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Blowup Behavior of Radial Solutions to Jäger-Luckhaus System in High Dimensional Domains

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Abstract. In this paper, we will consider the blowup and the time-global existence of radial solutions to a simplified system of the so called Keller-Segel model in a domain of three or more dimensional Euclidean space. In the two dimensional case, the blowup solutions have a delta function singularity at each blowup point (see [13]). However, in the three dimensional case, Herrero, Medina and Velázquez [7] show that there exist self-similar blowup solutions and that the solutions have a singularity which is different from a delta function singularity. In this paper, we will consider the blowup criterion and the blowup self-similar solutions in the three or more dimensional cases.

Keywords and Phrases. Chemotaxis, Keller-Segel model, Blowup, Self-similar solution.

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1. Introduction

In this paper, we consider the blowup behavior of radial solutions to the following system

$$\begin{cases} u_t = \nabla \cdot (\nabla u - u \nabla v) & \text{in } \Omega \times (0, T), \\ 0 = \Delta v - u + \mu & \text{in } \Omega \times (0, T). \end{cases} \tag{1}$$

Here, μ is a nonnegative constant, and $\Omega = \{x \in \boldsymbol{R}^N \mid |x| < L\}$ $(0 < L \leq \infty$, $N = 1, 2, 3, \dots)$. In particular, we consider the following case.

$$\mu = \begin{cases} \dfrac{\lambda}{|\Omega|} & \text{if } 0 < L < \infty, \\ 0 & \text{if } L = \infty. \end{cases}$$

In the case where $0 < L < \infty$, the initial condition and the boundary condition

$$\begin{cases} \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u_0 & \text{in } \Omega \end{cases} \tag{2}$$

are imposed on the solutions. Here u_0 is radial, smooth, nonnegative and nontrivial in $\overline{\Omega}$, and ν is the outer normal unit vector. Moreover, $\lambda = \|u_0\|_1$ and $|\Omega| = \int_\Omega 1\, dx$.

Here and henceforth, we use the notation $\|\cdot\|_p$ for the $L^p(\Omega)$ norms of functions.

Keller and Segel [9] introduced a system to describe the aggregation of cellular slime molds. We refer to the system introduced by Keller and Segel as the Keller-Segel model. The system (1) is introduced by Jäger and Luckhaus [8] as a simplified Keller-Segel model. We refer to the system (1) as the Jäger-Luckhaus system.

Firstly, we describe the results for the system (1) and (2) with $L \in (0, \infty)$ and $\mu = \lambda/|\Omega|$.

In Theorems 1 and 2, we describe the criterion of the blowup of the solutions. Concerning this aspect, the following results are shown.

In the case where $N = 1$, the blowup cannot occur. That is to say, the solution to (1) and (2) exists and is bounded globally in time.

In the case where $N = 2$, the blowup can occur. Jäger and Luckhaus [8] find blowup solutions for $\lambda \gg 1$. Here, we say that the solution u blows up if $\limsup_{t\to T}\|u(\cdot, t)\|_\infty = \infty$ for some $T \in (0, \infty)$. Nagai [10] introduces a system replacing the second equation (1) with $0 = \Delta v - v + u$. We refer to this system as the Nagai system. Moreover, Nagai [10] shows that the radial solutions to the Nagai system with (2) cannot blow up if $\lambda < 8\pi$, and that the radial solution blows up if $\lambda > 8\pi$ and $\int_\Omega |x|^2 u_0(x)dx \ll 1$. Nagai [11] and the author and Suzuki [14] show similar results for the non-radial solutions to the Nagai system. For non-radial and blowup solutions to the Nagai system with (2), the author and Suzuki [13] show that there are a set $\mathscr{B}$ of finite points in $\overline{\Omega}$ and a L^1-function f satisfying

$$u(\cdot, t) \rightharpoonup \sum_{q\in\mathscr{B}} m(q)\delta_q + f \quad \text{in } \mathscr{M}(\overline{\Omega}) \qquad \text{as } t \to T_{max}.$$

Here and henceforth, T_{max} is the maximal existence time of the classical solution and

$$m(q) \geq \begin{cases} 8\pi & \text{if } q \in \Omega, \\ 4\pi & \text{if } q \in \partial\Omega. \end{cases}$$

By using a similar argument as the one in [10, 11, 14, 13], we can show that the solutions to (1) and (2) have the same properties as those of the solutions to the Nagai system with (2).

Then, in the two dimensional case, the blowup solution has a delta function singularity at each blowup point $q \in \mathscr{B}$.

In the case where $N \geq 3$, Nagai [10] finds blowup solutions to the Nagai system with (2) for each $\lambda > 0$.

Secondly, we describe the result for the system (1) with $L = \infty$. For any $C > 0$, Herrero, Medina and Velázquez [6] find radial and blowup solutions to (1) with $N = 3$, $L = \infty$ and $\mu = 1$ satisfying

$$\int_{|x|<r} u(x, T_{max})dx \to C \qquad r \to 0.$$

Moreover, they [7] find radial and blowup solutions u to (1) with $N = 3$, $L = \infty$ and $\mu = 0$ satisfying

$$u(x, T_{max}) \sim \frac{C}{|x|^2} \qquad \text{as } |x| \to 0 \tag{3}$$

for a constant $C > 0$, and there is a function $\bar{u}$ such that

$$u(x, t) = \frac{1}{T_{max} - t}\bar{u}\left(\frac{|x|}{\sqrt{T_{max} - t}}\right). \tag{4}$$

We refer to the function $\bar{u}$ in (4) as the self-similar solution of (1).

Therefore, in the three or more dimensional cases, we can expect that several singularities appear in blowup solutions.

In this paper, we treat Jäger-Luckhaus system in the three or more dimensional domain. In Theorems 1 and 2, we treat the criterion of the blowup and the time-global existence. That is different from the one in [10]. In Theorem 3, we consider the blowup solutions having a singularity similar to the one in (3), and self-similar solutions to (1) with $\mu = 0$ in $\boldsymbol{R}^N$ $(N \geq 3)$.

Our results are the following.

In Theorems 1 and 2, we assume that u_0 is radial and that Ω is a bounded open ball.

Theorem 1. *Let $\Omega = \{x \in \boldsymbol{R}^N \mid |x| < L\}$, $0 < L < \infty$, $N \geq 3$ and $\mu = \lambda/|\Omega|$. For some $k > 0$ and $\ell \in (0, \ell^*]$, suppose that u_0 satisfies*

$$\int_{|x|<r} u_0(x)dx \leq \frac{\ell \omega_N k r^N}{1 + kr^2} \qquad \textit{for } 0 < r < L,$$

where

$$\ell^* = \frac{2N + 4 + 2(N - 2)kL^2}{N + (N - 2)kL^2} \in \left(2, \frac{2N + 4}{N}\right).$$

Then, the solution u to (1) and (2) exists globally in time and satisfies

$$\sup_{0 \le t} \|u(\cdot\,, t)\|_\infty < \infty.$$

Here, ω_N is the area of the unite sphere in $\boldsymbol{R}^N$.

Theorem 2. *Let $\Omega = \{x \in \boldsymbol{R}^N \mid |x| < L\}$, $0 < L < \infty$, $N \ge 3$, $\mu = \lambda/|\Omega|$ and $\ell > (2N+4)/N$. There exist $k > 0$ and $\delta \in (0, L)$ such that*

$$\int_{|x|<r} u_0(x)dx \ge \frac{\ell \omega_N k r^N}{1 + kr^2} \qquad \text{for } 0 < r \le \delta$$

implies that $T_{max} < \infty$ for the solution to (1) *and* (2). *Then, the solution u to* (1) *and* (2) *blows up.*

In Theorem 1, we observe that $\ell^* \to (2N+4)/N$ as $kL^2 \to 0$. Then, we can regard that $(2N+4)/N$ in Theorem 2 is the best constant.

In the following theorem, we describe the existence of self-similar solutions $\{\bar{u}_j\}_{j\ge1}$ to (1).

We denote

$$V_j(s) = \frac{1}{\omega_N e^{(N-2)s}} \int_0^{e^s} \bar{u}_j(\xi)\xi^{N-1}\, d\xi \qquad \text{and} \qquad H_j(s) = V_j'(s)$$

for $j = 1, 2, 3, \ldots$.

Theorem 3. *Let $\Omega = \boldsymbol{R}^N$, $N \ge 3$ and $\mu = 0$.*

There exists a radial and positive self-similar solution $\bar{u}_1$ to (1) *satisfying*

$$V_1(s) > 0 \qquad \text{and} \qquad H_1(s) > 0 \qquad \text{for } s \in \boldsymbol{R},$$

$\lim_{s\to\infty} V_1(s) = 4$ *and* $\lim_{s\to\infty} H_1(s) = 0$.

