frac{t_j}{u_l}\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-\frac{t_j}{v_l}\frac{\mathrm{d}v_l}{\mathrm{d}t_j}-\frac{(\alpha_j-2)t_j}{t_n-t_j}+\frac{t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}}{u_l-t_n}-k_1-k_2\right]\\
&+\frac{k_1+k_2}{2}\frac{\mathscr{T}'(t_j)}{t_j\mathscr{L}(t_j)}\left(\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{\alpha_i t_i}{t_j-t_i}+\sum_{l=1}^{N}\frac{\alpha_j t_j}{u_l-t_j}\right)+\sum_{l=1}^{N}\frac{t_j\mathscr{T}(u_l)v_l}{(t_n-t_j)\mathscr{L}'(u_l)(u_l-t_j)}\\
&\times\left[v_l-\frac{\theta}{u_l}-\sum_{i=0}^{N}\frac{\alpha_i-\delta_{i,j}}{u_l-t_i}\right]+\frac{t_nk_1k_2}{t_n-t_j}+\frac{k_1^2+k_2^2}{2}\left(1+\frac{\mathscr{T}'(t_j)}{t_j\mathscr{L}(t_j)}\right)\\
&+\frac{1}{2}\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)[2u_lv_l-k_1-k_2]}{\mathscr{L}'(u_l)u_l(u_l-t_j)^2}\left(\sum_{\substack{i=1\\ i\neq j}}^{N}\frac{\alpha_jt_j+\alpha_it_i}{t_j-t_i}+\sum_{m=1}^{N}\frac{\alpha_jt_j}{u_m-t_j}-1\right)\\
&+\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\left[\left(\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)[-u_lv_l+k_1]}{\mathscr{L}'(u_l)u_l(u_l-t_j)^2}\right)\left(\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)[-u_lv_l+k_2]}{\mathscr{L}'(u_l)u_l(u_l-t_j)^2}\right)\right.\\
&\qquad\left.+\left(\sum_{l=1}^{N}\frac{(t_n-t_j)\mathscr{T}(u_l)v_l}{(t_n-u_l)\mathscr{L}'(u_l)(u_l-t_j)^2}\right)^2\right].
\end{aligned}
$$

This equation contains the terms of $\mathrm{d}^2u_l/\mathrm{d}t_j^2$'s, and we want to eliminate these terms. Differentiating equation (26), we have

$$
\begin{aligned}
&\sum_{l=1}^{N}\left(\frac{t_j^2\frac{\mathrm{d}^2u_l}{\mathrm{d}t_j^2}}{u_l-t_j}-\frac{t_j^2\left(\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-1\right)\left(\frac{\mathrm{d}u_l}{\mathrm{d}t_j}+\alpha_0-1\right)}{(u_l-t_j)^2}\right)-\sum_{\substack{i=0\\ l\neq j}}^{N}\left(t_j\frac{(\alpha_j-1)t_j+\alpha_it_i}{(t_j-t_i)^2}+\frac{\alpha_it_i}{t_j-t_i}\right)\\
&=\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\sum_{l=1}^{N}\frac{2\mathscr{T}(u_l)v_l}{\mathscr{L}'(u_l)(u_l-t_j)^2}\left[\sum_{\substack{m=1\\ m\neq l}}^{N}\frac{t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-t_j\frac{\mathrm{d}u_m}{\mathrm{d}t_j}}{u_l-u_m}-\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}}{u_l-t_i}+\frac{t_j\left(\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-1\right)}{u_l-t_j}\right.\\
&\left.-\frac{t_j}{u_l}\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-\frac{t_j}{v_l}\frac{\mathrm{d}v_l}{\mathrm{d}t_j}\right]+\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\left(\theta+\sum_{i=0}^{N}\alpha_i-1-\sum_{l=1}^{N}\frac{2\mathscr{T}(u_l)v_l}{(t_j-u_l)^2\mathscr{L}'(u_l)}\right)
\end{aligned}
$$

$$\times\left(\sum_{l=1}^{N}\frac{t_j\left(\frac{\mathrm{d}u_l}{\mathrm{d}t_j}-1\right)}{u_l-t_j}-\sum_{\substack{i=0\\i\neq j}}^{N}\frac{t}{t-t_i}\right)-\sum_{\substack{i=0\\i\neq j}}^{N}\frac{(\alpha_j-1)t_j+\alpha_i t_i}{t_j-t_i}-\sum_{l=1}^{N}\frac{(\alpha_j-1)t_j+t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}}{u_l-t_j}.$$

