Funkcialaj Ekvacioj, **48** (2005) 231–246

Special Polynomials Associated with the Noumi-Yamada System of Type $A_5^{(1)}$

By

Tetsu MASUDA

(Kobe University, Japan)

Abstract. A determinant formula for algebraic solutions to the Noumi-Yamada system of type $A_5^{(1)}$ is presented. This expression is regarded as a special case of the universal characters. The entries of the determinant are given by the Laguerre polynomials. Degeneration to the rational solutions to the Painlevé IV equation is discussed.

Keywords and Phrases. Noumi-Yamada system, Special polynomials, Universal characters.

2000 *Mathematics Subject Classification Numbers.* 33E17, 33C45, 34M55.

1. Introduction

Noumi and Yamada have generalized the Painlevé equations from the viewpoint of symmetry and presented higher order analogues of the Painlevé equations [7]. These systems admit the affine Weyl group symmetry of type $A_{n-1}^{(1)}$. When $n=3$ and $n=4$, the systems are nothing but the Painlevé IV equation (P_{IV}) and the Painlevé V equation (P_V), respectively. It is also known that they admit the special solutions expressible by the n-core Schur functions, which originates that the system can be derived from the n-reduced KP hierarchy [8, 6].

However, it is easy to see that the special polynomials which characterize the rational solutions to P_V cannot be understood in such a picture. The author has shown that they are expressed in terms of the universal characters [2], a kind of generalization of the Schur functions [5]. In the determinant formula of Jacobi-Trudi type, the entries are given by the Laguerre polynomials.

This is not an isolated result. In fact, it has been revealed that the universal characters appear associated with a class of algebraic solutions to P_{VI} and the Garnier systems [4, 11]. It is also known that the rational solutions to q-P_V are expressed in terms of a q-analogue of the universal characters [3].

Watching the construction of the rational solutions to P_V, one expects that the similar special polynomials appear for the cases of the Noumi-Yamada system of type $A_{2n-1}^{(1)}$ ($n \geq 3$). In this article, we consider the case of $n=3$ or the Noumi-Yamada system of type $A_5^{(1)}$ and show that the associated special polynomials are expressed in terms of the universal characters specified by two 3-core partitions.

2. The Noumi-Yamada system of type $A_5^{(1)}$

The Noumi-Yamada system [7] of type $A_5^{(1)}$ is a differential system for unknown functions $f_i = f_i(t)$ $(i = 0, 1, \ldots, 5)$ containing complex parameters α_i $(i = 0, 1, \ldots, 5)$ with a constraint

$$\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 1. \tag{2.1}$$

The explicit formula of f_0' is given by

$$\begin{aligned} f_0' &= f_0(f_1 f_2 + f_1 f_4 + f_3 f_4 - f_2 f_3 - f_2 f_5 - f_4 f_5) \\ &\quad + \left(\frac{1}{2} - \alpha_2 - \alpha_4\right) f_0 + \alpha_0(f_2 + f_4), \qquad {}' = t\frac{d}{dt}. \end{aligned} \tag{2.2}$$

Formulas of the other f_i' are obtained by the rotation of indices that are understood as elements of $\boldsymbol{Z}/6\boldsymbol{Z}$. The system is essentially fourth order since it has two trivial integrals

$$f_0 + f_2 + f_4 = f_1 + f_3 + f_5 = \sqrt{t}. \tag{2.3}$$

It is known that the system admits the extended affine Weyl group $\tilde{W}(A_5^{(1)}) = \langle s_0, \ldots, s_5, \pi \rangle$ as the symmetry of Bäcklund transformations. The actions are given by

$$\begin{aligned} &s_i(\alpha_i) = -\alpha_i, \quad s_i(\alpha_j) = \alpha_j + \alpha_i \ (j = i \pm 1), \qquad s_i(\alpha_j) = \alpha_j \ (j \neq i, i \pm 1), \\ &s_i(f_i) = f_i, \qquad s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i} \ (j = i \pm 1), \quad s_i(f_j) = f_j \ (j \neq i, i \pm 1), \end{aligned} \tag{2.4}$$

and the fundamental relations

$$\begin{aligned} &s_i^2 = 1, \qquad s_i s_j = s_j s_i \quad (j \neq i, i \pm 1), \qquad s_i s_j s_i = s_j s_i s_j \quad (j = i \pm 1), \\ &\pi^6 = 1, \qquad \pi s_i = s_{i+1}\pi, \end{aligned} \tag{2.5}$$

hold. Moreover, the actions can be lifted to the τ-functions τ_i by the formulas

$$s_i(\tau_j) = \tau_j \quad (i \neq j), \qquad s_i(\tau_i) = f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i}, \qquad \pi(\tau_i) = \tau_{i+1}. \tag{2.6}$$

From (2.4) and (2.6), one can derive the bilinear relations of Hirota-Miwa type

$$\begin{aligned} \tau_0 s_0 s_1(\tau_1) &= s_0(\tau_0) s_1(\tau_1) + \alpha_0 \tau_2 \tau_5, \\ \tau_1 s_1 s_0(\tau_0) &= s_0(\tau_0) s_1(\tau_1) - \alpha_1 \tau_2 \tau_5, \end{aligned} \tag{2.7}$$

and their rotations of indices.

Let us define the translation operators T_i $(i = 0, 1, \dots, 5)$ by $T_1 = \pi s_5 s_4 s_3 s_2 s_1$ and $\pi T_i = T_{i+1}\pi$, which commute with each other and satisfy $T_1 T_2 T_3 T_4 T_5 T_0 = 1$. The actions on the parameters α_i are given by

$$T_1(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) = (\alpha_0 + 1, \alpha_1 - 1, \alpha_2, \alpha_3, \alpha_4, \alpha_5), \tag{2.8}$$

and so on. Then a multi-index $\nu = (\nu_1, \dots, \nu_5, \nu_0) \in \boldsymbol{Z}^6$ uniquely corresponds to an arbitrary $w \in \tilde{W}(A_5^{(1)})$ by $w = T_1^{\nu_1} \dots T_5^{\nu_5} T_0^{\nu_0}$. Note that all the multi-indices $\nu + k = (\nu_1 + k, \dots, \nu_5 + k, \nu_0 + k)$ $(k \in \boldsymbol{Z})$ correspond to the same $w \in \tilde{W}(A_5^{(1)})$ due to $T_1 T_2 T_3 T_4 T_5 T_0 = 1$.