In particular, in the case where $3 \le N \le 9$, there exist radial and positive self-similar solutions $\{\bar{u}_j\}_{j\ge2}$ to (1) *satisfying* $\lim_{s\to\infty} V_j(s) \in (0, 2)$ *and* $\lim_{s\to\infty} H_j(s) = 0$ *and the following.*

For $j = 2$, there exists $s_1 \in \boldsymbol{R}$ such that

$$\begin{cases} V_2(s) > 0 \text{ and } H_2(s) > 0 & \text{for } s < s_1, \\ V_2(s_1) > 2 \text{ and } H_2(s_1) = 0, \\ V_2(s) > 0 \text{ and } H_2(s) < 0 & \text{for } s_1 < s. \end{cases}$$

For each $j \ge 3$, there exists $\{s_i\}_{i=1}^{2j-3} \subset \boldsymbol{R}$ such that

$$-\infty < s_1 < s_2 < \cdots < s_{2j-3} < \infty,$$

$$\begin{cases} V_j(s) > 0 \text{ and } H_j(s) > 0 & \text{for } s < s_1 \text{ and } s_{2i} < s < s_{2i+1}, \\ V_j(s_{2i-1}) > 2 \text{ and } H_j(s_{2i-1}) = 0, \\ V_j(s) > 0 \text{ and } H_j(s) < 0 & \text{for } s_{2i-1} < s < s_{2i} \text{ and } s_{2j-3} < s, \\ V_j(s_{2i}) \in (0, 2) \text{ and } H_j(s_{2i}) = 0 \end{cases}$$

for $i = 1, 2, \dots, j-2$ *and that*

$$V_j(s_{2j-3}) > 2 \qquad \textit{and} \qquad H_j(s_{2j-3}) = 0.$$

The self-similar solution $\bar{u}_j$ in Theorem 3 satisfies that

$$e^{2s}\bar{u}_j(e^s) = \omega_N\{(N-2)V_j(s) + H_j(s)\},$$

for each $T > 0$

$$u(x,t) = \frac{1}{T-t}\bar{u}\left(\frac{|x|}{\sqrt{T-t}}\right)$$

is a blowup solution to (1) with $\mu = 0$ and that $u(x,T) = C_j/|x|^2$ with

$$C_j = \lim_{y \to \infty} y^2 \bar{u}(y) = \omega_N(N-2) \lim_{s \to \infty} V_j(s).$$

2. Proof of Theorems 1 and 2

In this section, we prove Theorems 1 and 2 by using the comparison theorem and the following lemma.

Throughout this section, we assume that

$$\Omega = \{x \in \boldsymbol{R}^N \mid |x| < L\}, \; N = 3, 4, 5, \dots, \; 0 < L < \infty \text{ and } \mu = \frac{\lambda}{|\Omega|}.$$

The following proposition and lemmas are shown for solutions to the Nagai system with (2) in [10]. However, by using a similar argument as the one in [10], we can show the following lemma for solutions to (1) and (2). Hence, here we omit the proofs.

Proposition 1. *The system* (1) *and* (2) *has the unique classical solution* u *in* $\Omega \times (0, T_{max})$. *Moreover,* u *is positive in* $\overline{\Omega} \times (0, T_{max})$.

Then, the maximal existence time T_{max} of the classical solution is positive or infinite.

Lemma 2.1. *Let* u *be a solution to* (1) *and* (2). *If* $T_{max} < \infty$, *then* u *satisfies that*

$$\lim_{t \to T_{max}} \|u(\cdot, t)\|_\infty = \infty.$$

Lemma 2.2. *Let* u *be a solution to* (1) *and* (2). *Suppose that*

$$\sup_{0<t<T_{max}} \|\nabla v(\cdot, t)\|_\infty < \infty. \tag{5}$$

Then it holds that

$$\sup_{0<t<T_{max}} \|u(\cdot\,,t)\|_\infty < \infty.$$

By using Lemmas 2.1 and 2.2, if (5) holds, the solution exists globally in time and is bounded.

Putting

$$U(r,t) = \int_{|x|<r} u(x,t)dx, \qquad V(r,t) = \int_{|x|<r} v(x,t)dx,$$

U and V satisfy

$$(6) \qquad \begin{cases} U_t = r^{N-1}\dfrac{\partial}{\partial r}\left(\dfrac{1}{r^{N-1}}\dfrac{\partial U}{\partial r}\right) + \dfrac{1}{\omega_N r^{N-1}}\left(U - \dfrac{\lambda}{L^N}r^N\right)\dfrac{\partial U}{\partial r}, \\ 0 = r^{N-1}\dfrac{\partial}{\partial r}\left(\dfrac{1}{r^{N-1}}\dfrac{\partial V}{\partial r}\right) - \dfrac{\lambda}{L^N}r^N + U. \end{cases}$$

Putting $\Phi = U/(\omega_N r^{N-2})$, Φ satisfies

$$\Phi_t = \Phi_{rr} + \frac{N-3}{r}\Phi_r - \frac{2(N-2)}{r^2}\Phi + \frac{1}{r^2}\left(\Phi - \frac{\lambda r^2}{\omega_N L^N}\right)\{(N-2)\Phi + r\Phi_r\}.$$

Let

$$\mathscr{L}(f) = f_t - f_{rr} - \frac{N-3}{r}f_r + \frac{2(N-2)}{r^2}f - \frac{1}{r^2}\left(f - \frac{\lambda r^2}{\omega_N L^N}\right)\{(N-2)f + rf_r\}.$$

Proof of Theorem 1. Putting $\overline{\Phi} = \dfrac{\ell k r^2}{1+kr^2}$, $\overline{\Phi}$ satisfies

$$\overline{\Phi}_r = \frac{2\ell kr}{(1+kr^2)^2} \qquad \text{and} \qquad \overline{\Phi}_{rr} = \frac{2\ell k}{(1+kr^2)^2} - \frac{8\ell k^2 r^2}{(1+kr^2)^3}.$$

Since we observe that

$$\begin{aligned} \mathscr{L}(\overline{\Phi}) = -\left\{\frac{2\ell k}{(1+kr^2)^2} - \frac{8\ell k^2 r^2}{(1+kr^2)^3}\right\} - \frac{2(N-3)\ell k}{(1+kr^2)^2} + \frac{2(N-2)\ell k}{1+kr^2} \\ - \left(\overline{\Phi} - \frac{\lambda r^2}{\omega_N L^N}\right)\left\{\frac{(N-2)\ell k}{1+kr^2} + \frac{2\ell k}{(1+kr^2)^2}\right\}, \end{aligned}$$

then we have that

$$(7)\quad (1+kr^2)^3\mathscr{L}(\overline{\Phi})$$
$$\begin{aligned}
&= -2\ell k(1+kr^2) + 8\ell k^2r^2 - 2(N-3)\ell k(1+kr^2) \\
&\quad + 2(N-2)\ell k(1+kr^2)^2 - \ell kr^2\{(N-2)\ell k(1+kr^2) + 2\ell k\} \\
&\quad + \frac{\lambda r^2}{\omega_N L^N}\{(N-2)\ell k(1+kr^2)^2 + 2\ell k(1+kr^2)\} \\
&= \{-2\ell k - 2(N-3)\ell k + 2(N-2)\ell k\} \\
&\quad + kr^2\{-2\ell k + 8\ell k - 2(N-3)\ell k + 4(N-2)\ell k \\
&\qquad - \ell[(N-2)\ell k + 2\ell k]\} + k^2r^4\{2(N-2)\ell k - (N-2)\ell^2 k\} \\
&\quad + \frac{\lambda r^2}{\omega_N L^N}\{(N-2)\ell k(1+kr^2)^2 + 2\ell k(1+kr^2)\} \\
&= I + kr^2 II + k^2r^4 III + IV.
\end{aligned}$$

Combining (7) with

$$(8)\qquad I = 0, \qquad II = \ell k\{2N + 4 - \ell N\}, \qquad III = \ell k(N-2)\{2-\ell\}.$$

and

$$(9)\qquad IV \geq 0$$

implies that for $0 < \ell \leq \ell^*$

$$(10)\qquad \mathscr{L}(\overline{\Phi}) \geq \frac{\ell k^2 r^2}{(1+kr^2)^3}\{[2N + 4 + 2(N-2)kr^2] - [N + (N-2)kr^2]\ell\} \geq 0 \qquad \text{for } 0 < r < L.$$

From the assumption of this theorem, we have $\Phi(\cdot,0) \leq \overline{\Phi}$ in $(0,L)$. Combining this with (10), $\Phi(0,t) = \overline{\Phi}(0) = 0$ and $\Phi(L,t) = \Phi(L,0) \leq \overline{\Phi}(L)$, we have that $\Phi \leq \overline{\Phi}$ in $(0,L)\times(0,T_{max})$ or

$$U(r,t) \leq \frac{\ell k \omega_N r^N}{1+kr^2} \qquad \text{in } (0,L)\times(0,T_{max}),$$

in the comparison theorem.

From this, (6) and $\omega_N v_r = (r^{1-N}V_r)_r$, we observe that

$$(11)\qquad |\nabla v| = |v_r| \leq \frac{1}{\omega_N r^{N-1}}\left\{\frac{\lambda}{L^N}r^N + U\right\} \leq C_1 \qquad \text{in } (0,L)\times(0,T_{max}),$$

where C_1 is a positive constant.