Using this equation and (26), we can eliminate $\mathrm{d}^2u_l/\mathrm{d}t_j^2$'s terms and obtain

$$0=\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)v_l}{(t_n-u_l)\mathscr{L}'(u_l)(u_l-t_j)}\left[\frac{t_j}{v_l}\frac{\mathrm{d}v_l}{\mathrm{d}t_j}+\left(L_l-\frac{1}{u_l-t_j}\right)t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}+\sum_{\substack{i=1\\i\neq l}}^{N}\frac{t_j\frac{\mathrm{d}u_i}{\mathrm{d}t_j}}{u_l-u_i}\right.$$
$$\left.+(k_1+k_2)\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\right]+\frac{t_j}{t_n-t_j}\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)v_l}{\mathscr{L}'(u_l)(u_l-t_j)}\left[v_l-\frac{\theta}{u_l}-\sum_{i=0}^{N}\frac{\alpha_i+\delta_{i,j}}{u_l-t_i}\right]$$
$$+\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}\left(\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)v_l}{(t_n-u_l)(u_l-t_j)\mathscr{L}'(u_l)}\right)^2+\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)v_l}{(t_n-u_l)^2\mathscr{L}'(u_l)(u_l-t_j)}t_j\frac{\mathrm{d}u_l}{\mathrm{d}t_j}$$
$$+(\alpha_j-1)t_j\sum_{l=1}^{N}\frac{\mathscr{T}(u_l)v_l}{(t_n-u_l)\mathscr{L}'(u_l)(u_l-t_j)^2}+\left(\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}+\frac{t_j}{t_n-t_j}\right)k_1k_2,$$

where $L_l=1/u_l+\sum_{i=0}^{N}1/(u_l-t_i)-\sum_{m=1,m\neq l}^{N}1/(u_l-u_m)$.

Moreover the equation obtained is linear with respect to $\mathrm{d}v_l/\mathrm{d}t_j$'s. Considering the inverse of the matrix $M_{n,l}=1/(t_n-u_l)$, we can calculate $\mathrm{d}v_k/\mathrm{d}t_j$. By using (28a)–(28e) in Remark 3 and equation (25), we get

$$\frac{\mathscr{T}'(t_j)}{\mathscr{L}(t_j)}\frac{\mathrm{d}v_k}{\mathrm{d}t_j}=L_k\frac{\mathscr{T}(u_k)v_k^2}{\mathscr{L}'(u_k)(u_k-t_j)}-\frac{\mathscr{T}(u_k)v_k}{\mathscr{L}'(u_k)(u_k-t_j)^2}\left[\frac{\theta}{u_k}+\sum_{i=0}^{N}\frac{\alpha_i-\delta_{i,j}}{u_k-t_i}\right]$$
$$+\sum_{\substack{l=1\\l\neq k}}^{N}\frac{\mathscr{T}(u_l)}{\mathscr{L}'(u_l)(u_k-t_j)}\frac{v_l}{u_l-u_k}\left[v_l-\frac{\theta}{u_l}-\sum_{i=0}^{N}\frac{\alpha_i-\delta_{i,j}}{u_l-t_i}\right]$$
$$+\sum_{\substack{l=1\\l\neq k}}^{N}\frac{\mathscr{T}(u_l)}{\mathscr{L}'(u_l)(u_l-t_j)}\frac{v_k}{u_l-u_k}\left[\frac{\theta}{u_l}+\sum_{i=0}^{N}\frac{\alpha_i-\delta_{i,j}}{u_l-t_i}\right]$$
$$-\frac{(\alpha_j-1)\mathscr{T}'(t_j)}{(u_k-t_j)\mathscr{L}(t_j)}v_k-(k_1+k_2)v_k+\frac{k_1k_2}{u_k-t_j}.$$

Simplifying this equation, by using (28g) and (28h) in Remark 3, finally we obtain

$$\frac{\mathscr{T}'(t_j)}{\mathscr{L}(t_j)}\frac{\mathrm{d}v_k}{\mathrm{d}t_j}=\frac{\mathscr{T}(u_k)v_k}{\mathscr{L}'(u_k)(u_k-t_j)}\left[L_k\left(v_k-\frac{\theta}{u_k}-\sum_{i=0}^{N}\frac{\alpha_i-\delta_{ij}}{u_k-t_i}\right)\right.$$
$$\left.+\frac{\theta}{u_k^2}+\sum_{i=0}^{N}\frac{\alpha_i-\delta_{ij}}{(u_k-t_i)^2}\right]+\sum_{\substack{l=1\\ l\neq k}}^{N}\frac{\mathscr{T}(u_l)}{\mathscr{L}'(u_l)(u_k-t_j)}\frac{v_l}{u_l-u_k}$$
$$\times\left[v_l-\frac{\theta}{u_l}-\sum_{i=0}^{N}\frac{\alpha_i-\delta_{i,j}}{u_l-t_i}\right]+\frac{k_1k_2}{u_k-t_j}. \tag{27}$$

Equations (25) and (27) coincide with the Garnier system (3).

Remark 3. Here we list formulas used in the above calculus. These can be verified by investigating residues, or by decomposition of rational functions into partial fraction like Remark 1.

