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Optimal Consumption and Portfolio Choice with Stopping

By

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Abstract. We study the Bellman equation associated with the optimal consumption and portfolio choice problem with stopping times in a complete market. We establish the existence of a strong solution by using the viscosity solutions technique. The optimal policy is shown to exist from the optimality conditions in the variational inequality.

Keywords and Phrases. Variational inequality, Viscosity solution, Combined control, Consumption, Portfolio, Stopping time.

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1. Introduction

We consider the consumption and portfolio choice problem with stopping times of a single agent, with an initial wealth x, who attempts to maximize the expected utility of consumption. The agent can consume the wealth at rate $C(t)$, invest it in any of the 2 available assets, and stop freely before his bankruptcy, *i.e.*, his wealth falls to zero. We adopt the following standard model of a financial market [3, 5, 8, 10, 15]. Let $(p_0(t), p_1(t))$ be the vector of prices at time t of the assets and $W(t)$ be a standard 1-dimensional Brownian motion on a complete probability space $(\Omega, \mathscr{F}, P)$ endowed with the natural filtration $\mathscr{F}_t = \sigma(W(s); s \le t)$. We assume that

$$dp_0(t) = rp_0(t)dt \qquad \text{for } t > 0,$$
$$dp_1(t) = p_1(t)[b\, dt + \sigma\, dW(t)] \qquad \text{for } t > 0.$$

Here we denote by b and σ positive constants, and the interest rate $r > 0$ is also a constant with $b > r$.

Given a policy (C, π, τ), the agent's wealth process $X^0(\cdot)$ stopped at τ is a solution of the stochastic differential equation

$$dX^0(t) = 1_{\{t \le \tau\}}[\{rX^0(t) - C(t)\}dt + \pi(t)\{(b - r)dt + \sigma\, dW(t)\}]$$

and $X^0(0) = x > 0$, where $C(\cdot)$ is an $\mathscr{F}_t$-progressively measurable, nonnegative process with $\int_0^T C(t)dt < \infty$ *a.s.*, $\pi(\cdot)$ an $\mathscr{F}_t$-progressively measurable, realvalued portfolio process with $\int_0^T |\pi(t)|^2 dt < \infty$ *a.s.* for every $T > 0$, and τ a stopping time.

We say that (C, π) is admissible for x if the wealth process $X^0(t)$ is non-negative *a.s.* We denote by $\mathscr{A}^0$ the class of all such policies, and also by $\mathscr{S}$ the class of all stopping times.

For given $U_i : [0, \infty) \to \boldsymbol{R}$ $(i = 1, 2)$, we consider the total expected utility of $(C, \pi, \tau) \in \mathscr{A}^0 \times \mathscr{S}$ up to time $\theta^0 = \inf\{t \geq 0 \mid X^0(t) = 0\}$ defined by

$$(1.1) \quad J(x, C, \pi, \tau) = E\left[\int_0^{\theta^0 \wedge \tau} e^{-\alpha t} U_1(C(t)) dt + e^{-\alpha(\theta^0 \wedge \tau)} U_2(X^0(\theta^0 \wedge \tau))\right],$$

where $\alpha > 0$ is a discount factor.

The purpose of this paper is to find an optimal policy (C^*, π^*, τ^*) which maximizes $J(x, C, \pi, \tau)$ over (C, π, τ) in $\mathscr{A}^0 \times \mathscr{S}$.

To this end, we obtain enough regularity of the optimal total expected utility $v(x) = \sup_{(C, \pi, \tau) \in \mathscr{A}^0 \times \mathscr{S}} J(x, C, \pi, \tau)$, which is expected to be a viscosity solution of

$$(1.2) \qquad \max\{\mathscr{H}_\alpha(x, v(x), v'(x), v''(x)), U_2(x) - v(x)\} = 0 \qquad \text{in } (0, \infty).$$

Here, we set

$$\mathscr{H}_\alpha(x, s, p, X) = -\alpha s + rxp + \mathscr{M}(p, X) + \tilde{U}_1(p),$$

where

$$\mathscr{M}(p, X) = \sup_{\pi \in \boldsymbol{R}}\left\{\frac{\pi^2\sigma^2}{2} X + \pi(b - r)p\right\}, \qquad \text{and} \qquad \tilde{U}_1(p) = \sup_{c \geq 0}\{U_1(c) - cp\}.$$

We will use the following identity without mentioning it:

$$\mathscr{M}(p, X) = -\frac{(b - r)^2 p^2}{2\sigma^2 X} \qquad \text{provided } X < 0.$$

One of difficulties to treat (1.2) is that (assuming $v \in C^2$) we have to show that $v' > 0$ and $v'' < 0$ so that (1.2) make sense. In fact, under our assumption for U_1 (see (2.3) below), if $v' \leq 0$ or $v'' \geq 0$ at a point $x > 0$, then the left hand side of (1.2) becomes ∞ at x.