From this, Lemmas 2.1 and 2.2, we get this theorem. □

For a positive constant η, let $\bar{\varphi}_\eta$ be a smooth function in $[0,\infty)$ satisfying

$$0 \le \bar{\varphi}_\eta \le 1, \qquad \bar{\varphi}_\eta' \le 0$$

and

$$\bar{\varphi}_\eta(s) = \begin{cases} 0 & \text{if } 1+\eta \le s < \infty, \\ 1 & \text{if } 0 \le s \le 1. \end{cases}$$

For a positive constant ε, we put $\varphi_{\eta,\varepsilon}(x) = \bar{\varphi}_\eta^3(|x|/\varepsilon)$.

Lemma 2.3. *Let u be a radial solution to* (1) *and* (2). *For $0 < \varepsilon < L/2$ and $0 < \eta < 1$, it holds that*

$$\left|\frac{d}{dt}\int_\Omega u(x,t)\varphi_{\varepsilon,\eta}(x)dx\right| \le C_2 \qquad \text{for } 0 < t < T_{max},$$

where C_2 is a positive constant depending only on ε and $\|\bar{\varphi}_\eta\|_{C^1([0,\infty))}$.

Proof. Noticing that $\partial\varphi_{\varepsilon,\eta}/\partial\nu = 0$ in $\partial\Omega$, we observe that

$$\begin{aligned} \frac{d}{dt}\int_\Omega u\varphi_{\varepsilon,\eta}\,dx &= \int_\Omega \nabla\cdot(\nabla u - u\nabla v)\varphi_{\varepsilon,\eta}\,dx \\ &= \int_\Omega u\Delta\varphi_{\varepsilon,\eta}\,dx + \int_\Omega u\nabla v\cdot\nabla\varphi_{\varepsilon,\eta}\,dx. \end{aligned} \tag{12}$$

Since it holds that $0 \le U \le \lambda$ in Ω and that $\nabla\varphi_{\varepsilon,\eta} = \mathbf{0}$ for $|x| \ge (1+\eta)\varepsilon$, we have that

$$\begin{aligned} \left|\int_\Omega u\nabla v\cdot\nabla\varphi_{\varepsilon,\eta}\,dx\right| &= \left|\int_\Omega u\left(\nabla v\cdot\frac{x}{|x|}\right)\left(\frac{x}{|x|}\cdot\nabla\varphi_{\varepsilon,\eta}\right)dx\right| \\ &\le \int_\Omega \frac{u}{\omega_N|x|^{N-1}}\left(\lambda\frac{|x|^N}{L^N} + U\right)\frac{3}{\varepsilon}|\bar{\varphi}_\eta'|dx \le \frac{6\lambda^2}{\omega_N\varepsilon^N}\|\bar{\varphi}_\eta\|_{C^1([0,\infty))}. \end{aligned}$$

From this and (12), we have this lemma. □

Proof of Theorem 2. For $(2N+4)/N < \tilde{\ell} < \ell$ and $T > 0$, we put

$$\underline{\Phi}(r,t) = \frac{\tilde{\ell}k(t)r^2}{1+k(t)r^2} \qquad \text{and} \qquad k(t) = \frac{1}{(T-t)^2}.$$

Combining (7) with

$$\underline{\Phi}_t(r,t) = \frac{\tilde{\ell}k'(t)r^2}{(1+k(t)r^2)^2}$$

implies that

$$(1+k(t)r^2)^3\mathscr{L}(\underline{\Phi}) = \tilde{\ell}k'(t)r^2(1+k(t)r^2) + \tilde{I} + k(t)r^2\widetilde{II} + k(t)^2r^4\widetilde{III} + \widetilde{IV},$$

where $\tilde{I}$, $\widetilde{II}$, $\widetilde{III}$ and $\widetilde{IV}$ are the terms I, II, III and IV in (7) replacing k and ℓ with $k(t)$ and $\tilde{\ell}$, respectively. Then, it follows from this and (8) that

$$(1+k(t)r^2)^3\mathscr{L}(\underline{\Phi}) = k(t)r^2 VI + k(t)^2r^4 VII, \tag{13}$$

where

$$\begin{aligned} VI &= \tilde{\ell}\frac{k'(t)}{k(t)} + \frac{\lambda N\tilde{\ell}}{\omega_N L^N} + \widetilde{II} \\ &= \tilde{\ell}k(t)\left\{2\tilde{\ell}(T-t) + 2N + 4 + \frac{\lambda N}{\omega_N L^N}(T-t)^2 - \tilde{\ell}N\right\} \end{aligned} \tag{14}$$

and

$$\begin{aligned} VII &= \tilde{\ell}\frac{k'(t)}{k(t)} + \frac{\lambda(2N-2)\tilde{\ell}}{\omega_N L^N} + \frac{\lambda(N-2)\tilde{\ell}k(t)r^2}{\omega_N L^N} + \widetilde{III} \\ &= \tilde{\ell}(N-2)k(t)\left\{\frac{2(T-t)}{(N-2)} + \frac{2(N-1)\lambda}{(N-2)\omega_N L^N}(T-t)^2 + \frac{\lambda r^2}{\omega_N L^N} + 2 - \tilde{\ell}\right\}. \end{aligned} \tag{15}$$

We take $\delta > 0$ such that

$$-\frac{2}{N} + \frac{\lambda\delta^2}{\omega_N L^N} = 0.$$

Noticing $\tilde{\ell} > (2N+4)/N$, we obtain that $VI \le 0$ and $VII \le 0$ in $(0,\delta)\times(0,T)$ for any sufficiently small $T > 0$. Then, we have that

$$\mathscr{L}(\underline{\Phi}) \le 0 \qquad \text{in } (0,\delta)\times(0,T) \tag{16}$$

for any sufficiently small $T > 0$.

By Lemma 2.3, we have that

$$\begin{aligned} U((1+\eta)\varepsilon, t) - U(\varepsilon, 0) &\ge \int_\Omega u(x,t)\varphi_{\varepsilon,\eta}(x)dx - \int_\Omega u_0(x)\varphi_{\varepsilon,\eta}(x)dx \\ &= \int_0^t \frac{d}{ds}\int_\Omega u(x,s)\varphi_{\varepsilon,\eta}(x)dxds \ge -C_2 t. \end{aligned} \tag{17}$$

We assume that for $k_0 > 0$

(18) $$U(r,0) \geq \frac{2\omega_N \ell k_0 r^N}{1+2k_0 r^2} \qquad \text{for } 0 < r \leq \delta.$$

Putting $\theta = \sqrt{\ell/\tilde{\ell}} > 1$, we take $\eta > 0$ satisfying

$$\theta\tilde{\ell}(1+\eta)^N = \ell.$$

Then, for any $k_0 > 0$ and $0 < r < L$, it holds that

$$\frac{2\ell k_0 r^N}{1+2k_0 r^2} \geq \frac{\theta\tilde{\ell}k_0(1+\eta)^N r^N}{1+k_0(1+\eta)^2 r^2}.$$

Combining this with (17) and (18), for any $0 < \varepsilon \leq \delta(1+\eta)^{-1}$, we have that

(19) $$U((1+\eta)\varepsilon, t) \geq U(\varepsilon, 0) - C_2 t \geq \frac{\theta\tilde{\ell}k_0(1+\eta)^N \varepsilon^N}{1+k_0(1+\eta)^2\varepsilon^2} - C_2 t.$$

Since C_2 depends only on η and ε, then for such η, ε and C_2, we can take $T > 0$ such that

(20) $$\frac{\theta\tilde{\ell}T^{-2}(1+\eta)^N\varepsilon^N}{1+T^{-2}(1+\eta)^2\varepsilon^2} - C_2 T \geq \frac{\tilde{\ell}T^{-2}(1+\eta)^N\varepsilon^N}{1+T^{-2}(1+\eta)^2\varepsilon^2}.$$

Putting $k_0 = T^{-2}$, we observe that

(21) $$\Phi((1+\eta)\varepsilon, t) \geq \underline{\Phi}((1+\eta)\varepsilon, t) \qquad \text{for } 0 < t < \min(T_{max}, T)$$

by (18) and (19).

Therefore, taking $\delta > 0$, $\eta > 0$ and $\varepsilon > 0$ such that

$$-\frac{2}{N} + \frac{\lambda\delta^2}{\omega_N L^N} = 0, \qquad \sqrt{\ell\tilde{\ell}}(1+\eta)^N = \ell \qquad \text{and} \qquad 0 < \varepsilon \leq \frac{\delta}{1+\eta},$$

and take $T > 0$ satisfying (20), $VI \leq 0$ at $t = 0$ and that $VII \leq 0$ at $(r,t) = (\delta, 0)$.

For such δ and T, we assume (18) with $k_0 = 1/T^2$. Then, we have (16), (21), $\Phi(0,t) = 0 = \underline{\Phi}(0,t)$ in $(0, \min(T_{max}, T))$ and $\Phi(r,0) \geq \underline{\Phi}(r,0)$ in $(0,\delta)$.