In calculation of multiplying $N_{k,n}=\frac{\mathscr{T}(u_k)t_n(t_n-t_j)\mathscr{L}(t_n)}{\mathscr{T}'(t_n)u_k(u_k-t_j)(u_k-t_n)\mathscr{L}'(u_k)}$, it is useful to consider the following formulas:

$$\sum_{\substack{n=0\\ n\neq j}}^{N}\frac{t_n(t_n-t_j)\mathscr{L}(t_n)}{u_k(u_k-t_n)\mathscr{T}'(t_n)}\cdot\frac{1}{t_n-u_l}=\frac{\mathscr{L}'(u_k)(u_k-t_j)}{\mathscr{T}(u_k)}\delta_{k,l}, \tag{28a}$$

$$\sum_{\substack{n=0\\ n\neq j}}^{N}\frac{t_n(t_n-t_j)\mathscr{L}(t_n)}{u_k(u_k-t_n)\mathscr{T}'(t_n)}\cdot\frac{1}{t_n-t_j}=-\frac{1}{u_k(u_k-t_j)}\frac{t_j\mathscr{L}(t_j)}{\mathscr{T}'(t_j)}, \tag{28b}$$

$$\sum_{\substack{n=0\\ n\neq j}}^{N}\frac{t_n(t_n-t_j)\mathscr{L}(t_n)}{u_k(u_k-t_n)\mathscr{T}'(t_n)}=-\frac{1}{u_k}, \tag{28c}$$

$$\sum_{\substack{n=0\\ n\neq j}}^{N}\frac{t_n(t_n-t_j)\mathscr{L}(t_n)}{u_k(u_k-t_n)\mathscr{T}'(t_n)}\cdot\frac{1}{(t_n-u_l)^2}=\frac{(t_j-u_l)\mathscr{L}'(u_l)}{(u_k-u_l)\mathscr{T}(u_l)}\frac{u_l}{u_k},\qquad(l\neq k), \tag{28d}$$

$$\sum_{\substack{n=0\\ n\neq j}}^{N}\frac{t_n(t_n-t_j)\mathscr{L}(t_n)}{u_k(u_k-t_n)\mathscr{T}'(t_n)}\cdot\frac{1}{(t_n-u_k)^2}=\frac{(u_k-t_j)\mathscr{L}'(u_k)}{\mathscr{T}(u_k)}\left(\frac{1}{u_k}+\frac{1}{u_k-t_j}-L_k\right), \tag{28e}$$

where $L_k=1/u_k+\sum_{i=0}^{N}1/(u_k-t_i)-\sum_{m=1,m\neq k}^{N}1/(u_k-u_m)$.

Moreover we used the following formula to get equation (25):

$$\frac{\mathscr{T}'(t_j)}{t_j\mathscr{L}(t_j)}\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{t_i\mathscr{L}(t_i)}{(t_j-t_i)\mathscr{T}'(t_i)}=\sum_{\substack{i=0\\ i\neq j}}^{N}\frac{1}{t_i-t_j}+\sum_{l=1}^{N}\frac{1}{t_j-u_l}. \tag{28f}$$

In the last calculation to get equation (27) we used the following formulas:

$$\sum_{\substack{l=1\\ l\neq k}}^{N}\frac{\mathscr{T}(u_l)}{\mathscr{L}'(u_l)u_l(u_l-t_j)}\frac{1}{u_l-u_k}=\frac{\mathscr{T}(u_k)}{\mathscr{L}'(u_k)u_k(u_k-t_j)}\left(\frac{1}{u_k}+\frac{1}{u_k-t_j}-L_k\right)+1, \tag{28g}$$

$$\sum_{\substack{l=1\\ l\neq k}}^{N}\frac{\mathscr{T}(u_l)}{\mathscr{L}'(u_l)u_l(u_l-t_j)^2}\frac{1}{u_l-u_k} \tag{28h}$$

$$=\frac{\mathscr{T}(u_k)}{\mathscr{L}'(u_k)u_k(u_k-t_j)^2}\left(\frac{1}{u_k}+\frac{2}{u_k-t_j}-L_k\right)+\frac{\mathscr{T}'(t_j)}{t_j\mathscr{L}(t_j)(u_k-t_j)}. \qquad \square$$

6. q-Garnier system of birational form

In Section 4 we calculated the compatibility, (19), but the determination of the time evolution needs a process of solving algebraic equations of degree N. In order to describe this system in the form of birational mappings, we change parameterization for $A(x)$.