Let us introduce the τ-functions on the lattice as

$$\tau_\nu = T^\nu(\tau_0), \qquad T^\nu = T_1^{\nu_1} \dots T_5^{\nu_5} T_0^{\nu_0}. \tag{2.9}$$

Then τ_ν are expressed in the form

$$\tau_\nu = \phi_\nu \tau_0 \left(\frac{\tau_1}{\tau_0}\right)^{\nu_1} \cdots \left(\frac{\tau_5}{\tau_4}\right)^{\nu_5} \left(\frac{\tau_0}{\tau_5}\right)^{\nu_0}, \tag{2.10}$$

where ϕ_ν are polynomials in α_i, f_i with coefficients in $\boldsymbol{Z}$ and expressed by the determinant formula of Jacobi-Trudi type.

3. Construction of special polynomials

The similar formulation to the previous section for the Noumi-Yamada system of type $A_{n-1}^{(1)}$ is given in [7, 8, 6]. Starting with the fixed point with respect to the transformation π, one obtain a solution

$$\alpha_i = \frac{1}{n}, \qquad f_i = \frac{x}{n} \qquad (i = 0, 1, \dots, n-1). \tag{3.1}$$

The polynomials ϕ_ν with the specialization of (3.1) are known to be expressed in terms of the n-core Schur functions. In the case of $n = 3$ or P_{IV}, these polynomials coincide with the Okamoto polynomials up to multiplication by non-zero constants [9, 10].

As we mentioned above, the special polynomials which characterize the rational solutions to P_V cannot be understood in such a picture and these polynomials are expressed in terms of the universal characters. Note that such special polynomials are constructed by starting with the fixed points with respect to the transformation π^2 (not π). It is meaningful to consider the fixed points of π^2 only for the cases of $A_{2n-1}^{(1)}$. This is the reason why we investigate the Noumi-Yamada system of type $A_5^{(1)}$ (or $A_{2n-1}^{(1)}$ more generally) in this article.

It is obvious that the Noumi-Yamada system of type $A_5^{(1)}$ has a solution

$$\alpha_{2i} = \frac{1}{3} - s, \qquad \alpha_{2i+1} = s \qquad (i = 0, 1, 2), \tag{3.2}$$

$$f_i = \frac{\sqrt{t}}{3} \qquad (i = 0, 1, \dots, 5),$$

on the fixed points with respect to the transformation π^2. Applying the Bäcklund transformations to the above solution, we observe that ϕ_ν are expressed in the form

$$\phi_\nu = \left(\frac{\sqrt{t}}{3}\right)^{\tilde{\nu}(\tilde{\nu}-1)/2} U_{\nu_-,\nu_+}, \tag{3.3}$$

$$\tilde{\nu} = \nu_1 - \nu_2 + \nu_3 - \nu_4 + \nu_5 - \nu_0, \qquad \nu_- = (\nu_1, \nu_3, \nu_5), \qquad \nu_+ = (\nu_2, \nu_4, \nu_0),$$

where $U_{\nu_-,\nu_+} = U_{\nu_-,\nu_+}(t, s)$ are some polynomials in t and s. The solutions to the system are written as

(3.4)

$$f_0 = \frac{\sqrt{t}}{3} \frac{U_{(0,0,0)_-(0,0,0)_+} U_{(1,0,0)_-(0,0,-1)_+}}{U_{(1,1,1)_-(1,1,0)_+} U_{(1,0,0)_-(0,0,0)_+}}, \qquad f_1 = \frac{\sqrt{t}}{3} \frac{U_{(1,0,0)_-(0,0,0)_+} U_{(0,0,0)_-(1,0,0)_+}}{U_{(0,0,0)_-(0,0,0)_+} U_{(1,0,0)_-(1,0,0)_+}},$$

$$f_2 = \frac{\sqrt{t}}{3} \frac{U_{(1,0,0)_-(1,0,0)_+} U_{(1,1,0)_-(0,0,0)_+}}{U_{(1,0,0)_-(0,0,0)_+} U_{(1,1,0)_-(1,0,0)_+}}, \qquad f_3 = \frac{\sqrt{t}}{3} \frac{U_{(1,1,0)_-(1,0,0)_+} U_{(1,0,0)_-(1,1,0)_+}}{U_{(1,0,0)_-(1,0,0)_+} U_{(1,1,0)_-(1,1,0)_+}},$$

$$f_4 = \frac{\sqrt{t}}{3} \frac{U_{(1,1,0)_-(1,1,0)_+} U_{(0,0,0)_-(0,-1,-1)_+}}{U_{(1,1,0)_-(1,0,0)_+} U_{(1,1,1)_-(1,1,0)_+}}, \qquad f_5 = \frac{\sqrt{t}}{3} \frac{U_{(1,1,1)_-(1,1,0)_+} U_{(0,0,-1)_-(0,0,0)_+}}{U_{(1,1,0)_-(1,1,0)_+} U_{(0,0,0)_-(0,0,0)_+}},$$

and

$$\begin{aligned} \alpha_0 &= \frac{1}{3} - s - \nu_0 + \nu_1, \qquad & \alpha_1 &= s - \nu_1 + \nu_2, \\ \alpha_2 &= \frac{1}{3} - s - \nu_2 + \nu_3, \qquad & \alpha_3 &= s - \nu_3 + \nu_4, \\ \alpha_4 &= \frac{1}{3} - s - \nu_4 + \nu_5, \qquad & \alpha_5 &= s - \nu_5 + \nu_0, \end{aligned} \tag{3.5}$$

where we denote $(\nu_1 + n_1, \nu_3 + n_3, \nu_5 + n_5)$ and $(\nu_2 + n_2, \nu_4 + n_4, \nu_0 + n_0)$ as $(n_1, n_3, n_5)_-$ and $(n_2, n_4, n_0)_+$, respectively.

We introduce some notations in order to present an explicit expression of the polynomials U_{ν_-,ν_+}. Let $M_{\nu_\pm}$ be the Maya diagrams determined by

$$\begin{aligned} M_{\nu_-} &= (3\boldsymbol{Z}_{<\nu_1} + 1) \cup (3\boldsymbol{Z}_{<\nu_3} + 2) \cup (3\boldsymbol{Z}_{<\nu_5} + 3), \\ M_{\nu_+} &= (3\boldsymbol{Z}_{<\nu_2} + 1) \cup (3\boldsymbol{Z}_{<\nu_4} + 2) \cup (3\boldsymbol{Z}_{<\nu_0} + 3). \end{aligned} \tag{3.6}$$