Combining those with the comparison theorem, it implies that

$$\Phi \geq \underline{\Phi} \qquad \text{in } (0,(1+\eta)\varepsilon) \times (0, \min(T_{max}, T))$$

or

$$U(r,t) \geq \omega_N r^{N-2}\underline{\Phi}(r,t) \qquad \text{in } (0,(1+\eta)\varepsilon) \times (0, \min(T_{max}, T)).$$

Putting $r(t) = (1+\eta)\varepsilon\sqrt{1-(t/T)}$, it holds that

$$\|u(\cdot,t)\|_\infty \geq \frac{NU(r(t),t)}{\omega_N r(t)^N} = \frac{N\Phi(r(t),t)}{r(t)^2} \geq \frac{N\underline{\Phi}(r(t),t)}{r(t)^2}$$
$$= \frac{N\tilde{\ell}k(t)}{1+k(t)r(t)^2} = \frac{N\tilde{\ell}T}{T(T-t)^2+(1+\eta)^2\varepsilon^2(T-t)}.$$

From this, we obtain that $0 < T_{max} \leq T$. Here, we finish the proof of this theorem. □

3. Proof of Theorem 3

In this section, we treat (1) with $\mu = 0$ and $\Omega = \boldsymbol{R}^N$ $(N \geq 3)$.
Recall that

$$\Phi(r,t) = \frac{1}{\omega_N r^{N-2}} U(r,t), \qquad U(r,t) = \int_{|x|<r} u(x,t)dx.$$

For $T > 0$, let as put

$$y = \frac{r}{\sqrt{T-t}}, \qquad \tau = -\log(T-t) \qquad \text{and} \qquad \Psi(y,\tau) = \Phi(r,t).$$

Then, Ψ satisfies that

$$\mathscr{M}(\Psi) = \Psi_\tau - \Psi_{yy} - \left(\frac{N-3}{y} - \frac{y}{2}\right)\Psi_y + \frac{2(N-2)}{y^2}\Psi$$
$$-\frac{\Psi}{y^2}\{(N-2)\Psi + y\Psi_y\} = 0 \qquad \text{in } (0,\infty)\times(-\log T,\infty)$$

with

$$\Psi(0,\cdot) = \Psi_y(0,\cdot) = 0 \qquad \text{in } (-\log T,\infty).$$

The self-similar solution satisfies $\Psi_\tau = 0$ and $\mathscr{M}(\Psi) = 0$. Then, we consider the problem

$$\begin{cases} \Psi_{yy} + \left(\dfrac{N-3}{y} - \dfrac{y}{2}\right)\Psi_y - \dfrac{2(N-2)}{y^2}\Psi \\ \qquad + \dfrac{\Psi}{y^2}\{(N-2)\Psi + y\Psi_y\} = 0 \quad \text{in } (0,\infty), \\ \Psi(0) = \Psi_y(0) = 0. \end{cases} \tag{22}$$

For $S \in \boldsymbol{R}$ and a solution of (22), putting $s = \log y - S$, $V(s) = \Psi(y)$ and $H(s) = V'(s)$, then the problem (22) is equivalent to the problem

$$(23)\qquad \begin{cases} V' = H \quad \text{in } \boldsymbol{R}, \\ H' = -(N-4)H + 2(N-2)V \\ \qquad\quad + \dfrac{1}{2}e^{2(s+S)}H - V\{(N-2)V + H\} \quad \text{in } \boldsymbol{R}, \\ \lim_{s\to-\infty} V(s) = 0, \quad \lim_{s\to-\infty} H(s) = 0. \end{cases}$$

Let $\varepsilon \in (0,1)$ be positive and sufficiently small and let $S < (1/2)\log\varepsilon + \log(N-2)$.

Firstly, we prove the existence and uniqueness of the solution to (23) in the set

$$(24)\qquad \mathcal{O}_\varepsilon = \{(V,H) \in C^1((-\infty,0])^2 \mid V > 0, H > 0, V(0) = \varepsilon,$$
$$\|V\|_{\mathcal{O},1} = \sup_{s\le 0} e^{-(N-2)s}|V(s)| \le 2\varepsilon,$$
$$\|H\|_{\mathcal{O},1} = \sup_{s\le 0} e^{-(N-2)s}|H(s)| \le (N-1)\varepsilon\}.$$

Let $\|(V,H)\|_{\mathcal{O}} = \|V\|_{\mathcal{O},1} + \|H\|_{\mathcal{O},1}$.

For $(V_0,H_0) \in \mathcal{O}_\varepsilon$, we define $(\tilde{V},\tilde{H})$ as the solution to

$$\begin{cases} \tilde{V}' = \tilde{H}, \\ \tilde{H}' = -(N-4)\tilde{H} + 2(N-2)\tilde{V} + \dfrac{1}{2}e^{2(s+S)}H_0 - V_0\{(N-2)V_0 + H_0\} \\ \lim_{s\to-\infty} \tilde{V}(s) = 0, \quad \lim_{s\to-\infty} \tilde{H}(s) = 0, \quad \tilde{V}(0) = \varepsilon. \end{cases}$$

Then, we define $(\tilde{V},\tilde{H}) = \mathcal{F}_S(V_0,H_0)$.

Lemma 3.1. *For any sufficiently small $\varepsilon > 0$ and $S < (1/2)\log\varepsilon + \log(N-2)$, $\mathcal{F}_S$ satisfies that $\mathcal{F}_S\mathcal{O}_\varepsilon \subset \mathcal{O}_\varepsilon$ and has a unique fixed point (V,H) in $\mathcal{O}_\varepsilon$ satisfying*

$$V(s) > 0, \qquad \textit{and} \qquad H(s) > 0 \qquad \textit{for } s \in (-\infty, 0].$$

Proof. Putting

$$F(s) = \frac{1}{2}e^{2(s+S)}H_0 - V_0\{(N-2)V_0 + H_0\},$$

we have that

$$\tilde{V}(s) = C_3 e^{(N-2)s} + \frac{1}{N}e^{(N-2)s}\int_{-\infty}^{s} e^{-(N-2)\xi}F(\xi)d\xi$$
$$-\frac{1}{N}e^{-2s}\int_{-\infty}^{s} e^{2\xi}F(\xi)d\xi = C_3 e^{(N-2)s} + VIII(s) - IX(s).$$

Then, it holds that $\varepsilon = \tilde{V}(0) = C_3 + VIII(0) - IX(0)$,

$$|VIII(s)| \le \frac{1}{N} e^{(N-2)s}\left(\frac{N-1}{4}\varepsilon^2 e^{2s} + \frac{2(3N-5)}{N-2}\varepsilon^2 e^{(N-2)s}\right)$$

and that

$$|IX(s)| \le \frac{1}{N} e^{(N-2)s}\left(\frac{N-1}{2(N+2)}\varepsilon^2 e^{2s} + \frac{3N-5}{N-1}\varepsilon^2 e^{(N-2)s}\right).$$

Therefore, for any sufficiently small $\varepsilon > 0$, we can obtain that $C_3 < 3\varepsilon/2$, $\|V\|_{\mathcal{O},1} \le 2\varepsilon$ and that V is positive in $(-\infty, 0)$.

We have that

$$\tilde{H}(s) = \tilde{V}'(s) = (N-2)C_3 e^{(N-2)s} + \frac{(N-2)}{N} e^{(N-2)s}\int_{-\infty}^{s} e^{-(N-2)\xi}F(\xi)d\xi + \frac{2}{N}e^{-2s}\int_{-\infty}^{s} e^{2\xi}F(\xi)d\xi.$$

Taking the smaller $\varepsilon > 0$, if necessary, and using an argument similar to the above, we have that $\tilde{H}(s) > 0$ for $-\infty < s \le 0$ and that $\|\tilde{H}\|_{\mathcal{O},1} \le (N-1)\varepsilon$.

Therefore, if $\mathscr{F}_S$ has a fixed point $(\tilde{V}, \tilde{H})$, then we obtain that $\tilde{V}(s) > 0$ and $\tilde{H}(s) > 0$ for $s \in (-\infty, 0]$ and that $(\tilde{V}, \tilde{H}) \in \mathcal{O}_\varepsilon$. For any sufficiently small $\varepsilon > 0$ and any $S < (1/2)\log\varepsilon + \log(N-2)$ we define

$$(\tilde{V}, \tilde{H}) = \mathscr{F}_S(V_0, H_0) \qquad \text{for } (V_0, H_0) \in \mathcal{O}_\varepsilon.$$

Then, for any sufficiently small $\varepsilon > 0$ and any $S < (1/2)\log\varepsilon + \log(N-2)$, $\mathscr{F}_S$ is a map on $\mathcal{O}_\varepsilon$.