Define y_i, z_i and w_i $(i=1,2,\dots,2N+2)$ by

$$A(a_i)=y_i\begin{pmatrix}1\\ w^{-1}z_i\end{pmatrix}(w_i \quad w) \qquad (i=1,2,\dots,2N+2). \tag{29}$$

(Notice that $\det A(a_i)=0$.) Then there are $3(2N+2)$ parameters and it is redundant. If we take distinct $N+1$ elements of i's, which is denoted by $\{i_0,i_1,\dots,i_N\}$, we can write down $A(x)$ by y_{i_k}, z_{i_k} and w_{i_k} $(k=0,1,\dots,N)$. These are Lagrange's interpolations.

$$A(x)=\begin{pmatrix}\kappa_1W(x) & \kappa_2wL(x)\\ \kappa_1w^{-1}X(x) & \kappa_2Z(x)\end{pmatrix},$$

$$W(x)=\frac{\Theta(x)}{\kappa_1}\left(1+\sum_{k=0}^{N}\frac{w_{i_k}y_{i_k}}{\Theta'(a_{i_k})(x-a_{i_k})}\right),\quad L(x)=\frac{\Theta(x)}{\kappa_2}\left(\sum_{k=0}^{N}\frac{y_{i_k}}{\Theta'(a_{i_k})(x-a_{i_k})}\right),$$

$$X(x)=\frac{\Theta(x)}{\kappa_1}\left(\sum_{k=0}^{N}\frac{w_{i_k}z_{i_k}y_{i_k}}{\Theta'(a_{i_k})(x-a_{i_k})}\right),\quad Z(x)=\frac{\Theta(x)}{\kappa_2}\left(1+\sum_{k=0}^{N}\frac{z_{i_k}y_{i_k}}{\Theta'(a_{i_k})(x-a_{i_k})}\right),$$

where $\Theta(x)=\prod_{k=0}^{N}(x-a_{i_k})$. In particular,

$$y_l=\Theta(a_l)\left(\sum_{k=0}^{N}\frac{y_{i_k}}{\Theta'(a_{i_k})(a_l-a_{i_k})}\right),\qquad z_l=\frac{1+\sum_{k=0}^{N}\frac{z_{i_k}y_{i_k}}{\Theta'(a_{i_k})(a_l-a_{i_k})}}{\sum_{k=0}^{N}\frac{y_{i_k}}{\Theta'(a_{i_k})(a_l-a_{i_k})}}, \tag{30}$$

$$w_l=\frac{1+\sum_{k=0}^{N}\frac{w_{i_k}y_{i_k}}{\Theta'(a_{i_k})(a_l-a_{i_k})}}{\sum_{k=0}^{N}\frac{y_{i_k}}{\Theta'(a_{i_k})(a_l-a_{i_k})}} \qquad (l\neq i_k).$$

Furthermore we obtain a linear equation for w_{i_k}:

$$z_j - \sum_{k=0}^{N} \frac{z_j - z_{i_k}}{a_j - a_{i_k}} \frac{y_{i_k}}{\Theta'(a_{i_k})} w_{i_k} = 0 \qquad (j \neq i_k), \tag{31}$$

from the equation $w_j y_j z_j = \kappa_1 X(a_j) = \kappa_1 W(a_j) z_j$. Let R be the inverse of the matrix $\left(\frac{z_{j_l} - z_{i_k}}{a_{j_l} - a_{i_k}}\right)_{l,k}$ (where $\{j_0, \dots, j_N\} \cup \{i_0, \dots, i_N\} = \{1, 2, \dots, 2N+2\}$). We get

$$w_{i_m} = \frac{\Theta'(a_{i_m})}{y_{i_m}} \sum_{l=0}^{N} R_{m,l} z_{j_l}.$$

In the end we add two relations. The first is

$$\sum_{k=0}^{N} \frac{y_{i_k}}{\Theta'(a_{i_k})} = \kappa_2, \tag{32}$$

because the gauge w normalizes the leading term of $L(x)$ to 1. The second comes from the trace of $A(0)$:

$$2 - \sum_{k=0}^{N} \frac{(z_{i_k} + w_{i_k}) y_{i_k}}{a_{i_k} \Theta'(a_{i_k})} = \frac{\theta_1 + \theta_2}{\Theta(0)}. \tag{33}$$

Finally we can identify the space of the coefficient $A(x)$ as

$$\begin{aligned} &\mathbf{C}(a_i, \kappa_h, \theta_h)(z_i, y_i, w_i)_{i=1,\dots,2N+2; h=1,2} \,|\, (17), (30), (31), (32), (33) \\ &\quad = \mathbf{C}(a_i, \kappa_h, \theta_h)(z_{i_k}, y_{i_k})_{k=0,1,\dots N}^{i=1,\dots,2N+1; h=1,2;} \,|\, (32), (33) \\ &\quad = \mathbf{C}(a_i, \kappa_h, \theta_h)(z_{i_0}, z_{i_1}, \dots, z_{i_N}; y_{i_1}, \dots, y_{i_N})_{i=1,\dots,2N+1; h=1,2} \,|\, (33). \end{aligned}$$

Here the symbol $K|(A)$ stands for a field K with a relation (A).