To each Maya diagram $M = \{\dots, m_3, m_2, m_1\}$ $(\cdots < m_3 < m_2 < m_1)$, one can associate a unique partition $\lambda = (\lambda_1, \lambda_2, \dots,)$ such that $m_i - m_{i+1} = \lambda_i - \lambda_{i+1} + 1$ for $i = 1, 2, \dots$. Note that all the Maya diagrams $M + k = \{\dots, m_2 + k, m_1 + k\}$ $(k \in \boldsymbol{Z})$ obtained from $M = \{\dots, m_3, m_2, m_1\}$ by shifting define the same partition by this correspondence. We assign the partitions λ'_- and λ_+ to the Maya diagrams M_{ν_-} and M_{ν_+}, respectively. Note that M_{ν_-} corresponds to the conjugate λ'_- and that the partitions of the form $\lambda_\pm$ are called 3-core. Let $H_{\lambda_\pm}$ be the products of the hook-length of the Young diagrams corresponding to the partitions $\lambda_\pm$, which are expressed as

$$H_{\lambda_\pm} = \prod_{i \in M_{\nu_\pm},\, j \in M^c_{\nu_\pm};\, i > j} (i - j). \tag{3.7}$$

For a pair of partitions $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_l)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m)$, the universal character $S_{\lambda,\mu}[x, y]$ is a polynomial in $x = (x_1, x_2, \dots)$ and $y = (y_1, y_2, \dots)$ defined as follows [2]:

$$S_{\lambda,\mu}[x, y] = \det \begin{bmatrix} q_{\mu_{m-i+1}+i-j}(y), & 1 \le i \le m \\ p_{\lambda_{i-m}-i+j}(x), & m+1 \le i \le l+m \end{bmatrix}_{1 \le i,j \le l+m}, \tag{3.8}$$

where $p_k(x)$ and $q_k(y)$ are the elementary Schur polynomials

$$\sum_{k \in \boldsymbol{Z}} p_k(x) \eta^k = \exp\left(\sum_{j=1}^{\infty} x_j \eta^j\right), \qquad \sum_{k \in \boldsymbol{Z}} q_k(y) \eta^k = \exp\left(\sum_{j=1}^{\infty} y_j \eta^j\right). \tag{3.9}$$

Then we have the following Theorem.

Theorem 3.1. *The polynomials U_{ν_-,ν_+} are expressed as*

$$U_{\nu_-,\nu_+} = 3^{-|\lambda_-|}(-3)^{-|\lambda_+|} H_{\lambda_-} H_{\lambda_+} S_{\lambda_-,\lambda_+}, \tag{3.10}$$

where $S_{\lambda_-,\lambda_+} = S_{\lambda_-,\lambda_+}[x^-, x^+]$ are the universal characters specified by the partitions λ_- and λ_+. The variables $x^- = (x_1^-, x_2^-, \dots)$ and $x^+ = (x_1^+, x_2^+, \dots)$ are specialized as

$$x_j^- = \frac{t}{3} + \frac{3s - \tilde{\nu}}{j}, \qquad x_j^+ = -\frac{t}{3} + \frac{3s - \tilde{\nu}}{j}. \tag{3.11}$$

Example. Let us take the case of $\nu_- = (1, 2, 0)$, $\nu_+ = (3, 2, 0)$. The corresponding Maya diagrams are

$$M_{\nu_-} = \{\dots, 0, 1, 2, 5\}, \qquad M_{\nu_+} = \{\dots, 1, 2, 4, 5, 7\}. \tag{3.12}$$

Hence we have

$$\lambda_- = (1, 1) = \begin{array}{|c|} \hline 2 \\ \hline 1 \\ \hline \end{array}, \qquad \lambda_+ = (2, 1, 1) = \begin{array}{|c|c|} \hline 4 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1} 1 & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}, \tag{3.13}$$

and

$$H_{\lambda_-} = 2 \cdot 1 = 2, \qquad H_{\lambda_+} = 4 \cdot 2 \cdot 1^2 = 8. \tag{3.14}$$

Thus the polynomial U_{ν_-,ν_+} is expressed as

$$U_{\nu_-,\nu_+} = 3^{-2} \times (-3)^{-4} \times 2 \times 8 \times S_{\lambda_-,\lambda_+}. \tag{3.15}$$

The determinant formula of Jacobi-Trudi type for S_{λ_-,λ_+} is given by

$$S_{\lambda_-,\lambda_+} = \begin{vmatrix} q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} & q_{-2}^{(r)} & q_{-3}^{(r)} \\ q_2^{(r)} & q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} & q_{-2}^{(r)} \\ p_0^{(r)} & p_1^{(r)} & p_2^{(r)} & p_3^{(r)} & p_4^{(r)} \\ p_{-2}^{(r)} & p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} & p_2^{(r)} \\ p_{-3}^{(r)} & p_{-2}^{(r)} & p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} \end{vmatrix}, \tag{3.16}$$

where $p_k^{(r)}$ and $q_k^{(r)}$ are defined by

$$\sum_{k=0}^{\infty} p_k^{(r)} \eta^k = (1-\eta)^{-r} \exp\left(-\frac{x\eta}{1-\eta}\right), \qquad p_k^{(r)} = 0 \quad \text{for } k < 0, \tag{3.17}$$

$$q_k^{(r)}(x) = p_k^{(r)}(-x),$$

with $x = t/3$ and $r = 3s - \tilde{\nu} = 3s + 2$. We remark that $p_k^{(r)}$ and $q_k^{(r)}$ are nothing but the Laguerre polynomials.

Outline of the proof of Theorem 3.1 is given as follows. Define $R_{\lambda_-,\lambda_+}^{(r)}$ by

$$R_{\lambda_-,\lambda_+}^{(r)}(x) = S_{\lambda_-,\lambda_+}(t,s), \qquad x = \frac{t}{3},\ r = 3s - \tilde{\nu}. \tag{3.18}$$

Then the bilinear relations (2.7) are reduced to

$$\begin{aligned} &\sigma R^{(r)}_{\lambda_-(1,0,0)\lambda_+(1,0,0)} R^{(r-1)}_{\lambda_-(1,1,1)\lambda_+(1,1,0)} \\ &\quad = -R^{(r)}_{\lambda_-(0,0,0)\lambda_+(0,0,0)} R^{(r-1)}_{\lambda_-(1,0,0)\lambda_+(1,0,-1)} \\ &\qquad + R^{(r-1)}_{\lambda_-(1,0,0)\lambda_+(0,0,0)} R^{(r)}_{\lambda_-(0,0,0)\lambda_+(1,0,-1)}, \\ &-|3\nu_2 - 3\nu_0 + 1| R^{(r)}_{\lambda_-(0,0,0)\lambda_+(0,0,0)} R^{(r-1)}_{\lambda_-(1,0,0)\lambda_+(1,0,-1)} \\ &\quad = x R^{(r-2)}_{\lambda_-(1,0,0)\lambda_+(0,0,-1)} R^{(r+1)}_{\lambda_-(0,0,0)\lambda_+(1,0,0)} \\ &\qquad + (1 - r - \tilde{\nu} - 3\nu_0 + 3\nu_1) R^{(r)}_{\lambda_-(1,0,0)\lambda_+(1,0,0)} R^{(r-1)}_{\lambda_-(1,1,1)\lambda_+(1,1,0)}, \end{aligned} \tag{3.19}$$