For $(V_{0i}, H_{0i}) \in \mathcal{O}_\varepsilon$ $(i = 1, 2)$, putting

$$(\tilde{V}_i, \tilde{H}_i) = \mathscr{F}_S(V_{0i}, H_{0i}),$$

$$F_i = \frac{1}{2}e^{2(s+S)}H_{0i} - V_{0i}(H_{0i} + (N-2)V_{0i}),$$

$$C_{3,i} = \varepsilon - \frac{1}{N}\int_{-\infty}^{0} e^{-(N-2)\xi}F_i(\xi)d\xi + \frac{1}{N}\int_{-\infty}^{0} e^{2\xi}F_i(\xi)d\xi,$$

we observe that

$$\tilde{V}_2(s) - \tilde{V}_1(s) = (C_{3,2} - C_{3,1})e^{(N-2)s} + \frac{1}{N}e^{(N-2)s}\int_{-\infty}^{s} e^{-(N-2)\xi}(F_2(\xi) - F_1(\xi))d\xi - \frac{1}{N}e^{-2s}\int_{-\infty}^{s} e^{2\xi}(F_2(\xi) - F_1(\xi))d\xi,$$

$$\tilde{H}_2(s) - \tilde{H}_1(s) = (N-2)(C_{3,2} - C_{3,1})e^{(N-2)s} + \frac{(N-2)}{N} e^{(N-2)s} \int_{-\infty}^{s} e^{-(N-2)\xi}(F_2(\xi) - F_1(\xi))d\xi + \frac{2}{N} e^{-2s} \int_{-\infty}^{s} e^{2\xi}(F_2(\xi) - F_1(\xi))d\xi.$$

$$C_{3,2} - C_{3,1} = -\frac{1}{N}\int_{-\infty}^{0} e^{-(N-2)\xi}(F_2(\xi) - F_1(\xi))d\xi + \frac{1}{N}\int_{-\infty}^{0} e^{2\xi}(F_2(\xi) - F_1(\xi))d\xi.$$

Combining those with

$$\|F_2 - F_1\|_{\mathcal{O},1} \le 3\varepsilon\|H_{02} - H_{01}\|_{\mathcal{O},1} + (3N-5)\varepsilon\|V_{02} - V_{01}\|_{\mathcal{O},1}$$

implies that

$$\|\tilde{V}_2 - \tilde{V}_1\|_{\mathcal{O},1} \le C_4\varepsilon(\|V_{02} - V_{01}\|_{\mathcal{O},1} + \|H_{02} - H_{01}\|_{\mathcal{O},1}),$$

$$\|\tilde{H}_2 - \tilde{H}_1\|_{\mathcal{O},1} \le C_5\varepsilon(\|V_{02} - V_{01}\|_{\mathcal{O},1} + \|H_{02} - H_{01}\|_{\mathcal{O},1}),$$

Here and henceforth, C_i $(i = 4, 5, 6, 7, 8)$ are positive constants depending only on N. Then, we obtain that

$$\|(\tilde{V}_2 - \tilde{V}_1, \tilde{H}_2 - \tilde{H}_1)\|_{\mathcal{O}} \le C_6\varepsilon\|(V_{02} - V_{01}, H_{02} - H_{01})\|_{\mathcal{O}}$$

and that $\mathscr{F}_S$ is a contraction map on $\mathcal{O}_\varepsilon$ for any sufficiently small $\varepsilon > 0$ and any $S < (1/2)\log\varepsilon + \log(N-2)$. Then, $\mathscr{F}_S$ has a unique fixed point (V, H) in $\mathcal{O}_\varepsilon$. Thus, we finish the proof of this lemma. □

From the definition of $\mathscr{F}_S$, the fixed point of $\mathscr{F}_S$ is the solution to (23) with $V(0) = \varepsilon$.

For any sufficiently small $\varepsilon > 0$ and any $S < (1/2)\log\varepsilon + \log(N-2)$, we define the fixed point of $\mathscr{F}_S$ in $\mathcal{O}_\varepsilon$ as (V_S, H_S). Moreover, we define

$$F_S(s) = \frac{1}{2}e^{2(s+S)}H_S(s) - V_S(s)(H_S(s) + (N-2)V_S(s)),$$

$$C_3(S) = \varepsilon - \frac{1}{N}\int_{-\infty}^{0} e^{-(N-2)\xi}F_S(\xi)d\xi + \frac{1}{N}\int_{-\infty}^{0} e^{2\xi}F_S(\xi)d\xi.$$

Then, the following holds.

Lemma 3.2. *For any sufficiently small* $\varepsilon > 0$, (V_S, H_S) *is continuous in* $\mathcal{O}_\varepsilon$ *with respect to* $S \in (-\infty, (1/2)\log\varepsilon + \log(N-2))$.

Proof. We consider only any sufficiently small $\varepsilon > 0$ such that Lemma 3.1 holds. Let $S_i < (1/2)\log\varepsilon + \log(N-2)$ for $i = 1, 2$. We assume that $S_1 < S_2$ without loss of generality. For $i = 1, 2$, putting

$$V_i = V_{S_i}, \qquad H_i = H_{S_i}, \qquad \mathscr{F}_i = \mathscr{F}_{S_i} \qquad \text{and}$$

$$F_i = \frac{1}{2}e^{2(s+S_i)}H_i - V_i(H_i + (N-2)V_i),$$

we observe that

$$\begin{aligned}|F_2(s) - F_1(s)| \le \frac{1}{2}e^{2s}(e^{2S_2} - e^{2S_1})H_1(s) + \frac{1}{2}e^{2(s+S_2)}|H_2(s) - H_1(s)| \\ + |V_2(s) - V_1(s)|(H_2(s) + (N-2)V_2(s)) \\ + V_1(s)(|H_2(s) - H_1(s)| + (N-2)|V_2(s) - V_1(s)|)\end{aligned}$$

and that

$$\begin{aligned}|C_3(S_1) - C_3(S_2)| \le \frac{1}{N}\left|\int_{-\infty}^{0} e^{-(N-2)\xi}(F_2(\xi) - F_1(\xi))d\xi\right| \\ + \frac{1}{N}\left|\int_{-\infty}^{0} e^{2\xi}(F_2(\xi) - F_1(\xi))d\xi\right|\end{aligned}$$

Then, by using a calculation similar to the one in Lemma 3.1, we have that

$$\|(V_2 - V_1, H_2 - H_1)\|_{\mathscr{O}} \le C_7|e^{2S_1} - e^{2S_2}| + C_8\varepsilon\|(V_2 - V_1, H_2 - H_1)\|_{\mathscr{O}}.$$

That implies the continuity of (V_S, H_S) in $\mathscr{O}_\varepsilon$ with respect to S for any sufficiently small $\varepsilon > 0$ and $S < (1/2)\log\varepsilon + \log(N-2)$. Thus, we have this lemma. □

Here and henceforth, we consider only $\varepsilon \in (0, 1)$ such that Lemmas 3.1 and 3.2 hold.

Let $(V_{-\infty}, H_{-\infty})$ be the fixed point of $\mathscr{F}_{-\infty}$. Then, $(V_{-\infty}, H_{-\infty})$ satisfies the following system.

$$(25) \qquad \begin{cases} V' = H \quad \text{in } \boldsymbol{R}, \\ H' = -(N-4)H + 2(N-2)V - V\{(N-2)V + H\} \quad \text{in } \boldsymbol{R}, \\ \lim_{s\to-\infty} V(s) = 0, \quad \lim_{s\to-\infty} H(s) = 0, \quad V(0) = \varepsilon. \end{cases}$$

Biler, Hilhorst and Nadzieja [1] investigate the properties of the solutions to (25). Then, we have the following lemma.

Lemma 3.3. *The system* (25) *has a unique solution* $(V_{-\infty}, H_{-\infty})$.

(1) $\lim_{s\to\infty} V_{-\infty}(s) = 2$, $\lim_{s\to\infty} H_{-\infty}(s) = 0$.

(2) *In the case where* $3 \le N \le 9$, *there exists a sequence* $\{s_j\}_{j=1}^{\infty}$ *satisfying*

$$\begin{cases} -\infty < s_j < s_{j+1} < \infty, \\ V_{-\infty}(s) > 0 \ \text{and} \ H_{-\infty}(s) > 0 \quad \text{for} \ s < s_1 \ \text{and} \ s_{2j} < s < s_{2j+1}, \\ V_{-\infty}(s_{2j-1}) > 2 \ \text{and} \ H_{-\infty}(s_{2j-1}) = 0, \\ V_{-\infty}(s) > 0 \ \text{and} \ H_{-\infty}(s) < 0 \quad \text{for} \ s_{2j-1} < s < s_{2j}, \\ V_{-\infty}(s_{2j}) \in (0,2) \ \text{and} \ H_{-\infty}(s_{2j}) = 0 \end{cases}$$

for $j = 1, 2, 3, \ldots$.

(3) *In the case where* $N \ge 10$, *it holds that* $V(s) > 0$ *and* $H(s) > 0$ *for* $s \in \boldsymbol{R}$.

Here and henceforth, we regard (V_S, H_S) as the function in $(-\infty, s_{max})$, where s_{max} is the maximal existence time of (V_S, H_S). Then, it holds that $s_{max} \in (0, \infty]$.

The following lemmas are shown by using Lemma 3.2 and the continuity of the solution to

$$\begin{cases} V' = H \quad \text{in } (0,\infty), \\ H' = -(N-4)H + 2(N-2)V \\ \qquad + \dfrac{1}{2}e^{2(s+S)}H - V\{(N-2)V + H\} \quad \text{in } (0,\infty) \end{cases}$$

with respect to $V(0)$, $H(0)$ and S.

Lemma 3.4. *For any* $\tilde{S} < (1/2)\log\varepsilon + \log(N-2)$ *and any* $\tilde{s} < s_{max} = s_{max}(\tilde{S})$, *there exists a positive constant* δ *such that* $\tilde{s} < s_{max}(S)$ *for* $S \in (\tilde{S}-\delta, \tilde{S}+\delta)$ *and that* (V_S, H_S) *is continuous with respect to* $(S, s) \in (\tilde{S}-\delta, \tilde{S}+\delta) \times [0, \tilde{s}]$.