These parameters z_i, y_i and w_i are written by λ_k and μ_k. Namely,

$$w_i = \frac{\kappa_1}{\kappa_2} \frac{W(a_i)}{L(a_i)} = \frac{\kappa_1}{\kappa_2} \left(a_i - \alpha + \sum_{l=1}^{N} \frac{\mu_l^{(1)}}{a_i - \lambda_l} \right), \tag{34}$$

$$y_i = \kappa_2 L(a_i) = \kappa_2 \prod_{l=1}^{N} (a_i - \lambda_l), \qquad z_i = \frac{Z(a_i)}{L(a_i)} = a_i - \beta + \sum_{l=1}^{N} \frac{\mu_l^{(2)}}{a_i - \lambda_l}.$$

See the variables $\mu_l^{(1)}$, $\mu_l^{(2)}$, α and β in Section 4. The field $\mathbf{C}(a_i, \kappa_h, \theta_h)(\lambda_k, \mu_k)_{k=1,\dots N}^{i=1,\dots,2N+1; h=1,2;}$ is an algebraic extension of $\mathbf{C}(a_i, \kappa_h, \theta_h) \cdot (z_{i_0}, z_{i_1}, \dots, z_{i_N}; y_{i_1}, \dots, y_{i_N})_{i=1,\dots,2N+1; h=1,2} \,|\, (33)$.

Now we calculate the compatibility condition (19). Before we see the proof of Theorem 3, we rewrite the compatibility in this parameterization.

Theorem 4. *The q-Garnier system* (7) *is equivalent to system* (8):

$$\frac{\overline{z_{n_l}}}{z_r z_s}\left(\frac{z_{n_l}-z_r}{a_{n_l}-a_r}-\frac{z_{n_l}-z_s}{a_{n_l}-a_s}\right)=\left(1-\frac{(1-q\kappa_1/\kappa_2)(a_r-a_s)}{z_r-z_s}\right)\left(\frac{z_{n_l}-z_r}{a_{n_l}-a_r}\frac{1}{z_r}-\frac{z_{n_l}-z_s}{a_{n_l}-a_s}\frac{1}{z_s}\right),$$

$$\begin{aligned}&\overline{y_{n_l}}\left(1-\frac{(1-q\kappa_1/\kappa_2)(a_r-a_s)}{z_r-z_s}\right)\\&\quad=-y_{n_l}\frac{(a_{n_l}-qa_r)(a_{n_l}-qa_s)}{(z_r-z_s)^2}\left(\frac{z_{n_l}-z_r}{a_{n_l}-a_r}-\frac{z_{n_l}-z_s}{a_{n_l}-a_s}\right)\left(\frac{w_{n_l}+z_r}{a_{n_l}-qa_r}-\frac{w_{n_l}+z_s}{a_{n_l}-qa_s}\right),\end{aligned}$$

$(l=0,1\dots,N$ *and* $n_l\in\{1,2,\dots,2N+2\}\backslash\{r,s\})$.

Remark 4. Equation (21) is written in this parameterization as follows:

$$\frac{\overline{w}}{w}=\frac{\overline{w_r}-\overline{w_s}}{z_r-z_s}. \quad \square \tag{35}$$

In system (8), w_{n_l}, z_r and z_s are written by the rational functions of $(y_{n_l},z_{n_l})_{l=0,1,\dots,N}$ (see (30) and (31)). Moreover we have two relations (32) and (33).

These equations define the action of time evolution $T_{r,s}$ on the space of coefficients $A(x)$. Time evolution $(\overline{y_i},\overline{z_i},\overline{w_i})=T_{r,s}(y_i,z_i,w_i)$ is expressed by the system:

$$\frac{\overline{w_r}}{z_s}=\frac{\overline{w_s}}{z_r}=\frac{(1-q\kappa_1/\kappa_2)(a_r-a_s)}{z_r-z_s}-1, \tag{36a}$$

$$\begin{aligned}\overline{y_r}\,\overline{w_r}=\frac{q(a_r-a_s)z_s}{(z_r-z_s)^2}\Bigg[&\left(\frac{1}{(q-1)a_r}-\frac{1}{qa_r-a_s}\right)(\kappa_1X(qa_r)+\kappa_2z_rZ(qa_r))\\&-\left(\frac{z_r}{(q-1)a_r}-\frac{z_s}{qa_r-a_s}\right)(\kappa_1W(qa_r)+\kappa_2z_rL(qa_r))\Bigg],\end{aligned} \tag{36b}$$

$$\begin{aligned}\overline{z_r}\,\overline{y_r}=\frac{q(a_r-a_s)z_rz_s}{(z_r-z_s)^2}\Bigg[&\left(\frac{1}{(q-1)a_r}-\frac{1}{qa_r-a_s}\right)(\kappa_1W(qa_r)+\kappa_2z_rL(qa_r))\\&-\left(\frac{1}{(q-1)a_r}\frac{1}{z_r}-\frac{1}{qa_r-a_s}\frac{1}{z_s}\right)(\kappa_1X(qa_r)+\kappa_2z_rZ(qa_r))\Bigg],\end{aligned} \tag{36c}$$

($\overline{y_s},\overline{z_s}$ are expressed by the equations obtained by the replacement $(r,s)\mapsto(s,r)$.)