In order to obtain regularity of solutions of obstacle problems (1.2), we will use the standard penalized equation associeted with (1.2). Moreover, to impose enough regularity on solutions of the penalized problem, we will also introduce approximate equations corresponding to bounded control problems with respect to (C, π) and elliptic regularization.

We refer to [5, 8, 12, 15] for the consumption/investment problem without stopping, and [1] for the penalty method in the theory of variational inequalities.

We also mention the work [7] for the utility maximization problem with stopping discussed from a point of view of the duality method in the finite horizon case.

The present paper is organized as follows. In section 2, we consider the approximate problems and obtain the unique viscosity solution of those. Section 3 is devoted to the regularity of viscosity solutions of (1.2).

We finally present a synthesis of optimal policies for the optimization problem in section 4.

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2. Penalized problem

Throughout this paper, we fix

$$0 < \gamma < 1.$$

We then introduce the Banach space $\mathscr{B}$ by

$$\mathscr{B} = \left\{ h \in C([0,\infty), \boldsymbol{R}) \,\middle|\, \begin{array}{l} \text{for } \forall\rho > 0,\ \exists C_\rho > 0 \text{ such that } |h(x) - h(y)| \\ \leq C_\rho |x-y|^\gamma + \rho(1 + x^\gamma + y^\gamma) \text{ for } x, y \geq 0 \end{array} \right\}$$

with its norm $\|h\|_{\mathscr{B}} = \sup_{x \geq 0}\{|h(x)|/(1+x^\gamma)\}$.

Now, we shall present a list of our hypotheses.

(2.1) $$0 < r < \alpha \wedge b,$$

(2.2) $$\exists \alpha_0 \in (r, \alpha) \text{ such that } \mathscr{H}_{\alpha_0}(x, x^\gamma, \gamma x^{\gamma-1}, \gamma(\gamma-1)x^{\gamma-2}) \leq 0 \text{ in } (0,\infty).$$

We remark that (2.2) yields

(2.3) $$\left\{ -\alpha_0 + r\gamma - \frac{(b-r)^2\gamma}{2\sigma^2(\gamma-1)} \right\} x^\gamma + \tilde{U}_1(\gamma x^{\gamma-1}) \leq 0 \qquad \text{for } x > 0.$$

Thus, when $U_1, U_2 \in \mathscr{B}$, under (2.2), we can find $C_0 > 1$ such that $\zeta(x) := C_0(1 + x^\gamma)$ is a supersolution of (1.2);

$$\max\{\mathscr{H}_\alpha(x, \zeta(x), \zeta'(x), \zeta''(x)), U_2(x) - \zeta(x)\} \leq 0, \qquad \text{in } (0,\infty).$$

We remark that this ζ is also a supersolution of approximate equations (2.6) below. When U_1, U_2 are nonnegative, which we will suppose, $\zeta_0 \equiv 0$ is a subsolution of (2.6). Thus, it is natural to seek an approximate solution for (1.2) in $\mathscr{B}_\zeta$ defined by

$$\mathscr{B}_\zeta = \{h \in \mathscr{B} \mid 0 \le h \le \zeta \ \text{in} \ (0,\infty), \ \text{and} \ h(0) = 0\}.$$

Notice that $\mathscr{H}_\alpha(x, \zeta_0, \zeta_0', \zeta_0'') = \infty$ if $\tilde{U}_1(0) = \infty$.

It is easy to verify that (2.2) is fulfilled in case when $U_2(x) = x^{\hat{\gamma}}$ for $\hat{\gamma} \in (0, \gamma]$, and $U_1(c) = c^\gamma$ for a suitable $\alpha_0 > 0$ since $\tilde{U}_1(\gamma x^{\gamma-1}) = (1-\gamma)x^\gamma$.

We will also use the following hypotheses:

$$U_1, U_2 \in \mathscr{B}_\zeta \cap C^2(0,\infty), \tag{2.4}$$

$$U_1(0) = \lim_{x\to\infty} U_1'(x) = 0, \qquad \lim_{x\to+0} U_1'(x) = \lim_{x\to\infty} U_1(x) = \infty. \tag{2.5}$$

In order to introduce penalty equations of (1.2), we use some notations. First, we choose $\beta \in C^2(\boldsymbol{R})$ satisfying that

$$-1 \le \beta' \le 0, \beta'' \ge 0 \quad \text{in } \boldsymbol{R}, \qquad \text{and} \qquad \beta(s) = \begin{cases} 0 & \text{for } s \ge 0, \\ -s-1 & \text{for } s \le -2. \end{cases}$$

Then, we set

$$\beta_\varepsilon(s) = \frac{\beta(s)}{\varepsilon} \qquad \text{for } \varepsilon \in (0,1).$$