Here and henceforth, $s_j(S)$ $(j = 1, 2, 3, \ldots)$ and $s_{max}(S)$ denote s_j and the maximal existence time of (V_S, H_S), respectively.

Lemma 3.5. *As* $S \to -\infty$, $s_{max}(S)$ *tends to infinity and* (V_S, H_S) *uniformly converges to* $(V_{-\infty}, H_{-\infty})$ *in* $(-\infty, \tilde{s}]$ *for any constant* $\tilde{s}$.

Proof of Theorem 3. For $N \ge 3$, we obtain that

$$\Psi(y) = \frac{4y^2}{2(N-2) + y^2}$$

satisfies (22). Since there exists a self-similar solution corresponding to Ψ, then for $N \ge 10$ we get this theorem.

Therefore, we consider only the case where $3 \le N \le 9$.

Putting

$$\varepsilon = \frac{4e^{2S(1)}}{2(N-2) + e^{2S(1)}},$$

$$V_{S(1)}(s) = \frac{2e^{2(s+S(1))}/(N-2)}{1 + e^{2(s+S(1))}/(2N-4)} \qquad \text{and} \qquad H_{S(1)}(s) = V'_{S(1)}(s),$$

$S(1)$ satisfies $S(1) < (1/2)\log\varepsilon + \log(N-2)$ and $(V_{S(1)}, H_{S(1)})$ is the fixed point of $\mathscr{F}_{S(1)}$ in $\mathscr{O}_\varepsilon$.

We shall show the existence of $\{S(j)\}_{j=1}^{\infty} \subset (-\infty, (1/2)\log\varepsilon + \log(N-2))$ such that $(V_{S(j)}, H_{S(j)})$ is the fixed point of $\mathscr{F}_{S(j)}$ in $\mathscr{O}_\varepsilon$.

Step 1. For any sufficiently small S, there exist s_1 and s_2 with $s_1 < s_2$ such that

$$\begin{cases} V_S(s) > 0, \quad H_S(s) > 0, \quad \text{for } s < s_1, \\ V_S(s_1) > 2, \quad H_S(s_1) = 0, \\ V_S(s) > 0, \quad H_S(s) < 0 \quad \text{for } s_1 < s < s_2, \\ V_S(s_2) \in (0,2), \quad H_S(s_2) = 0, \end{cases} \tag{26}$$

by Lemmas 3.3 and 3.5. Let

$$\mathscr{S}_2 = \{S < S(1) \mid (V_S, H_S) \text{ has } s_1 \text{ and } s_2 \text{ satisfying (26)}\}.$$

Since any sufficiently small S is in $\mathscr{S}_2$, the unbounded connected component of $\mathscr{S}_2$ exists. Let the supremum of the unbounded connected component of $\mathscr{S}_2$ be $S(2)$.

In order to prove $S(2) \notin \mathscr{S}_2$, we assume that $S(2) \in \mathscr{S}_2$. Since it holds that

$$H'_{S(2)}(s_2) = (N-2)V_{S(2)}(s_2)\{2 - V_{S(2)}(s_2)\} > 0,$$

we observe that $s_2 < s_{max}$ and that

$$V_{S(2)}(s) > 0 \qquad \text{and} \qquad H_{S(2)}(s) > 0 \qquad \text{for } 0 < s - s_2 \ll 1.$$

Combining this with Lemma 3.4 implies that (V_S, H_S) satisfies (26) for $0 < S - S(2) \ll 1$. It contradicts the definition of $S(2)$. Then, we have $S(2) \notin \mathscr{S}_2$ and that the unbounded connected component of $\mathscr{S}_2$ is $(-\infty, S(2))$.

In order to prove $S(2) < S(1)$, we assume that $S(2) = S(1)$.

For any $S < S(2)$, (V_S, H_S) has a constant $s_1 = s_1(S)$ and $s_2 = s_2(S)$ satisfying (26) by the definition of $S(2)$. By using $S(2) = S(1)$, we shall show that

$$s_1(S) \to \infty \qquad \text{as } S \nearrow S(1). \tag{27}$$

In order to prove (27), we assume that $\liminf_{S \nearrow S(1)} s_1(S) < \infty$.

Since there exists $\{S_n\} \subset (-\infty, S(1))$ such that

$$\lim_{n\to\infty} S_n = S(1), \qquad \lim_{n\to\infty} s_1(S_n) = s_{1*} < \infty,$$

we have that

$$0 = \lim_{n\to\infty} H_{S_n}(s_1(S_n)) = H_{S(1)}(s_{1*}),$$

by Lemma 3.4. It contradicts $H_{S(1)} > 0$ in $\boldsymbol{R}$. Then, we get (27).

Let $V_{S(1)}(\infty) = \lim_{s\to\infty} V_{S(1)}(s)$,

$$\ell_1 = \frac{1}{2}(V_{S(1)}(\infty) + 2) \qquad \text{and} \qquad \ell_2 = -(N-2)^{1/2}\left(\frac{4}{3} - \ell_1^2 + \frac{1}{3}\ell_1^3\right)^{1/2}.$$

For $S < S(1)$ with $|S - S(1)| \ll 1$, we can find $s < s_1$ such that $V_S(s) \geq \ell_1$ by Lemma 3.4 and $V_{S(1)}(\infty) = 4$. It holds that $V_S(s_1) > \ell_1$ for $S < S(1)$ with $|S - S(1)| \ll 1$, since $V_S'(s) = H_S(s) > 0$ for $s < s_1$.

By doing this and (27), we can find $\tau_1 \in (s_1, s_2)$ satisfying

(28)
$$\frac{1}{4}e^{2(\tau_1+S)}\ell_2 + 2(N-2) < 0, \quad \frac{1}{4}e^{2(\tau_1+S)} - (N-4) - \ell_1 > 0 \quad \text{and} \quad V_S(\tau_1) = \ell_1$$

for any $S < S(1)$ with $|S - S(1)| \ll 1$, since $V_S' = H_S < 0$ in (s_1, s_2).

From (28), we get that

$$H_S' \leq \frac{1}{4}e^{2(s+S)}H_S + 2(N-2)V_S - (N-2)V_S^2, \qquad V_S' = H_S \qquad \text{in } [\tau_1, s_2]. \tag{29}$$

Multiplying $H_S = V_S'$ for the first equation of the above system and integrating over $[\tau_1, s]$, we have that for $s \in [\tau_1, s_2)$

$$(V_S'(s))^2 \geq (N-2)\left\{\left(V_S^2(s) - \frac{1}{3}V_S^3(s)\right) - \left(V_S^2(\tau_1) - \frac{1}{3}V_S^3(\tau_1)\right)\right\}.$$

Since $H_S < 0$ in (s_1, s_2) and $V_S(s_2) \in (0, 2)$ for $S < S(1)$ with $|S - S(1)| \ll 1$, then we can find $\tau_2 \in (\tau_1, s_2)$ such that $V_S(\tau_2) = 2$. Then, (V_S, H_S) satisfies $(H_S(\tau_2))^2 = (V_S'(\tau_2)) \geq \ell_2^2$. Combining this with (28) implies that

$$\frac{1}{4}e^{2(\tau_2+S)}H_S(\tau_2) + (N-2) < 0. \tag{30}$$

Since $V_S(s) \in (0, 2]$ for $s \in [\tau_2, s_2]$ and (29), we have that

$$H_S'(s) \leq \frac{1}{4}e^{2(s+S)}H_S(s) + (N-2) \qquad \text{for } s \in [\tau_2, s_2]. \tag{31}$$

Then, we have that $H_S'(s) < 0$ in $[\tau_2, s_2)$. In fact, we assume that

$$\tau_3 = \sup\{s \in [\tau_2, s_2) \mid H_S'(\xi) < 0 \text{ for } \xi \in [\tau_2, s)\} < s_2.$$

Since $H_S'(\tau_2) < 0$, by (30) and (31), then we have that $\tau_3 > \tau_2$. That is to say, H_S decreases in $[\tau_2, \tau_3]$. This means that

$$H_S'(\tau_3) \le \frac{1}{4} e^{2(\tau_3+S)} H_S(\tau_3) + (N-2) \le \frac{1}{4} e^{2(\tau_2+S)} H_S(\tau_2) + (N-2) < 0.$$

It contradicts the definition of τ_3. Then, we have that $H_S'(s) < 0$ in $[\tau_2, s_2)$. This means that $H_S(s_2) \le \ell_2 < 0$. It contradicts the definition of s_2. Therefore, we get $S(2) < S(1)$.

Step 2. In order to prove that the orbit $\{(V_{S(2)}(s), H_{S(2)}(s))\}_{s<s_{max}}$ crosses the right half V-axis, we assume that the orbit does not cross the right half V-axis.