where

$$\sigma = \begin{cases} +1 & \nu_2 \geq \nu_0 \\ -1 & \nu_2 < \nu_0 \end{cases}, \tag{3.20}$$

and we denote the partitions corresponding to the multi-indices $\nu_- = (\nu_1 + n_1, \nu_3 + n_3, \nu_5 + n_5)$ and $\nu_+ = (\nu_2 + n_2, \nu_4 + n_4, \nu_0 + n_0)$ as $\lambda_-(n_1, n_3, n_5)$ and $\lambda_+(n_2, n_4, n_0)$, respectively. The bilinear relations (3.19) can be proved in terms of the contiguity relations

$$p_k^{(r)} - p_{k-1}^{(r)} = p_k^{(r-1)}, \qquad q_k^{(r)} - q_{k-1}^{(r)} = q_k^{(r-1)}, \tag{3.21}$$

and

$$(k+1)p_{k+1}^{(r)} = rp_k^{(r+1)} - xp_k^{(r+2)}, \qquad (k+1)q_{k+1}^{(r)} = rq_k^{(r+1)} + xq_k^{(r+2)}, \tag{3.22}$$

by using the same technique as in [5]. For avoiding complication, we just illustrate with the case of $\nu_- = (1, 2, 0)$, $\nu_+ = (2, 1, 0)$ in Appendix.

By construction of $R_{\lambda_-, \lambda_+}^{(r)}$ we have the bilinear relations obtained from (3.19) by replacing ν_i and $R_{\lambda_-(n_1, n_3, n_5), \lambda_+(n_2, n_4, n_0)}^{(r)}$ with ν_{i+2} and $R_{\lambda_-(n_5+1, n_1, n_3), \lambda_+(n_0+1, n_2, n_4)}^{(r)}$, respectively. Due to the symmetry

$$R_{\lambda_-, \lambda_+}^{(r)}(-x) = R_{\lambda_+, \lambda_-}^{(r)}(x), \tag{3.23}$$

we also have the bilinear relations obtained by replacing x and $R_{\lambda_-(n_1, n_3, n_5), \lambda_+(n_2, n_4, n_0)}^{(r)}$ by $-x$ and $R_{\lambda_-(n_2, n_4, n_0), \lambda_+(n_1, n_3, n_5)}^{(r)}$, respectively. These bilinear relations are equivalent to those derived from the rotations of indices in (2.7).

4. Degeneration to the Okamoto polynomials

In this section, we show that the special polynomials obtained in the previous section degenerate to the Okamoto polynomials which characterize the rational solutions to P_{IV}. This degeneration process is achieved by putting

$$x = 3\varepsilon^{-2}\left(1 - \frac{\varepsilon}{3}z\right), \qquad r = 3\varepsilon^{-2}, \tag{4.1}$$

and taking the limit of $\varepsilon \to 0$. One can put $\lambda_- = \emptyset$ without losing generality in this limiting procedure. Then we consider the degeneration of the solutions in the form

$$(4.2)\qquad f_0=\sqrt{\frac{x}{3}}\,\frac{R^{(r)}_{\lambda_+(0,0,0)}R^{(r-2)}_{\lambda_+(0,0,-1)}}{R^{(r-1)}_{\lambda_+(1,1,0)}R^{(r-1)}_{\lambda_+(0,0,0)}},\qquad f_1=\sqrt{\frac{x}{3}}\,\frac{R^{(r-1)}_{\lambda_+(0,0,0)}R^{(r+1)}_{\lambda_+(1,0,0)}}{R^{(r)}_{\lambda_+(0,0,0)}R^{(r)}_{\lambda_+(1,0,0)}},$$

$$f_2=\sqrt{\frac{x}{3}}\,\frac{R^{(r)}_{\lambda_+(1,0,0)}R^{(r-2)}_{\lambda_+(0,0,0)}}{R^{(r-1)}_{\lambda_+(0,0,0)}R^{(r-1)}_{\lambda_+(1,0,0)}},\qquad f_3=\sqrt{\frac{x}{3}}\,\frac{R^{(r-1)}_{\lambda_+(1,0,0)}R^{(r+1)}_{\lambda_+(1,1,0)}}{R^{(r)}_{\lambda_+(1,0,0)}R^{(r)}_{\lambda_+(1,1,0)}},$$

$$f_4=\sqrt{\frac{x}{3}}\,\frac{R^{(r)}_{\lambda_+(1,1,0)}R^{(r-2)}_{\lambda_+(0,-1,-1)}}{R^{(r-1)}_{\lambda_+(1,0,0)}R^{(r-1)}_{\lambda_+(1,1,0)}},\qquad f_5=\sqrt{\frac{x}{3}}\,\frac{R^{(r-1)}_{\lambda_+(1,1,0)}R^{(r+1)}_{\lambda_+(0,0,0)}}{R^{(r)}_{\lambda_+(1,1,0)}R^{(r)}_{\lambda_+(0,0,0)}},$$

and

$$(4.3)\qquad \alpha_0=\frac{1}{3}-s-v_0,\qquad \alpha_1=s+v_2,$$

$$\alpha_2=\frac{1}{3}-s-v_2,\qquad \alpha_3=s+v_4,$$

$$\alpha_4=\frac{1}{3}-s-v_4,\qquad \alpha_5=s+v_0,$$

with $r=3s+(v_2+v_4+v_0)$. Let us investigate the degeneration of the polynomials $R^{(r)}_{\lambda_+}$. Putting

$$(4.4)\qquad \eta\to\varepsilon\eta,\qquad \bar{p}^{(r)}_k=\varepsilon^k p^{(r)}_k,$$

in (3.17), we have

$$(4.5)\qquad \sum_{k=0}^{\infty}\bar{p}^{(r+j)}_k\eta^k=\exp\left(z\eta-\frac{3}{2}\eta^2\right)[1+\varepsilon(j\eta+z\eta^2-2\eta^3)+O(\varepsilon^2)].$$

This implies

$$(4.6)\qquad \bar{p}^{(r+j)}_k=p_k+\varepsilon jp_{k-1}+\varepsilon(zp_{k-2}-2p_{k-3})+O(\varepsilon^2),$$

where $p_k=p_k(z)$ are the polynomials defined by

$$(4.7)\qquad \sum_{k=0}^{\infty}p_k\eta^k=\exp\left(z\eta-\frac{3}{2}\eta^2\right),\qquad p_k=0\quad\text{for } k<0.$$

Then we obtain

$$(4.8)\qquad R^{(r+j)}_{\lambda_+}=\varepsilon^{|\lambda_+|}\left[R_{\lambda_+}+\varepsilon\left(j\frac{dR_{\lambda_+}}{dz}+Q_{\lambda_+}\right)+O(\varepsilon^2)\right],$$

by using the relation

$$\frac{dp_k}{dz} = p_{k-1}, \tag{4.9}$$

where we denote the contribution of the third term of (4.6) as Q_{λ_+}. The polynomials $R_{\lambda_+} = R_{\lambda_+}(z)$ coincide with the Okamoto polynomials up to multiplication by non-zero constants.