Next, to consider bounded control problems, for $L, R \ge 1$, we set

$$\mathscr{M}^L(p, X) = \max_{|\pi| \le L} \left\{ \frac{\pi^2\sigma^2}{2} X + \pi(b-r)p \right\}, \qquad \text{and} \qquad \tilde{U}_1^R(p) = \max_{0 \le c \le R} \{U_1(c) - pc\}.$$

Using the viscosity solution theory for combined control [14], we shall show that (1.2) admits a strong solution v as the limit of the solution $u_{\varepsilon,\mu,L,R}$ of the following approximate problem: For $\varepsilon, \mu \in (0,1]$ and $L, R \ge 1$,

(2.6)

$$-\alpha u + rxu' + \frac{\mu^2 x^2}{2} u'' + \mathscr{M}^L(u', u'') + \tilde{U}_1^R(u') + \beta_\varepsilon(u - U_2(x)) = 0 \qquad \text{in } (0,\infty)$$

under $u(0) = 0$.

We note that (2.6) is related to the optimization problem (1.1) with $\theta = \theta_{C,\pi,x} := \inf\{t \ge 0 \mid X_{C,\pi,x}(t) = 0\}$ in place of θ^0 for the process $X(\cdot) = X_{C,\pi,x}(\cdot)$ given by

$$dX(t) = \{rX(t) - C(t)\}dt + \pi(t)\{(b-r)dt + \sigma\, dW(t)\} + \mu X(t) d\tilde{W}(t), \tag{2.7}$$

and $X(0) = x \ge 0$, over the class $\mathscr{A} \times \mathscr{S}$, where

$$\mathscr{A} = \mathscr{A}^{L,R} = \{(C,\pi) \in \mathscr{A}^0 \mid |\pi(t)| \le L, \ 0 \le C(t) \le R \ \text{for all} \ t \ge 0\},$$

and $\tilde{W}(t)$ is a 1-dimensional Brownian motion independent of $W(t)$. However, we notice that the total expected utility cannot be simply expressed unlike for the original problem because we have a semilinear term β_ε. Thus, giving up finding an explicit formula, we first show the existence of solutions of (2.6).

For this purpose, it is convenient to use an equivalent equation to (2.6);

$$-\left(\alpha+\frac{1}{\varepsilon}\right)u+rxu'+\frac{\mu^2x^2}{2}u''+\mathscr{M}^L(u',u'')+\tilde{U}_1^R(u')+\hat{\beta}_\varepsilon(x,u)=0 \qquad \text{in } (0,\infty),$$

under the boundary condition $u(0)=0$. Here, we set $\hat{\beta}_\varepsilon=\varepsilon^{-1}\hat{\beta}$, where

$$\hat{\beta}(x,s)=s+\beta(s-U_2(x)) \qquad \text{for } (x,s)\in[0,\infty)\times\boldsymbol{R}.$$

If there exists a solution u of (2.6), then we may expect u to satisfy that for $x\geq 0$,

$$u(x)=\sup_{(C,\pi)\in\mathscr{A}^{L,R}} E\left[\int_0^{\theta_{C,\pi,x}} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t)) + \hat{\beta}_\varepsilon(X_{C,\pi,x}(t),u(X_{C,\pi,x}(t)))\}dt\right]. \tag{2.8}$$

In this section, we shall simply write $\mathscr{A}$, $X(t)$ and θ for $\mathscr{A}^{L,R}$, $X_{C,\pi,x}(t)$ and $\theta_{C,\pi,x}$, respectively when there is no confusion.

2.1. Existence of solutions of the penalty equation

By a usual localization argument, we may assume that $X(t)$ is bounded, if necessary. Applying Ito's formula, for $\alpha'\geq\alpha_0$, $\phi\in C^2(0,\infty)$ with $\phi''<0$ and $\tau_1,\tau_2\in\mathscr{S}$ such that $\theta\wedge\tau_2\geq\tau_1$, we have

$$\begin{aligned}
&E[e^{-\alpha'(\theta\wedge\tau_2)}\phi(X(\theta\wedge\tau_2))-e^{-\alpha'\tau_1}\phi(X(\tau_1))]\\
&\quad=E\Bigg[\int_{\tau_1}^{\theta\wedge\tau_2}e^{-\alpha't}\Big\{(rX(t)-C(t)+\pi(t)(b-r))\phi'(X(t))\\
&\qquad+\frac{\pi(t)^2\sigma^2+\mu^2X(t)^2}{2}\phi''(X(t))-\alpha'\phi(X(t))\Big\}dt\Bigg]\\
&\quad\leq E\Bigg[\int_{\tau_1}^{\theta\wedge\tau_2}e^{-\alpha't}\{(\alpha_0-\alpha')\phi(X(t))-U_1(C(t))\\
&\qquad+\mathscr{H}_{\alpha_0}(X(t),\phi(X(t)),\phi'(X(t)),\phi''(X(t)))\}dt\Bigg],
\end{aligned}\tag{2.9}$$

which we will use in the proof of Theorem 2.1 below.