Combining this with $V_{S(2)}'(s) = H_{S(2)}(s) > 0$ for $s \ll -1$ implies that

$$V_{S(2)}'(s) = H_{S(2)}(s) > 0 \qquad \text{for } s < s_{max} \tag{32}$$

or

$$\lim_{s \nearrow s_{max}} V_{S(2)}(s) \in (0, \infty]. \tag{33}$$

We assume that $\lim_{s\nearrow s_{max}} V_{S(2)}(s) = V_{S(2)}(s_{max}) \le 2$. Then, we have that $s_{max} = \infty$ and that $\liminf_{s\to\infty} H_{S(2)}(s) = 0$. It follows from (32) that

$$H_{S(2)}'(s) \ge (N-2) V_{S(2)}(s)(2 - V_{S(2)}(s)) \qquad \text{for any sufficiently large } s.$$

Then, it holds that $H_{S(2)}'(s) > 0$ for any sufficiently large s. Combining this with $H_{S(2)}(s) > 0$ in $\boldsymbol{R}$ implies that $\lim_{s\to\infty} H_{S(2)}(s) > 0$. It contradicts $\liminf_{s\to\infty} H_{S(2)}(s) = 0$. Therefore, we have that $V_{S(2)}(s_{max}) \in (2, \infty]$.

By using this and the argument of Step 1 replacing $V_{S(1)}$ and ℓ_1 by $V_{S(2)}$ and $\min((V_{S(2)}(s_{max}) + 2)/2, 3)$, respectively, we get that

$$H_S(s_2) \le \ell_2 < 0 \qquad \text{for } 0 < S - S(2) \ll 1.$$

It is a contradiction.

Therefore, we obtain that the orbit $\{(V_{S(2)}(s), H_{S(2)}(s))\}_{s<s_{max}}$ crosses the right half V-axis. That is to say, there exists $s_1 = s_1(S(2))$ satisfying

$$\begin{cases} V_{S(2)}(s_1) > 2, \quad H_{S(2)}(s_1) = 0, \\ V_{S(2)}(s) > 0, \quad H_{S(2)}(s) > 0 \quad \text{for } s < s_1. \end{cases}$$

Step 3. It holds that

$$(34)\qquad \begin{cases} s_{max} = s_{max}(S(2)) = \infty, \\ V_{S(2)}(s) > 0 \text{ and } H_{S(2)}(s) < 0 \quad \text{for } s > s_1. \end{cases}$$

We shall show (34). As is mentioned in Step 1, it holds that $S(2) \notin \mathscr{S}_2$. Then, $(V_{S(2)}, H_{S(2)})$ does not have $s_2 \in (s_1, s_{max})$ satisfying

$$V_{S(2)}(s) > 0, \qquad H_{S(2)}(s) < 0 \qquad \text{for } s_1 < s < s_2,$$

$$V_{S(2)}(s_2) \in (0,2), \qquad H_{S(2)}(s_2) = 0.$$

If it holds that for $\tau \in (s_1, s_{max})$

$$V_{S(2)} > 0, \qquad H_{S(2)} < 0 \qquad \text{in } (s_1, \tau),$$

then it holds that $0 < V_{S(2)}(s) < V_{S(2)}(s_1)$ for $s \in (s_1, \tau)$ and that

$$H'_{S(2)}(s) \geq \frac{1}{2} e^{2(s+S(2))} H_{S(2)}(s) - (N-2) V_{S(2)}(s_1)^2 \qquad \text{for } s_1 < s < \tau.$$

Then, putting

$$\tau_4 = \sup\{\tau \in (s_1, s_{max}) \mid V_{S(2)}(s) > 0 \text{ and } H_{S(2)}(s) < 0 \text{ for } s_1 < s < \tau\}$$

and assuming $\tau_4 < s_{max} < \infty$, $(V_{S(2)}, H_{S(2)})$ satisfies

$$V_{S(2)}(\tau_4) = 0, \qquad H_{S(2)}(\tau_4) \leq 0.$$

Since $(V_{S(2)}, H_{S(2)}) = (0,0)$ is the equilibrium point, then it holds that $\tau_4 = s_{max} = \infty$. It contradicts $\tau_4 < s_{max} < \infty$. Therefore, we obtain that $H_{S(2)}(\tau_4) < 0$. That is to say, we have that

$$V_{S(2)}(s) < 0, \qquad H_{S(2)}(s) < 0 \qquad \text{for } s > \tau_4 \text{ with } 0 < s - \tau_4 \ll 1.$$

Combining this with Lemma 3.4, implies that for any $S < S(2)$ with $|S - S(2)| \ll 1$ (V_S, H_S) has a $\tau_4 = \tau_4(S) < s_{max}(S)$ satisfying

$$V_S(\tau_4) = 0, \qquad H_S(\tau_4) < 0,$$

$$V_S(s) > 0, \qquad H_S(s) < 0 \qquad \text{for } s_1 < s < \tau_4.$$

It contradicts the definition of $S(2)$. Then, we have that $\tau_4 = s_{max} = \infty$ or (34).

Step 4. We shall show

$$(35)\qquad \lim_{s\to\infty} V_{S(2)}(s) \in (0,2), \qquad \lim_{s\to\infty} H_{S(2)}(s) = 0.$$

By (34), $V_{S(2)}$ is positive and decreasing in (s_1, ∞). Then, there exists $\lim_{s\to\infty} V_{S(2)}(s)$ and the limit is nonnegative.

We assume that $\lim_{s\to\infty} V_{S(2)}(s) \geq 2$ or $V_{S(2)}(s) > 2$ for any $s \geq s_1$. Combining this with

$$H'_{S(2)}(s) = \left\{\frac{1}{2}e^{2(s+S(2))} - (N-4) - V_{S(2)}(s)\right\}H_{S(2)}(s)$$
$$+ (N-2)(2 - V_{S(2)}(s))V_{S(2)}(s)$$
$$< \frac{1}{4}e^{2(s+S(2))}H_{S(2)}(s) < 0 \qquad \text{for any sufficiently large } s$$

implies that $H_{S(2)}$ decreases for any sufficiently large s. By this and (34), we have that $\lim_{s\to\infty} H_{S(2)}(s) = -\infty$ and that

$$V'_{S(2)}(s) \le -1 \qquad \text{for any sufficiently large } s.$$

This implies that $V_{S(2)}(s) < 0$ for any sufficiently large s. It contradicts (34). Then, we obtain that $\lim_{s\to\infty} V_{S(2)}(s) \in [0,2)$. Combining this with (34) implies that $\limsup_{s\to\infty} H_{S(2)}(s) = 0$.

We assume that $\lim_{s\to\infty} V_{S(2)}(s) = 0$. Since it holds that

$$\text{(36)} \quad \frac{1}{4}e^{2(s+S(2))} \ge (N-3), \qquad V_{S(2)}(s) \in (0,1) \qquad \text{for any sufficiently large } s,$$

then for s satisfying (36) we have that

$$H'_{S(2)}(s) \le \frac{1}{4}e^{2(s+S(2))}H_{S(2)}(s) + 2(N-2)V_{S(2)}(s).$$

Putting $A = e^{2S(2)}/4$, $\xi = e^{2s}$ and $p(\xi) = V_{S(2)}(s)$, p satisfies that

$$\text{(37)} \qquad p_{\xi\xi}(\xi) - \left(\frac{A}{2} - \frac{1}{\xi}\right)p_\xi(\xi) \le \frac{(N-2)}{2\xi^2}p(\xi).$$

Then, for any ξ with $A/4 \ge 1/\xi$ and $\xi > e^{2s_1}$, it follows from $p_\xi(\xi) = e^{-2s}H_{S(2)}(s)/2 < 0$ that

$$p_{\xi\xi}(\xi) - \frac{A}{4}p_\xi(\xi) \le \frac{(N-2)}{2\xi^2}p(\xi).$$

Multiplying $e^{-A\xi/4}$, integrating this over $[\xi, \Xi]$ for $\xi < \Xi$ and using that $V_{S(2)}$ decreases in $[s_1, \infty)$, we get that for any sufficiently large ξ

$$e^{-A\Xi/4}p_\xi(\Xi) - e^{-A\xi/4}p_\xi(\xi) \le \frac{(N-2)}{2}\int_\xi^\Xi \frac{1}{\zeta^2}e^{-A\zeta/4}p(\zeta)d\zeta$$
$$\le \frac{(N-2)p(\xi)}{2\xi^2}\int_\xi^\Xi e^{-A\zeta/4}\,d\zeta \le \frac{2(N-2)}{A\xi^2}(e^{-A\xi/4} - e^{-A\Xi/4})p(\xi).$$

Combining this with $\limsup_{\Xi\to\infty} p_\xi(\Xi)\Xi = \limsup_{s\to\infty} H_{S(2)}(s)/2 = 0$ implies that

$$-e^{-A\xi/4}p_\xi(\xi) \le \frac{2(N-2)}{A\xi^2}e^{-A\xi/4}p(\xi) \qquad \text{for any sufficiently large } \xi.$$

Then, for a sufficiently large ξ_0 it holds that

$$p(\xi) \ge \exp\left(\frac{2(N-2)}{A}(\xi^{-1} - \xi_0^{-1})\right)p(\xi_0) \qquad \text{for } \xi \ge \xi_0$$

or

$$\lim_{s\to\infty} V_{S(2)}(s) = \lim_{\xi\to\infty} p(\xi) \ge \exp\left(-\frac{2(N-2)}{A\xi_0}\right)p(\xi_0) > 0.$$