Next we investigate the degeneration of the solutions and equations. It is easy to see that f_i are expressed in the form

(4.10)

$$f_0 = \varepsilon^{-1} + \frac{d}{dz}\log\frac{R_{\lambda_+(0,0,0)}}{R_{\lambda_+(1,1,0)}} - \frac{z}{6} + O(\varepsilon), \quad f_1 = \varepsilon^{-1} + \frac{d}{dz}\log\frac{R_{\lambda_+(1,0,0)}}{R_{\lambda_+(0,0,0)}} - \frac{z}{6} + O(\varepsilon),$$

$$f_2 = \varepsilon^{-1} + \frac{d}{dz}\log\frac{R_{\lambda_+(1,0,0)}}{R_{\lambda_+(0,0,0)}} - \frac{z}{6} + O(\varepsilon), \quad f_3 = \varepsilon^{-1} + \frac{d}{dz}\log\frac{R_{\lambda_+(1,1,0)}}{R_{\lambda_+(1,0,0)}} - \frac{z}{6} + O(\varepsilon),$$

$$f_4 = \varepsilon^{-1} + \frac{d}{dz}\log\frac{R_{\lambda_+(1,1,0)}}{R_{\lambda_+(1,0,0)}} - \frac{z}{6} + O(\varepsilon), \quad f_5 = \varepsilon^{-1} + \frac{d}{dz}\log\frac{R_{\lambda_+(0,0,0)}}{R_{\lambda_+(1,1,0)}} - \frac{z}{6} + O(\varepsilon).$$

Put

$$\begin{aligned} &f_0 = \varepsilon^{-1} + g_1 - \frac{z}{2}, && f_1 = \varepsilon^{-1} + g_2 - \frac{z}{2}, \\ &f_2 = \varepsilon^{-1} + g_2 - \frac{z}{2}, && f_3 = \varepsilon^{-1} + g_0 - \frac{z}{2}, \\ &f_4 = \varepsilon^{-1} + g_0 - \frac{z}{2}, && f_5 = \varepsilon^{-1} + g_1 - \frac{z}{2}, \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} &\alpha_0 = -\varepsilon^{-2} + \frac{2\beta_0 + \beta_1}{3}, && \alpha_1 = \varepsilon^{-2} + \frac{\beta_0 - \beta_1}{3}, \\ &\alpha_2 = -\varepsilon^{-2} + \frac{2\beta_1 + \beta_2}{3}, && \alpha_3 = \varepsilon^{-2} + \frac{\beta_1 - \beta_2}{3}, \\ &\alpha_4 = -\varepsilon^{-2} + \frac{2\beta_2 + \beta_0}{3}, && \alpha_5 = \varepsilon^{-2} + \frac{\beta_2 - \beta_0}{3}. \end{aligned} \tag{4.12}$$

Then we find that the equations

$$\begin{aligned} (f_0 f_1)' &= \sqrt{t}[f_0 f_1 (f_1 - f_0) + \alpha_0 f_1 + \alpha_1 f_0], \\ (f_2 f_3)' &= \sqrt{t}[f_2 f_3 (f_3 - f_2) + \alpha_2 f_3 + \alpha_3 f_2], \\ (f_4 f_5)' &= \sqrt{t}[f_4 f_5 (f_5 - f_4) + \alpha_4 f_5 + \alpha_5 f_4], \end{aligned} \tag{4.13}$$

are reduced to

$$g_0' = g_0(g_1 - g_2) + \beta_0, \tag{4.14}$$

$$g_1' = g_1(g_2 - g_0) + \beta_1, \qquad ' = \frac{d}{dz},$$

$$g_2' = g_2(g_0 - g_1) + \beta_2,$$

with

$$\beta_0 + \beta_1 + \beta_2 = 1, \qquad g_0 + g_1 + g_2 = z, \tag{4.15}$$

in the degeneration limit. This is nothing but the symmetric form of P_{IV} or the Noumi-Yamada system of type $A_2^{(1)}$. The solutions are reduced to

$$g_0 = \frac{d}{dz} \log \frac{R_{\lambda_+(1,1,0)}}{R_{\lambda_+(1,0,0)}} + \frac{z}{3}, \qquad g_1 = \frac{d}{dz} \log \frac{R_{\lambda_+(0,0,0)}}{R_{\lambda_+(1,1,0)}} + \frac{z}{3}, \tag{4.16}$$

$$g_2 = \frac{d}{dz} \log \frac{R_{\lambda_+(1,0,0)}}{R_{\lambda_+(0,0,0)}} + \frac{z}{3},$$

and

$$(\beta_0, \beta_1, \beta_2) = \left(\frac{1}{3} - v_0 + v_2, \frac{1}{3} - v_2 + v_4, \frac{1}{3} - v_4 + v_0\right), \tag{4.17}$$

which are nothing but the rational solutions to P_{IV} [9, 1].

5. A conjecture for the case of $A_{2n-1}^{(1)}$

It is easy to propose a conjecture with respect to the special polynomials associated with the Noumi-Yamada system of type $A_{2n-1}^{(1)}$ from the discussion in the previous sections. We start with a particular solution

$$\alpha_{2i} = \frac{1}{n} - s, \qquad \alpha_{2i+1} = s \qquad (i = 0, 1, \dots, n-1), \tag{5.1}$$

$$f_i = \frac{\sqrt{t}}{n} \qquad (i = 0, 1, \dots, 2n-1),$$

to the system on the fixed points with respect to the transformation π^2.