Theorem 2.1. *Assume that* (2.1), (2.2), (2.4), (2.5) *hold. Then, there exists a unique* $u\in\mathscr{B}_\zeta$ *satisfying of* (2.8).

Proof. Define

$$(2.10)\quad \mathscr{T}h(x) = \sup_{(C,\pi)\in\mathscr{A}} E\left[\int_0^{\theta} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t)) + \hat{\beta}_\varepsilon(X(t), h(X(t)))\}dt\right]$$

for any $h \in \mathscr{B}_\zeta$. It suffices to show that

$$(2.11)\qquad \mathscr{T} \text{ is a contraction mapping on } \mathscr{B}_\zeta.$$

By the definition, it is clear that $\mathscr{T}h(0) = 0$. In view of (2.9) with $\alpha' = \alpha + \varepsilon^{-1}$, $\phi = \zeta$, $\tau_1 = 0$ and $\tau_2 = \theta$, we have

$$E\left[\int_0^{\theta} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t)) + \hat{\beta}_\varepsilon(X(t), \zeta(X(t)))\}dt + e^{-(\alpha+1/\varepsilon)\theta}\zeta(X(\theta))\right] \le \zeta(x),$$

from which it follows that

$$0 \le \mathscr{T}h(x) \le \mathscr{T}\zeta(x) \le \zeta(x) \le \|\zeta\|_{\mathscr{B}}(1 + x^\gamma).$$

Next, for $x \ge y \ge 0$, we set $Y(\cdot) = X_{C,\pi,y}(\cdot)$ and $\hat{\theta} = \theta_{C,\pi,y}$. We note that $X(t) - Y(t) = e^{rt+\mu\tilde{W}(t)-\mu^2t/2}(x-y) \ge 0$, and $\theta \ge \hat{\theta}$, $E[(e^{\mu\tilde{W}(t)-\mu^2t/2})^\gamma] \le 1$.

We shall estimate the $\mathscr{T}_i := \mathscr{T}_i^{C,\pi}$ $(i = 0, 1, 2)$ given by

$$\mathscr{T}_0 = E\left[\int_{\hat{\theta}}^{\theta} e^{(-\alpha+1/\varepsilon)t}U_1(C(t))dt\right],$$

$$\mathscr{T}_1 = E\left[\int_{\hat{\theta}}^{\theta} e^{-(\alpha+1/\varepsilon)t}\hat{\beta}(X(t), h(X(t)))dt\right],$$

$$\mathscr{T}_2 = E\left[\int_0^{\hat{\theta}} e^{-(\alpha+1/\varepsilon)t}|\hat{\beta}(X(t), h(X(t))) - \hat{\beta}(Y(t), h(Y(t)))|dt\right].$$

We apply (2.9) with $\alpha' = \alpha$, $\phi(x) = x^\gamma$, $\tau_1 = \hat{\theta}$ and $\tau_2 = \theta$ to get

$$(2.12)\qquad \mathscr{T}_0 \le E\left[\int_{\hat{\theta}}^{\theta} e^{-\alpha t}U_1(C(t))dt\right] \le E[e^{-\alpha\hat{\theta}}X(\hat{\theta})^\gamma - e^{-\alpha\theta}X(\theta)^\gamma]$$
$$= E[e^{-\alpha\hat{\theta}}\{X(\hat{\theta})^\gamma - Y(\hat{\theta})^\gamma\}1_{\{\hat{\theta}<\theta\}}]$$
$$\le E[e^{-\alpha\hat{\theta}}|X(\hat{\theta}) - Y(\hat{\theta})|^\gamma] \le |x-y|^\gamma.$$

Furthermore, we have

$$E\left[\int_{\hat{\theta}}^{\theta} e^{-\alpha t}X(t)^\gamma dt\right] \le \frac{E[e^{-\alpha\hat{\theta}}X(\hat{\theta})^\gamma - e^{-\alpha\theta}X(\theta)^\gamma]}{\alpha - \alpha_0} \le \frac{|x-y|^\gamma}{\alpha - \alpha_0}.$$

Since $\hat{\beta}(\cdot, h(\cdot)) \in \mathscr{B}_\zeta$, for $\rho > 0$, we find $C_\rho > 0$ such that

$$\mathscr{T}_1 \le E\left[\int_{\hat{\theta}}^{\theta} e^{-\alpha t}\{\rho + C_\rho X(t)^\gamma\}dt\right] \le \frac{\rho}{\alpha} + C_\rho \frac{|x-y|^\gamma}{\alpha - \alpha_0}. \tag{2.13}$$