It contradicts $\lim_{s\to\infty} V_{S(2)}(s) = 0$. Then, we have that

$$\lim_{s\to\infty} V_{S(2)} \in (0,2). \tag{38}$$

In order to prove $\lim_{s\to\infty} H_{S(2)}(s) = 0$, we assume that

$$\mathscr{H}_\infty = \liminf_{s\to\infty} H_{S(2)}(s) < 0$$

or that there exits a sequence η_k with $\lim_{k\to\infty} \eta_k = \infty$ satisfying

$$H_{S(2)}(\eta_k) \le \frac{1}{2}\mathscr{H}_\infty \qquad \text{for } k \ge 1. \tag{39}$$

From this and the second equation of (23), we have that

$$\begin{aligned} H'_{S(2)}(\eta_K) &< \frac{1}{4}e^{2(\eta_K+S(2))}\mathscr{H}_\infty + 2(N-2)\sup_{0\le s<\infty} V_{S(2)}(s) \\ &< \frac{1}{8}e^{2(\eta_K+S(2))}\mathscr{H}_\infty \qquad \text{for any sufficiently large } K. \end{aligned}$$

We shall show that

$$H'_{S(2)}(s) < \frac{1}{8}e^{2(\eta_K+S(2))}\mathscr{H}_\infty \qquad \text{for } s \ge \eta_K. \tag{40}$$

In fact, if this is not the case, then $s > \eta_K$ such that $H'_{S(2)}(s) = e^{2(\eta_K+S(2))}\mathscr{H}_\infty/8$. Let η_* be the minimum of such s. Since $H_{S(2)}$ decreases in $[\eta_K, \eta_*]$, we have that $H_{S(2)}(s) \le \mathscr{H}_\infty/2$ for $s \in [\eta_N, \eta_*]$ by (39). Then, we obtain that $H_{S(2)}$ satisfies (40) at η_*. It contradicts the definition of η_*. Then, we get (40). This implies that $V_{S(2)}(s) = 0$ for some $s \ge \eta_K$. It contradicts (34). Then, we have that $\lim_{s\to\infty} H_{S(2)}(s) = 0$. By this and (38), we have (35).

Step 5. By Lemmas 3.3 and 3.5, for any sufficiently small $S(V_S, H_S)$ has s_i $(i = 1,2,3,4)$ satisfying

$$(41)\qquad \begin{cases} -\infty < s_1 < s_2 < s_3 < s_4 < \infty, \\ V_S(s) > 0 \text{ and } H_S(s) > 0 \quad \text{for } s < s_1 \text{ and } s_2 < s < s_3, \\ V_S(s_i) > 2 \text{ and } H_S(s_i) = 0 \quad \text{for } i = 1, 3, \\ V_S(s) > 0, \quad H_S(s) < 0 \quad \text{for } s_1 < s < s_2 \text{ and } s_3 < s < s_4, \\ V_S(s_i) \in (0,2) \text{ and } H_S(s_i) = 0 \quad \text{for } i = 2, 4. \end{cases}$$

Then, we define $\mathscr{S}_3$ and $S(3)$ as

$$\{S < S(2) \mid (V_S, H_S) \text{ has } s_i \ (i = 1,2,3,4) \text{ satisfying (41)}\}$$

and the supremum of the unbounded connected component of $\mathscr{S}_3$, respectively. By using similar arguments as those in Steps 1, 2, 3 and 4, we have that the unbounded connected component of $\mathscr{S}_3$ is $(-\infty, S(3))$, $S(3) < S(2)$, $s_{max} = \infty$, $\lim_{s\to\infty} V_{S(3)}(s) \in (0,2)$ and that $\lim_{s\to\infty} H_{S(3)}(s) = 0$. Moreover, $(V_{S(3)}, H_{S(3)})$ has s_i $(i = 1,2,3)$ satisfying

$$\begin{cases} -\infty < s_1 < s_2 < s_3 < \infty, \\ V_{S(3)}(s) > 0 \text{ and } H_{S(3)}(s) > 0 \quad \text{for } s < s_1 \text{ and } s_2 < s < s_3, \\ V_{S(3)}(s_i) > 2 \text{ and } H_{S(3)}(s_i) = 0 \quad \text{for } i = 1, 3, \\ V_{S(3)}(s) > 0 \text{ and } H_{S(3)}(s) < 0 \quad \text{for } s_1 < s < s_2 \text{ and } s_3 < s, \\ V_{S(3)}(s_2) \in (0,2) \text{ and } H_{S(3)}(s_2) = 0. \end{cases}$$

Step 6. By using arguments similar to the above, we can find $\{S(j)\}_{j=4}^{\infty}$ with $S(j) < S(j-1)$ $(j = 4,5,6,\dots)$ satisfying the following properties.

For each $j \geq 4$, $(V_{S(j)}, H_{S(j)})$ satisfies $s_{max}(S(j)) = \infty$, $\lim_{s\to\infty} V_{S(j)}(s) \in (0,2)$ and $\lim_{s\to\infty} H_{S(j)}(s) = 0$ and has $\{s_i\}_{i=1}^{2j-3}$ satisfying

$$-\infty < s_1 < s_2 < \cdots < s_{2j-3} < \infty,$$

$$\begin{cases} V_{S(j)}(s) > 0 \text{ and } H_{S(j)}(s) > 0 \quad \text{for } s < s_1 \text{ and } s_{2i} < s < s_{2i+1}, \\ V_{S(j)}(s_{2i-1}) > 2 \text{ and } H_{S(j)}(s_{2i-1}) = 0, \\ V_{S(j)}(s) > 0 \text{ and } H_{S(j)}(s) < 0 \quad \text{for } s_{2i-1} < s < s_{2i} \text{ and } s_{2j-3} < s, \\ V_{S(j)}(s_{2i}) \in (0,2) \text{ and } H_{S(j)}(s_{2i}) = 0 \end{cases}$$

for $i = 1,2,3,\dots j-2$, and

$$V_{S(j)}(s_{2j-3}) > 2 \qquad \text{and} \qquad H_{S(j)}(s_{2j-3}) = 0.$$

Then, we can find the self-similar solutions corresponding to $\{(V_{S(j)}, H_{S(j)})\}_{j=1}^{\infty}$. Since $(V_{S(j)}, H_{S(j)}) \neq (V_{S(j')}, H_{S(j')})$ for $j \neq j'$, we can find infinite self-similar solutions.

Step 7. We shall show that the corresponding self-similar solution $\bar{u}_j$ to $(V_{S(j)}, H_{S(j)})$ is positive. Recall $y = r/\sqrt{T-t}$ and $s = \log y - S(j)$. Then, it holds that

$$V_{S(j)}(s) = \frac{1}{\omega_N y^{N-2}} \int_0^y \bar{u}_j(\eta)\eta^{N-1}\, d\eta$$

or

$$H_{S(j)}(s) + (N-2)V_{S(j)}(s) = \frac{y^2}{\omega_N}\bar{u}_j(y).$$

In order to prove the positivity of $\bar{u}_j$, we assume that the existence of τ_5 is satisfying

$$H_{S(j)}(\tau_5) + (N-2)V_{S(j)}(\tau_5) = 0.$$

Since it holds that $\lim_{s\to\infty} V_{S(j)}(s) \in (0,2)$ and that $\lim_{s\to\infty} H_{S(j)}(s) = 0$, we observe that

$$H_{S(j)}(s) + (N-2)V_{S(j)}(s) > 0 \qquad \text{for any sufficiently large } s.$$

Then, we can define $\tau_6 \in \boldsymbol{R}$ as

$$\sup\{s \in \boldsymbol{R} \mid H_{S(j)}(s) + (N-2)V_{S(j)}(s) \le 0\}.$$

Since

(42) $$H_{S(j)}(\tau_6) + (N-2)V_{S(j)}(\tau_6) = 0$$

and $H_{S(j)}(s) + (N-2)V_{S(j)}(s) > 0$ for $s > \tau_6$, then it holds that

(43) $$H_{S(j)}(\tau_6) < 0, \qquad V_{S(j)}(\tau_6) > 0,$$

(44) $$H'_{S(j)}(\tau_6) + (N-2)V'_{S(j)}(\tau_6) \ge 0.$$

and that

(45) $$H'_{S(j)}(\tau_6) = \left(\frac{1}{2}e^{2(\tau_6+S(j))} - (N-2)\right)H_{S(j)}(\tau_6),$$

by (42) and the second equation of (23). By (44), (45) and $V'_{S(j)} = H_{S(j)}$, we have that

$$0 \le H'_{S(j)}(\tau_6) + (N-2)V'_{S(j)}(\tau_6) = \frac{1}{2}e^{2(\tau_6+S(j))}H_{S(j)}(\tau_6)$$

or $H_{S(j)}(\tau_6) \ge 0$. It contradicts (43). Then, we have the positivity of the self-similar solutions $\{\bar{u}_j\}_{j=1}^{\infty}$.

Thus, we conclude the proof of this theorem. □

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