(36d) $$\overline{z_n}\left(\frac{z_n - z_r}{a_n - a_r} - \frac{z_n - z_s}{a_n - a_s}\right) + \left(\frac{z_n - z_r}{a_n - a_r}\overline{w_r} - \frac{z_n - z_s}{a_n - a_s}\overline{w_s}\right) = 0,$$

(36e) $$w_n\left(\frac{\overline{w_n} - \overline{w_r}}{a_n - qa_r} - \frac{\overline{w_n} - \overline{w_s}}{a_n - qa_s}\right) + \left(\frac{\overline{w_n} - \overline{w_r}}{a_n - qa_r}z_r - \frac{\overline{w_n} - \overline{w_s}}{a_n - qa_s}z_s\right) = 0,$$

(36f) $$\frac{\overline{y_n}\,\overline{w_r}}{y_n z_s} = \frac{(a_n - qa_r)(a_n - qa_s)}{(z_r - z_s)^2}\left(\frac{z_n - z_r}{a_n - a_r} - \frac{z_n - z_s}{a_n - a_s}\right)\left(\frac{w_n + z_r}{a_n - qa_r} - \frac{w_n + z_s}{a_n - qa_s}\right),$$

$$(n \in \{1, 2, \dots, 2N+2\} \backslash \{r, s\}).$$

We also call this system q-Garnier system. This system is a dynamical system on the $2N$-dimensional space. But we may consider this as a dynamical system on the $3(2N+2)$-dimensional space, which has $4N+6$ integrals defined by relations (30)–(33).

Proof of Theorems 3 *and* 4. Now we calculate the compatibility, (19), in the variables (y_i, z_i, w_i) and deduce that it is equivalent to the system (8). Theorem 3 can be shown from the discussion below, by substituting (34).

The compatibility (19) is written as

(37) $$q(x - a_r)(x - a_s)\bar{A}(x)(xI + B^0_{r,s}) - (x - qa_r)(x - qa_s)(qxI + B^0_{r,s})A(x) = 0.$$

The left hand side is a polynomial of degree $N+4$. We look at this condition at the 4 special points: $x = a_r, a_s, qa_r, qa_s$, then we have

(38) $$\bar{A}(qa_h)(qa_h I + B^0_{r,s}) = 0, \qquad (qa_h I + B^0_{r,s})A(a_h) = 0 \qquad (h = r, s).$$

So the determinant of $\bar{A}(qa_h)$ $(h = r, s)$ is zero and we can parameterize as

$$\bar{A}(qa_h) = \overline{y_h}\begin{pmatrix} 1 \\ \overline{w}^{-1}\overline{z_h} \end{pmatrix}(\overline{w_h} \quad \overline{w}), \qquad (h = r, s).$$

Equations (38) are equivalent to equation (35), the equation

(39) $$z_r\overline{w_r} = z_s\overline{w_s}$$

and the parameterization of $B^0_{r,s}$:

(40) $$B_{r,s}(x) = \frac{x(xI + B^0_{r,s})}{(x - qa_r)(x - qa_s)} = \frac{x}{\overline{w}(z_r - z_s)}(B_r - B_s),$$

where

$$B_h = \frac{1}{x - qa_h}\begin{pmatrix} \overline{w} \\ -\overline{w_h} \end{pmatrix}(z_h \quad -w) \qquad (h = r, s).$$

Substituting the matrix $B^0_{r,s}$ to the compatibility condition (37), we can write $\bar{A}(x)$ by the parameterization of $A(x)$ as follows:

$$\bar{A}(x) = \frac{(x - qa_r)(x - qa_s)}{q(x - a_r)(x - a_s)}(qxI + B^0_{r,s})A(x)(xI + B^0_{r,s})^{-1}.$$

Then $\bar{A}(x)$ is a polynomial. (Although possible simple poles appear at $x = a_h$, it is shown that $(x - a_h)\bar{A}(x)|_{x=a_h} = 0$ $(h = r, s)$.) Because the determinant of $xI + B^0_{r,s}$ is $(x - qa_r)(x - qa_s)$, we have

$$\det \bar{A}(x) = \kappa_1\kappa_2 \frac{(x - qa_r)(x - qa_s)}{(x - a_r)(x - a_s)} \prod_{i=1}^{2N+2} (x - a_i).$$

Moreover we know the leading term of $\bar{A}(x)$ is $\mathrm{diag}(\kappa_1, \kappa_2)$ and $\bar{A}(x)$ have $q\theta_1$ and $q\theta_2$ as the eigenvalues at $x = 0$. Consequently $\bar{A}(x)$ can be parameterized by $\overline{z_j}$, $\overline{y_j}$, etc. as

$$\overline{A(a_j)} = \overline{y_j}\begin{pmatrix} 1 \\ \overline{w}^{-1}\overline{z_j} \end{pmatrix}(\overline{w_j} \quad \overline{w}).$$

They satisfies the over-lined version of (30), (31), (32) and (33).