Again, by (2.9) with $\alpha' = \alpha$, $\phi(x) = x^\gamma$, $\tau_1 = 0$ and $\tau_2 = \hat{\theta}$, we have

$$E[e^{-\alpha_0(\hat{\theta}\wedge t)} X(\hat{\theta} \wedge t)^\gamma] \le x^\gamma,$$

and hence,

$$\begin{aligned} \mathscr{T}_2 &\le E\left[\int_0^{\hat{\theta}} e^{-\alpha t}\{C_\rho |X(t) - Y(t)|^\gamma + \rho(1 + X(t)^\gamma + Y(t)^\gamma)\}dt\right] \\ &\le \int_0^\infty e^{-\alpha t}\{C_\rho e^{\gamma r t}|x-y|^\gamma + e^{\alpha_0 t}\rho(1 + x^\gamma + y^\gamma)\}dt \\ &= C_\rho \frac{|x-y|^\gamma}{\alpha - r\gamma} + \rho \frac{1 + x^\gamma + y^\gamma}{\alpha - \alpha_0}. \end{aligned} \tag{2.14}$$

Since $|\mathscr{T}h(x) - \mathscr{T}h(y)| \le \sup_{(C,\pi)\in\mathscr{A}}\{\mathscr{T}_0 + \varepsilon^{-1}(\mathscr{T}_1 + \mathscr{T}_2)\}$, (2.12), (2.13) and (2.14) yield $\mathscr{T}h \in \mathscr{B}_\zeta$.

To complete the proof, we only need to observe that for $h_1, h_2 \in \mathscr{B}_\zeta$,

$$\begin{aligned} &|\mathscr{T}h_1(x) - \mathscr{T}h_2(x)| \\ &\quad \le \frac{1}{\varepsilon} \sup_{(C,\pi)\in\mathscr{A}} E\left[\int_0^\infty e^{-(\alpha+1/\varepsilon)t}|\hat{\beta}(X(t), h_1(X(t))) - \hat{\beta}(X(t), h_2(X(t)))|dt\right] \\ &\quad \le \frac{1}{\varepsilon} \sup_{(C,\pi)\in\mathscr{A}} E\left[\int_0^\infty e^{-(\alpha+1/\varepsilon)t}\|h_1 - h_2\|_{\mathscr{B}}(1 + X(t)^\gamma)dt\right] \\ &\quad \le \frac{1}{(\alpha - \alpha_0)\varepsilon + 1}\|h_1 - h_2\|_{\mathscr{B}}(1 + x^\gamma). \quad \square \end{aligned}$$

2.2. Viscosity solutions

We shall show that the solution of the penalty equation (2.8) is indeed a viscosity solution of (2.6). We shall recall the definition of viscosity solutions of

$$F(x, u, u', u'') = 0 \qquad \text{in an open interval} \qquad J \subset \boldsymbol{R}. \tag{2.15}$$

Definition 2.2. We call $u \in C(J)$ a viscosity subsolution (resp., supersolution) of (2.15) if, whenever for $\phi \in C^2(J)$, $u - \phi$ attains its local maximum (resp., minimum) at $x \in J$, then

$$F(x, u(x), \phi'(x), \phi''(x)) \ge 0 \qquad (\text{resp., } \le 0).$$

We call $u \in C(J)$ a viscosity solution of (2.15) if it is both a viscosity sub- and supersolution of (2.15).

Remarks. (1) We will use the equivalent definition from [2]:

$$F(x, u(x), p, X) \geq 0 \qquad \text{for } x \in J \text{ and } (p, X) \in \bar{J}^{2,+}u(x)$$
$$(\text{resp., } F(x, u(x), p, X) \leq 0 \text{ for } x \in J \text{ and } (p, X) \in \bar{J}^{2,-}u(x)).$$

We refer to [2] for the definition of $\bar{J}^{2,\pm}u(x)$.

(2) Since $\{x \geq 0 \mid J^{2,-}u(x) \neq \emptyset\}$ is dense in $(0, \infty)$ for $u \in C(0, \infty)$, we see that $U_2 \leq u$ in $(0, \infty)$ when $u \in C(0, \infty)$ is a viscosity supersolution of (1.2).

Theorem 2.3. *Assume that* (2.1), (2.2), (2.4), (2.5) *hold. Then, the solution* $u \in \mathscr{B}_\zeta$ *of* (2.8) *is a viscosity solution of* (2.6).