Conjecture 5.1. *The functions ϕ_v with the above specialization are expressed in the form*

$$\phi_v = \left(\frac{\sqrt{t}}{n}\right)^{\tilde{v}(\tilde{v}-1)/2} U_{v_-, v_+}, \tag{5.2}$$

$$\tilde{v} = \sum_{i=1}^{2n} (-1)^{i-1} v_i, \qquad v_- = (v_1, \dots, v_{2n-1}), \qquad v_- = (v_2, \dots, v_{2n}).$$

The polynomials $U_{v_-,v_+} = U_{v_-,v_+}(t,s)$ *are expressed in terms of the universal characters as*

$$U_{v_-,v_+} = n^{-|\lambda_-|}(-n)^{-|\lambda_+|} H_{\lambda_-} H_{\lambda_+} S_{\lambda_-,\lambda_+}, \tag{5.3}$$

where λ'_- *and* λ_+ *are partitions of n-core corresponding to the Maya diagrams*

$$\begin{aligned} M_{v_-} &= (n\boldsymbol{Z}_{<v_1}+1) \cup (n\boldsymbol{Z}_{<v_3}+2) \cup \cdots \cup (n\boldsymbol{Z}_{<v_{2n-1}}+n), \\ M_{v_+} &= (n\boldsymbol{Z}_{<v_2}+1) \cup (n\boldsymbol{Z}_{<v_4}+2) \cup \cdots \cup (n\boldsymbol{Z}_{<v_{2n}}+n), \end{aligned} \tag{5.4}$$

respectively. The variables x^- *and* x^+ *are specialized as*

$$x_j^- = \frac{t}{n} + \frac{ns-\tilde{v}}{j}, \qquad x_j^+ = -\frac{t}{n} + \frac{ns-\tilde{v}}{j}, \tag{5.5}$$

which means that the entries in the determinant formula of Jacobi-Trudi type are also the Laguerre polynomials.

6. Remarks and Discussions

It is natural to ask what kind of solutions to the Noumi-Yamada system of type $A_5^{(1)}$ one can get by starting with the fixed points with respect to the transformation π^3. In this setting, we find that the system is reduced to

$$\begin{aligned} f_0' &= \sqrt{t}[f_0(f_1-f_2)+\alpha_0], \\ f_1' &= \sqrt{t}[f_1(f_2-f_3)+\alpha_1], \\ f_2' &= \sqrt{t}[f_2(f_3-f_0)+\alpha_2], \end{aligned} \tag{6.1}$$

$$\alpha_0+\alpha_1+\alpha_2 = \frac{1}{2}, \qquad f_0+f_1+f_2 = \sqrt{t}, \tag{6.2}$$

which is equivalent to P_{IV} or the Noumi-Yamada system of type $A_2^{(1)}$. It is obvious that putting $\alpha_4 = \alpha_5 = 0$ and $f_4 = f_5 = 0$ in the system of type $A_5^{(1)}$ we have

$$\begin{aligned} f_0' &= f_0 f_2(f_1-f_3) + \left(\frac{1}{2}-\alpha_2\right) f_0 + \alpha_0 f_2, \\ f_1' &= f_1 f_3(f_2-f_0) + \left(\frac{1}{2}-\alpha_3\right) f_1 + \alpha_1 f_3, \\ f_2' &= f_2 f_0(f_3-f_1) + \left(\frac{1}{2}-\alpha_0\right) f_2 + \alpha_2 f_0, \\ f_3' &= f_3 f_1(f_0-f_2) + \left(\frac{1}{2}-\alpha_1\right) f_3 + \alpha_3 f_1, \end{aligned} \tag{6.3}$$

$$\alpha_0+\alpha_1+\alpha_2+\alpha_3=1,\qquad f_0+f_2=f_1+f_3=\sqrt{t},\tag{6.4}$$

which is equivalent to P_V or the Noumi-Yamada system of type $A_3^{(1)}$. These mean that the Noumi-Yamada system of type $A_5^{(1)}$ contains P_{IV} and P_V as special cases. Similarly, the solutions to the Noumi-Yamada system of type $A_{lm-1}^{(1)}$ $(l=3,4,\dots,m=1,2,\dots)$ on the fixed points with respect to the transformation π^l are subject to the system of type $A_{l-1}^{(1)}$.

As we mentioned above, the special polynomials associated with a class of algebraic solutions to P_{VI} and the Garnier systems can be also expressed in terms of the universal characters. It is interesting to clarify why the universal characters appear associated with these systems. Recently, Tsuda has constructed an integrable hierarchy whose τ-functions are given by the universal characters [12]. It is expected that the Noumi-Yamada system of type $A_{2n-1}^{(1)}$ is derived as some reduction from this integrable hierarchy.

A. Illustration of the proof

In the case of $v_-=(1,2,0)$, $v_+=(2,1,0)$, the first relation of (3.19) is written as

$$\begin{vmatrix} q_2^{(r-1)} & q_1^{(r-1)} \\ p_0^{(r-1)} & p_1^{(r-1)} \end{vmatrix}\begin{vmatrix} q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} & q_{-2}^{(r)} & q_{-3}^{(r)} & q_{-4}^{(r)} \\ q_2^{(r)} & q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} & q_{-2}^{(r)} & q_{-3}^{(r)} \\ p_2^{(r)} & p_3^{(r)} & p_4^{(r)} & p_5^{(r)} & p_6^{(r)} & p_7^{(r)} \\ p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} & p_2^{(r)} & p_3^{(r)} & p_4^{(r)} \\ p_{-3}^{(r)} & p_{-2}^{(r)} & p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} & p_2^{(r)} \\ p_{-4}^{(r)} & p_{-3}^{(r)} & p_{-2}^{(r)} & p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} \end{vmatrix}\tag{A.1}$$

$$=\begin{vmatrix} q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} \\ q_2^{(r)} & q_1^{(r)} & q_0^{(r)} \\ p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} \end{vmatrix}\begin{vmatrix} q_2^{(r-1)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} \\ p_3^{(r-1)} & p_4^{(r-1)} & p_5^{(r-1)} & p_6^{(r-1)} & p_7^{(r-1)} \\ p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} & p_3^{(r-1)} & p_4^{(r-1)} \\ p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} \\ p_{-3}^{(r-1)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} \end{vmatrix}$$

$$+\begin{vmatrix} q_2^{(r)} & q_1^{(r)} & q_0^{(r)} \\ p_2^{(r)} & p_3^{(r)} & p_4^{(r)} \\ p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} \end{vmatrix}\begin{vmatrix} q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} & q_{-3}^{(r-1)} \\ q_2^{(r-1)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} \\ p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} & p_3^{(r-1)} & p_4^{(r-1)} \\ p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} \\ p_{-3}^{(r-1)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} \end{vmatrix}.$$