Considering (37) at $x = a_n$ $(n \neq r, s)$, we get

$$\overline{y_n}\begin{pmatrix} 1 \\ \overline{z_n} \end{pmatrix}\left[\frac{\overline{w_n} - \overline{w_r}}{a_n - qa_r}(z_r \quad -1) - \frac{\overline{w_n} - \overline{w_s}}{a_n - qa_s}(z_s \quad -1)\right] + y_n\left[\frac{z_n - z_r}{a_n - a_r}\begin{pmatrix} 1 \\ -\overline{w_r} \end{pmatrix} - \frac{z_n - z_s}{a_n - a_2}\begin{pmatrix} 1 \\ -\overline{w_s} \end{pmatrix}\right](w_n \quad 1) = 0. \tag{41}$$

This is equivalent to equations (36d)–(36f). Notice that we have only to consider N points among $x = a_n$'s $(n \in \{1, 2, \dots, 2N + 2\}\backslash\{r, s\})$, because equation (37) is of degree $N + 4$. Hence N equations of (41) assure the rest of them.

Now we calculate equation (39). In equation (30) we set $i_0 = r$ and $s \notin \{i_0, \dots, i_N\}$. Eliminating $\overline{w_s}$ in (39), we get an equation about $\overline{y_{i_l}}$, $\overline{w_{i_l}}$ $(l = 1, \dots, N)$, $\overline{y_r}$ and $\overline{w_r}$. But we can eliminate $\overline{y_r}$ by equation (32). Using equations (36d), (36f) and (39), we obtain an equation about $\overline{z_r}$; it is equation (36a).

Eliminating $\overline{w_r}$ and $\overline{w_s}$ from (41), we obtain (8a)–(8b). Inversely, under the condition (8a)–(8b), we can deduce equations (36) and (39), so the proof is finished. □

We can calculate inverse transformation of the system (8a)–(8b). In particular,

$$\frac{z_r}{\overline{w_s}} = \frac{z_s}{\overline{w_r}} = \frac{(1 - q\kappa_1/\kappa_2)(a_r - a_s)}{\overline{w_s} - \overline{w_r}} - 1.$$

It follows that q-Garnier system with this parameterization defines a birational mapping.

7. Relations among the time evolutions

Till the previous section, we discussed about a time evolution $T_{r,s}$ for the fixed r, s ($r, s \in \{1, 2, \dots, 2N+2\}$). In this section we study on relations among several time evolutions. In the first place the next proposition is obvious.

Proposition 5.

$$T_{r,s} = T_{s,r}. \tag{42}$$

Next we see commutativities of time evolutions.

Proposition 6.

$$T_{r_1,s_1} \circ T_{r_2,s_2} = T_{\rho_1,\sigma_1} \circ T_{\rho_2,\sigma_2}, \tag{43}$$

where $\{\rho_1, \rho_2, \sigma_1, \sigma_2\} = \{r_1, r_2, s_1, s_2\}$ *and* r_1, s_1, r_2, s_2 *are distinct from each other.*

Proof. Since the time evolution is defined by the equation

$$T_{r,s}(A(x)) = B_{r,s}(qx)A(x)B_{r,s}(x)^{-1},$$

we have only to show $T_{r_2,s_2}(B_{r_1,s_1})B_{r_2,s_2} = T_{\rho_2,\sigma_2}(B_{\rho_1,\sigma_1})B_{\rho_2,\sigma_2}$. The matrix $T_{r_2,s_2}(B_{r_1,s_1})B_{r_2,s_2}$ is a rational function of the form

$$\frac{x^2(x^2 I + xB^1 + B^0)}{(x - qa_{r_1})(x - qa_{s_1})(x - qa_{r_2})(x - qa_{s_2})}.$$

Moreover this is uniquely determined by the equation

$$q^2(x - a_{r_1})(x - a_{s_1})(x - a_{r_2})(x - a_{s_2})T_{r_1,s_1} \circ T_{r_2,s_2}(A(x))(x^2 I + xB^1 + B^0)$$
$$= (x - qa_{r_1})(x - qa_{s_1})(x - qa_{r_2})(x - qa_{s_2})(q^2x^2 I + qxB^1 + B^0)A(x),$$

as in the proof of Theorems 3 and 4. But this equation is also satisfied by $T_{\rho_2,\sigma_2}(B_{\rho_1,\sigma_1})B_{\rho_2,\sigma_2}$. Uniqueness of solutions of the equation leads to the result. □

As can be seen from the proof above, time evolutions are uniquely determined by the information about the move of $(a_i)_{i=1,\dots,2N+2}$. Let $T_{(d_1,d_2,\dots,d_{2N+2})}$ be the time evolution associated with the transformation $(a_i) \mapsto (q^{d_i}a_i)$ ($\sum_{i=1}^{2N+2} d_i \equiv 0 \mod 2$). Then we don't need the assumption that r_1, s_1, r_2, s_2 are distinct from each other, in Proposition 6.