Proof. In view of the viscosity solution theory (e.g. [4]), it suffices to show that the dynamic programming principle holds;

$$(2.16) \qquad u(x) = \sup_{(C,\pi) \in \mathscr{A}} E\left[\int_0^{\theta\wedge\tau} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t)) + \hat{\beta}_\varepsilon(X(t), u(X(t)))\}dt + e^{-(\alpha+1/\varepsilon)(\theta\wedge\tau)}u(X(\theta\wedge\tau))\right]$$

for all $\tau \in \mathscr{S}$, which may depend on (C, π). Let $\tilde{u}(x)$ denote the right hand side of (2.16). In view of the Nisio semigroup [4], we have

$$\begin{aligned} u(x) &= \sup_{(C,\pi) \in \mathscr{A}} E\left[\left(\int_0^{\theta\wedge\tau} + \int_{\theta\wedge\tau}^{\theta}\right) e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t)) + \hat{\beta}_\varepsilon(X(t), u(X(t)))\}dt\right] \\ &= \sup_{(C,\pi) \in \mathscr{A}} E\left[\int_0^{\theta\wedge\tau} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t)) + \hat{\beta}_\varepsilon(X(t), u(X(t)))\}\right. \\ &\quad \left. + e^{-(\alpha+1/\varepsilon)(\theta\wedge\tau)} \int_0^{\theta-\theta\wedge\tau} e^{-(\alpha+1/\varepsilon)t}\{U_1(\tilde{C}_1(t)) + \hat{\beta}_\varepsilon(\tilde{X}(t), u(\tilde{X}(t)))\}dt\right] \\ &\leq \tilde{u}(x), \end{aligned}$$

where $\tilde{X}(t)$ is the process of (2.7) with $\tilde{X}(0) = X(\theta \wedge \tau)$ for $(\tilde{C}, \tilde{\pi}) \in \mathscr{A}$ given by $\tilde{C}(t) := C(t + \theta \wedge \tau)$ and $\tilde{\pi}(t) := \pi(t + \theta \wedge \tau)$.

To prove the reverse inequality, we set

$$\mathscr{T}^{C,\pi}u(x) = E\left[\int_0^{\theta} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t)) + \hat{\beta}_\varepsilon(X(t), u(X(t))\}dt\right].$$

As before, for any $\rho > 0$, there exists $C_\rho > 0$ such that

$$|\mathscr{T}^{C,\pi}u(x) - \mathscr{T}^{C,\pi}u(y)| \leq C_\rho|x - y|^\gamma + \rho(1 + x^\gamma + y^\gamma).$$

Take $0 < \delta < 1$ such that $C_\rho\delta^\gamma < \rho$. Then, we have for $|x - y| < \delta$,

$$(2.17) \qquad |u(x)-u(y)| \le \sup_{(C,\pi)\in\mathscr{A}} |\mathscr{T}^{C,\pi}u(x)-\mathscr{T}^{C,\pi}u(y)| \le \rho(2+x^\gamma+y^\gamma) < \Xi_\rho(x) := \rho(3+2x^\gamma).$$

Let $\{S_i\}$ be a sequence of disjoint subsets of $[0,\infty)$ such that $\mathrm{diam}(S_i)<\delta$ and $\bigcup_i S_i = [0,\infty)$. For any i, we take $x_i \in S_i$ and $(C_i,\pi_i)\in\mathscr{A}$ such that

$$(2.18) \qquad \mathscr{T}^{C_i,\pi_i}u(x_i) \ge u(x_i)-\rho.$$

Setting

$$(C^\tau,\pi^\tau)(t) = (C,\pi)(t)1_{\{t<\theta\wedge\tau\}} + (C_i,\pi_i)(t-\theta\wedge\tau)1_{\{t\ge\theta\wedge\tau\}}$$

for $X(\theta\wedge\tau)\in S_i$, by (2.17) and (2.18), we have

$$\begin{aligned}
\mathscr{T}^{C_i,\pi_i}u(X(\theta\wedge\tau)) &= \{\mathscr{T}^{C_i,\pi_i}u(X(\theta\wedge\tau)) - \mathscr{T}^{C_i,\pi_i}u(x_i)\} + \mathscr{T}^{C_i,\pi_i}u(x_i)\\
&\ge -\Xi_\rho(X(\theta\wedge\tau)) + \mathscr{T}^{C_i,\pi_i}u(x_i)\\
&\ge -\Xi_\rho(X(\theta\wedge\tau)) + u(x_i)-\rho\\
&\ge -\Xi_\rho(X(\theta\wedge\tau)) + \{u(X(\theta\wedge\tau)) - \Xi_\rho(X(\theta\wedge\tau))\}-\rho.
\end{aligned}$$