Using the contiguity relation (3.21), we have

(A.2)
$$\begin{vmatrix} q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} & q_{-2}^{(r)} & q_{-3}^{(r)} & q_{-4}^{(r)} \\ q_2^{(r)} & q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} & q_{-2}^{(r)} & q_{-3}^{(r)} \\ p_2^{(r)} & p_3^{(r)} & p_4^{(r)} & p_5^{(r)} & p_6^{(r)} & p_7^{(r)} \\ p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} & p_2^{(r)} & p_3^{(r)} & p_4^{(r)} \\ p_{-3}^{(r)} & p_{-2}^{(r)} & p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} & p_2^{(r)} \\ p_{-4}^{(r)} & p_{-3}^{(r)} & p_{-2}^{(r)} & p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} \end{vmatrix}$$
$$= \begin{vmatrix} -q_1^{(r)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} & q_{-3}^{(r-1)} \\ -q_2^{(r)} & q_2^{(r-1)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} \\ p_2^{(r)} & p_3^{(r-1)} & p_4^{(r-1)} & p_5^{(r-1)} & p_6^{(r-1)} & p_7^{(r-1)} \\ p_{-1}^{(r)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} & p_3^{(r-1)} & p_4^{(r-1)} \\ p_{-3}^{(r)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} \\ p_{-4}^{(r)} & p_{-3}^{(r-1)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} \end{vmatrix}.$$

Put

(A.3)
$$D = \begin{vmatrix} -q_1^{(r)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} & q_{-3}^{(r-1)} \\ -q_2^{(r)} & q_2^{(r-1)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} \\ p_2^{(r)} & p_3^{(r-1)} & p_4^{(r-1)} & p_5^{(r-1)} & p_6^{(r-1)} & p_7^{(r-1)} \\ p_{-1}^{(r)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} & p_3^{(r-1)} & p_4^{(r-1)} \\ p_{-3}^{(r)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} \\ p_{-4}^{(r)} & p_{-3}^{(r-1)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} \end{vmatrix}.$$

Then Jacobi's identity

(A.4)
$$D \cdot D\begin{bmatrix} 1 & 3 \\ 1 & 6 \end{bmatrix} = D\begin{bmatrix} 1 \\ 1 \end{bmatrix} D\begin{bmatrix} 3 \\ 6 \end{bmatrix} - D\begin{bmatrix} 1 \\ 6 \end{bmatrix} D\begin{bmatrix} 3 \\ 1 \end{bmatrix},$$

is reduced to (A.1). The second relation of (3.19) is written as

(A.5)
$$x\begin{vmatrix} q_2^{(r-2)} & q_1^{(r-2)} & q_0^{(r-2)} & q_{-1}^{(r-2)} \\ p_1^{(r-2)} & p_2^{(r-2)} & p_3^{(r-2)} & p_4^{(r-2)} \\ p_{-1}^{(r-2)} & p_0^{(r-2)} & p_1^{(r-2)} & p_2^{(r-2)} \\ p_{-2}^{(r-2)} & p_{-1}^{(r-2)} & p_0^{(r-2)} & p_1^{(r-2)} \end{vmatrix} \begin{vmatrix} q_1^{(r+1)} & q_0^{(r+1)} & q_{-1}^{(r+1)} & q_{-2}^{(r+1)} \\ q_2^{(r+1)} & q_1^{(r+1)} & q_0^{(r+1)} & q_{-1}^{(r+1)} \\ p_1^{(r+1)} & p_2^{(r+1)} & p_3^{(r+1)} & p_4^{(r+1)} \\ p_{-2}^{(r+1)} & p_{-1}^{(r+1)} & p_0^{(r+1)} & p_1^{(r+1)} \end{vmatrix}$$

$$
= -7 \begin{vmatrix} q_1^{(r)} & q_0^{(r)} & q_{-1}^{(r)} \\ q_2^{(r)} & q_1^{(r)} & q_0^{(r)} \\ p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} \end{vmatrix} \begin{vmatrix} q_2^{(r-1)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} \\ p_3^{(r-1)} & p_4^{(r-1)} & p_5^{(r-1)} & p_6^{(r-1)} & p_7^{(r-1)} \\ p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} & p_3^{(r-1)} & p_4^{(r-1)} \\ p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} \\ p_{-3}^{(r-1)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} \end{vmatrix}
$$

$$
+ (r-4) \begin{vmatrix} q_2^{(r)} & q_1^{(r)} & q_0^{(r)} \\ p_2^{(r)} & p_3^{(r)} & p_4^{(r)} \\ p_{-1}^{(r)} & p_0^{(r)} & p_1^{(r)} \end{vmatrix} \begin{vmatrix} q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} & q_{-3}^{(r-1)} \\ q_2^{(r-1)} & q_1^{(r-1)} & q_0^{(r-1)} & q_{-1}^{(r-1)} & q_{-2}^{(r-1)} \\ p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} & p_3^{(r-1)} & p_4^{(r-1)} \\ p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} & p_2^{(r-1)} \\ p_{-3}^{(r-1)} & p_{-2}^{(r-1)} & p_{-1}^{(r-1)} & p_0^{(r-1)} & p_1^{(r-1)} \end{vmatrix}.
$$

Using the contiguity relation (3.21), we get

$$
\text{(A.6)} \quad \begin{vmatrix} q_1^{(r+1)} & q_0^{(r+1)} & q_{-1}^{(r+1)} & q_{-2}^{(r+1)} \\ q_2^{(r+1)} & q_1^{(r+1)} & q_0^{(r+1)} & q_{-1}^{(r+1)} \\ p_1^{(r+1)} & p_2^{(r+1)} & p_3^{(r+1)} & p_4^{(r+1)} \\ p_{-2}^{(r+1)} & p_{-1}^{(r+1)} & p_0^{(r+1)} & p_1^{(r+1)} \end{vmatrix} = \begin{vmatrix} q_1^{(r-2)} & q_0^{(r-1)} & q_{-1}^{(r)} & q_{-2}^{(r+1)} \\ q_2^{(r-2)} & q_1^{(r-1)} & q_0^{(r)} & q_{-1}^{(r+1)} \\ -p_4^{(r-2)} & p_4^{(r-1)} & -p_4^{(r)} & p_4^{(r+1)} \\ -p_1^{(r-2)} & p_1^{(r-1)} & -p_1^{(r)} & p_1^{(r+1)} \end{vmatrix}.
$$