At the end we close this section with the next proposition.

Proposition 7.

$$T_{(1,1,\dots,1)}(A(x)) = q^{N+1}\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}^{-1} A(q^{-1}x)\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}. \tag{44}$$

Proof. In this case the deformation equation $T(Y(x)) = B(x)Y(x)$ is expressed by the matrix $B = B_{(1,1,\dots,1)}$ that is a rational function of the form

$$\frac{x^{N+1}(x^{N+1}I + x^N B^N + \cdots + B^0)}{\prod_{i=1}^{2N+2}(x - qa_i)}.$$

This is also uniquely determined by the compatibility

$$T_{(1,1,\dots,1)}(A(x))B_{(1,\dots,1)}(x) = B_{(1,\dots,1)}(qx)A(x).$$

This equation has

$$B_{(1,\dots,1)}(x) = (x/q)^{N+1}\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} A(q^{-1}x)^{-1},$$

$$T_{(1,1,\dots,1)}(A(x)) = q^{N+1}\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}^{-1} A(q^{-1}x)\begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}$$

as a solution. Uniqueness of solutions leads to the proposition. □

Equation (44) is written in the terms of (λ_k, μ_k) as

$$T_{(1,\dots,1)} = T_{1,2} \circ T_{3,4} \circ \cdots \circ T_{2N+1,2N+2} : (\lambda_k, \mu_k)_{k=1,\dots,N} \mapsto (q\lambda_k, q^{N+1}\mu_k)_{k=1,\dots,N}.$$

This is a kind of trivial transformations.

Acknowledgements. The author wishes to thank B. Grammaticos, Jimbo M., Ohyama Y., A. Ramani and Tsuda T. for discussions and interest.

References

[1] Birkhoff, G. D., The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference equations, Proc. Am. Acad. Arts and Sciences, **49** (1914), 521–568.

[2] Fuchs, R., Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singulären Stellen, Math. Ann., **63** (1907), 301–321.

[3] Garnier, R., Sur des équations différentielles du troisième ordre dont l'intégrale générale est uniforme et sur une classe d'équations nouvelles d'ordre supérieur dont l'intégrale générale a ses points critiques fixes, Ann. Sci. Ecole Norm. Sup., **29** (1912), 1–126.

[4] Grammaticos, B., Nijhoff, F. W., Papageorgiou, V. G., Ramani, A. and Satsuma, J., Linearization and solutions of the discrete Painlevé III equation, Phys. Lett. A, **185** (1994), 446–452.

[5] Grammaticos, B., Ramani, A. and Papageorgiou, V. G., Do integrable mappings have the Painlevé property?, Phys. Rev. Lett., **67** (1991), 1825–1828.

[6] Jimbo, M. and Sakai, H., A q-analog of the sixth Painlevé equation, Lett. Math. Phys., **38** (1996), 145–154.

[7] Kajiwara, K., Noumi, M. and Yamada, Y., q-Painlevé systems arising from q-KP hierarchy, Lett. Math. Phys., **62** (2002), 259–268.

[8] Kimura, H. and Okamoto, K., On the polynomial Hamiltonian structure of the Garnier systems, J. Math. Pure et Appl., **63** (1984), 129–146.
[9] Noumi, M. and Yamada, Y., Affine Weyl groups, discrete dynamical systems and Painlevé equations, Comm. Math. Phys., **199** (1998), no. 2, 281–295.
[10] Okamoto, K., Isomonodromic deformation and Painlevé equations, and the Garnier system, J. Fac. Sci. Univ. Tokyo Sec. IA, Math., **33** (1986), 575–618.
[11] Painlevé, P., Œuvres t.I,II,III, Centre National de la Recherche Scientifique, Paris, 1976.
[12] Papageorgiou, V. G., Nijhoff, F. W., Grammaticos, B. and Ramani, A., Isomonodromic deformation problems for discrete analogues of Painlevé equations, Phys. Lett. A, **164** (1992), 57–64.
[13] Ramani, A., Grammaticos, B. and Hietarinta, J., Discrete versions of the Painlevé equations, Phys. Rev. Lett., **67** (1991), 1829–1832.

nuna adreso:
Graduate School of Mathematical Sciences
University of Tokyo
Komaba, Tokyo 153-8914
Japan
E-mail: sakai@poisson.ms.u-tokyo.ac.jp

(Ricevita la 28-an de aprilo, 2004)
(Reviziita la 17-an de aŭgusto, 2004)