Since we can find $(C,\pi)\in\mathscr{A}$ such that

$$\tilde{u}(x)-\rho \le E\left[\int_0^{\theta\wedge\tau} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t))+\hat{\beta}_\varepsilon(X(t),u(X(t)))\}dt + e^{-(\alpha+1/\varepsilon)(\theta\wedge\tau)}u(X(\theta\wedge\tau))\right],$$

we get

$$\begin{aligned}
&\tilde{u}(x)-\rho\\
&\quad\le \sum_i E\left[\int_0^{\theta\wedge\tau} e^{-(\alpha+1/\varepsilon)t}\{U_1(C(t))+\hat{\beta}_\varepsilon(X(t),u(X(t)))\}dt\right.\\
&\qquad \left.+ e^{-(\alpha+1/\varepsilon)(\theta\wedge\tau)}\{\mathscr{T}^{C_i,\pi_i}u(X(\theta\wedge\tau)) + 2\Xi_\rho(X(\theta\wedge\tau))+\rho\} : X(\theta\wedge\tau)\in S_i\right]\\
&\quad\le \mathscr{T}^{C^\tau,\pi^\tau}u(x) + E[e^{-(\alpha+1/\varepsilon)(\theta\wedge\tau)}\{2\Xi_\rho(X(\theta\wedge\tau))+\rho\}]\\
&\quad\le u(x)+2\Xi_\rho(x)+\rho,
\end{aligned}$$

which follows from the same calculation as in (2.9). Letting $\rho\to 0$, we deduce $\tilde{u}(x)\le u(x)$, which gives (2.16). □

We present the comparison principle, which together with Theorem 2.3 implies that the solution of (2.8) is the unique viscosity solution of (2.6).

Theorem 2.4. *Assume that* (2.1), (2.2), (2.4), (2.5) *hold. If* f_1 *and* $f_2 \in \mathscr{B}$ *are, respectively a viscosity sub- and supersolution of* (2.6) *such that* $f_1(0) \leq f_2(0)$, *then* $f_1 \leq f_2$ *in* $[0, \infty)$.

Proof. We first claim that there exists $v_0 \in (\gamma, 1)$ such that

$$\mathscr{H}_\alpha(x, \xi_v(x), \xi_v'(x), \xi_v''(x)) \leq 0 \qquad \text{for } x \in (0, \infty) \text{ and } v \in (\gamma, v_0], \tag{2.19}$$

where $\xi_v(x) = x^v + 1$. Indeed, otherwise, we find a sequence $\{x_k\}$ such that

$$(\alpha - \alpha_0)\xi_v(x_k) < \mathscr{H}_{\alpha_0}(x_k, \xi_{\gamma+1/k}(x_k), \xi_{\gamma+1/k}'(x_k), \xi_{\gamma+1/k}''(x_k)) \qquad \text{for large } k \geq 1. \tag{2.20}$$

We observe that

$$b(k) := -\alpha_0 + r\left(\gamma + \frac{1}{k}\right) - \frac{(b-r)^2(\gamma + k^{-1})}{2\sigma^2(\gamma + k^{-1} - 1)} < \frac{\alpha - \alpha_0}{2} \qquad \text{for large } k \geq 1$$

because $\lim_{k\to\infty} b(k) \leq 0$ by (2.3). Thus, we have

$$\begin{aligned}
&(\alpha - \alpha_0)\xi_{\gamma+1/k}(x_k) \\
&\quad < -\alpha_0 + b(k)x_k^{\gamma+1/k} + \tilde{U}_1(\xi_{\gamma+1/k}'(x_k)) \\
&\quad < \frac{\alpha - \alpha_0}{2} x_k^{\gamma+1/k} + \sup_{c\geq 0}\{\|U_1\|_{\mathscr{B}}(1 + c^\gamma) - c\xi_{\gamma+1/k}'(x_k)\} \\
&\quad \leq \frac{\alpha - \alpha_0}{2} x_k^{\gamma+1/k} + \|U_1\|_{\mathscr{B}} + \frac{1-\gamma}{\gamma}(\gamma\|U_1\|_{\mathscr{B}})^{1/(1-\gamma)}\gamma^{1/(1-\gamma)}(1-\gamma)\gamma^{-1}x_k^{\gamma-\gamma/\{k(1-\gamma)\}}.
\end{aligned}$$

Hence, $\{x_k\}$ is bounded, and we find a subsequence of x_k which converges to $x_0 \geq 0$.

Suppose that $x_0 = 0$. Note that $\mathscr{M}(\xi_{\gamma+1/k}'(x_k), \xi_{\gamma+1/k}''(x_k)) = -(\gamma + 1/k) \cdot (b-r)^2 x_k^{\gamma+1/k}\{2\sigma^2(\gamma + 1/k - 1)\}$ and $\tilde{U}_1(\xi_{\gamma+1/k}'(x_k))$ converge to 0 as $k \to \infty$. Thus, sending $k \to \infty$ in (2.20), we have $\alpha - \alpha_0 \leq 0$, which is a contradiction to (2.2). Thus, we may suppose that $x_0 > 0$.

However, letting $k \to \infty$ in (2.20), we get

$$\mathscr{H}_{\alpha_0}(x_0, \xi_\gamma(x_0), \xi_\gamma'(x_0), \xi_\gamma''(x_0)) > 0,$$

which contradicts (2.2).