Similarly we have

$$
\text{(A.7)} \quad \begin{vmatrix} q_1^{(r+1)} & q_0^{(r+1)} & q_{-1}^{(r+1)} & q_{-2}^{(r+1)} \\ q_2^{(r+1)} & q_1^{(r+1)} & q_0^{(r+1)} & q_{-1}^{(r+1)} \\ p_1^{(r+1)} & p_2^{(r+1)} & p_3^{(r+1)} & p_4^{(r+1)} \\ p_{-2}^{(r+1)} & p_{-1}^{(r+1)} & p_0^{(r+1)} & p_1^{(r+1)} \end{vmatrix}
$$

$$
= \begin{vmatrix} q_1^{(r-4)} & q_0^{(r-3)} & q_{-1}^{(r-2)} & q_{-2}^{(r-1)} & q_{-3}^{(r)} & q_{-4}^{(r+1)} \\ q_2^{(r-4)} & q_1^{(r-3)} & q_0^{(r-2)} & q_{-1}^{(r-1)} & q_{-2}^{(r)} & q_{-3}^{(r+1)} \\ -p_6^{(r-4)} & p_6^{(r-3)} & -p_6^{(r-2)} & p_6^{(r-1)} & -p_6^{(r)} & p_6^{(r+1)} \\ -p_3^{(r-4)} & p_3^{(r-3)} & -p_3^{(r-2)} & p_3^{(r-1)} & -p_3^{(r)} & p_3^{(r+1)} \\ -p_1^{(r-4)} & p_1^{(r-3)} & -p_1^{(r-2)} & p_1^{(r-1)} & -p_1^{(r)} & p_1^{(r+1)} \\ -p_0^{(r-4)} & p_0^{(r-3)} & -p_0^{(r-2)} & p_0^{(r-1)} & -p_0^{(r)} & p_0^{(r+1)} \end{vmatrix}.
$$

Using the contiguity relation (3.22), we obtain

$$\text{(A.8)}\qquad \text{(A.7)} = x^{-5}(r-4)(r-3)\times 7\cdot 4\cdot 2\cdot 1$$

$$\times\begin{vmatrix} \dfrac{q_1^{(r-4)}}{r-4} & q_1^{(r-5)} & q_0^{(r-4)} & q_{-1}^{(r-3)} & q_{-2}^{(r-2)} & q_{-3}^{(r-1)} \\ \dfrac{q_2^{(r-4)}}{r-3} & q_2^{(r-5)} & q_1^{(r-4)} & q_0^{(r-3)} & q_{-1}^{(r-2)} & q_{-2}^{(r-1)} \\ -\dfrac{p_6^{(r-4)}}{7} & -p_7^{(r-5)} & p_7^{(r-4)} & -p_7^{(r-3)} & p_7^{(r-2)} & -p_7^{(r-1)} \\ -\dfrac{p_3^{(r-4)}}{4} & -p_4^{(r-5)} & p_4^{(r-4)} & -p_4^{(r-3)} & p_4^{(r-2)} & -p_4^{(r-1)} \\ -\dfrac{p_1^{(r-4)}}{2} & -p_2^{(r-5)} & p_2^{(r-4)} & -p_2^{(r-3)} & p_2^{(r-2)} & -p_2^{(r-1)} \\ -p_0^{(r-4)} & -p_1^{(r-5)} & p_1^{(r-4)} & -p_1^{(r-3)} & p_1^{(r-2)} & -p_1^{(r-1)} \end{vmatrix}.$$

Put

$$\text{(A.9)}\qquad D = \begin{vmatrix} \dfrac{q_1^{(r-4)}}{r-4} & q_1^{(r-5)} & q_0^{(r-4)} & q_{-1}^{(r-3)} & q_{-2}^{(r-2)} & q_{-3}^{(r-1)} \\ \dfrac{q_2^{(r-4)}}{r-3} & q_2^{(r-5)} & q_1^{(r-4)} & q_0^{(r-3)} & q_{-1}^{(r-2)} & q_{-2}^{(r-1)} \\ -\dfrac{p_6^{(r-4)}}{7} & -p_7^{(r-5)} & p_7^{(r-4)} & -p_7^{(r-3)} & p_7^{(r-2)} & -p_7^{(r-1)} \\ -\dfrac{p_3^{(r-4)}}{4} & -p_4^{(r-5)} & p_4^{(r-4)} & -p_4^{(r-3)} & p_4^{(r-2)} & -p_4^{(r-1)} \\ -\dfrac{p_1^{(r-4)}}{2} & -p_2^{(r-5)} & p_2^{(r-4)} & -p_2^{(r-3)} & p_2^{(r-2)} & -p_2^{(r-1)} \\ -p_0^{(r-4)} & -p_1^{(r-5)} & p_1^{(r-4)} & -p_1^{(r-3)} & p_1^{(r-2)} & -p_1^{(r-1)} \end{vmatrix}.$$

Then Jacobi's identity (A.4) is reduced to (A.5).

References

[1] Kajiwara, K. and Ohta, Y., Determinant structure of the rational solutions for the Painlevé IV equation, J. Phys. A: Math. Gen., **31** (1998), 2431–2446.

[2] Koike, K., On the decomposition of tensor products of the representations of the classical groups: by means of the universal characters, Adv. Math., **74** (1989), 57–86.

[3] Masuda, T., On the rational solutions of q-Painlevé V equation, Nagoya Math. J., **169** (2003), 119–143.

[4] Masuda, T., On a class of algebraic solutions to the Painlevé VI equation, its determinant formula and coalescence cascade, Funkcial. Ekvac., **46** (2003), 121–171.

[5] Masuda, T., Ohta, Y. and Kajiwara, K., A determinant formula for a class of rational solutions of Painlevé V equation, Nagoya Math. J., **168** (2002), 1–25.
[6] Noumi, M., *Painlevé equations through symmetry*, Translations of Mathematical Monographs Vol. 223, American Mathematical Society, 2004.
[7] Noumi, M. and Yamada, Y., Higher order Painlevé equations of type $A_l^{(1)}$, Funkcial. Ekvac., **41** (1998), 483–503.
[8] Noumi, M. and Yamada, Y., Affine Weyl groups, discrete dynamical systems and Painlevé equations, Commun. Math. Phys., **199** (1998), 281–295.
[9] Noumi, M. and Yamada, Y., Symmetries in the fourth Painlevé equation and Okamoto polynomials, Nagoya Math. J., **153** (1999), 53–86.
[10] Okamoto, K., Studies on the Painlevé equations III, second and fourth Painlevé equations, P_{II} and P_{IV}, Math. Ann., **275** (1986), 222–254.
[11] Tsuda, T., Toda equation and special polynomials associated with the Garnier system, preprint.
[12] Tsuda, T., Universal characters and an extension of the KP hierarchy, Comm. Math. Phys., **248** (2004), 501–526.

nuna adreso:
Department of Mathematics
Kobe University
Rokko, Kobe, 657-8501
Japan
E-mail: masuda@math.kobe-u.ac.jp

(Ricevita la 1-an de aprilo, 2004)
(Reviziita la 3-an de septembro, 2004)