Now, suppose that $\sup_{x\geq 0}\{f_1(x) - f_2(x)\} > 0$, and then will get a contradiction. For fixed $v \in (\gamma, v_0]$, we can find a constant $\delta > 0$ such that

$$\sup_{x\geq 0}\{f_1(x) - f_2(x) - 2\delta\xi_v(x)\} > 0.$$

Set

$$\Phi(x, y) = f_1(x) - f_2(y) - \frac{k}{2}|x - y|^2 - \delta(\xi_\nu(x) + \xi_\nu(y)).$$

Since

$$f_1(x) - f_2(x) - 2\delta\xi_\nu(x) \le (\|f_1\|_{\mathscr{B}} + \|f_2\|_{\mathscr{B}})(1 + x^\gamma) - 2\delta\xi_\nu(x) \to -\infty \tag{2.21}$$

as $x \to \infty$, there is an $\bar{x} \ge 0$ such that $\sup_{x \ge 0} \Phi(x, x) = \Phi(\bar{x}, \bar{x}) > 0$. Moreover,

$$\Phi(x, y) \le (\|f_1\|_{\mathscr{B}} + \|f_2\|_{\mathscr{B}})(1 + (x + y)^\gamma) - \delta(2 + (x + y)^\nu) \to -\infty$$
$$\text{as } x + y \to \infty.$$

Thus, we find $(x_k, y_k) \in [0, \infty) \times [0, \infty)$ such that

$$\Phi(x_k, y_k) = \sup_{x, y \ge 0} \Phi(x, y) \ge f_1(\bar{x}) - f_2(\bar{x}) - 2\delta\xi_\nu(\bar{x}) > 0. \tag{2.22}$$

Hence, we have

$$\begin{aligned}\frac{k}{2}|x_k - y_k|^2 &< f_1(x_k) - f_2(y_k) - \delta(\xi_\nu(x_k) + \xi_\nu(y_k)) \\ &\le \|f_1\|_{\mathscr{B}}(1 + x_k^\gamma) - \delta\xi_\nu(x_k) + \|f_2\|_{\mathscr{B}}(1 + y_k^\gamma) - \delta\xi_\nu(y_k),\end{aligned}$$

which is bounded from above since the right hand side of the above goes to $-\infty$ as $x_k + y_k \to \infty$. Thus, we deduce that the sequences $\{k|x_k - y_k|^2\}$, $\{x_k\}$ and $\{y_k\}$ are bounded by some constant $K > 0$. Since we may suppose that $\lim_{k\to\infty}(x_k, y_k) = (\tilde{x}, \tilde{y}) \in [0, K]^2$, noting $\Phi(x_k, y_k) \ge \Phi(x_k, x_k)$ by (2.22), we have

$$\begin{aligned}\frac{k}{2}|x_k - y_k|^2 &\le f_2(x_k) - f_2(y_k) + \delta(\xi_\nu(x_k) - \xi_\nu(y_k)) \\ &\le \sup\{|f_2(x) - f_2(y)| \mid k|x - y|^2 \le K, 0 \le x, y \le K\} + \delta\left(\frac{K}{k}\right)^{\nu/2}.\end{aligned}$$

Letting $k \to \infty$, we have

$$\lim_{k\to\infty} k|x_k - y_k|^2 = 0, \tag{2.23}$$

which yields $\tilde{x} = \tilde{y}$. Sending $k \to \infty$ in (2.22), we have

$$f_1(\tilde{x}) - f_2(\tilde{x}) - 2\delta\xi_\nu(\tilde{x}) > 0. \tag{2.24}$$

Since $f_1(0) \le f_2(0)$, we conclude $\tilde{x} > 0$.

Now, applying Ishii's lemma [2] to Φ, we obtain $X_1, X_2 \in \boldsymbol{R}$ such that

$$(p_1, \overline{X}_1) := (k(x_k - y_k) + \delta\xi_\nu'(x_k), X_1 + \delta\xi_\nu''(x_k)) \in \bar{J}^{2,+} f_1(x_k),$$
$$(p_2, \overline{X}_2) := (k(x_k - y_k) - \delta\xi_\nu'(y_k), X_2 - \delta\xi_\nu''(y_k)) \in \bar{J}^{2,-} f_2(y_k),$$

and

$$-3k\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \le \begin{pmatrix} X_1 & 0 \\ 0 & -X_2 \end{pmatrix} \le 3k\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{2.25}$$

By the equivalent definition in Remarks (1), we have

$$\begin{aligned} \alpha(f_1(x_k) - f_2(y_k)) \le r(x_k p_1 - y_k p_2) &+ \frac{\mu^2}{2}(x_k^2 \overline{X}_1 - y_k^2 \overline{X}_2) \\ &+ \mathscr{M}(p_1 - p_2, \overline{X}_1 - \overline{X}_2) + \sup_{c \ge 0}\{-c(p_1 - p_2)\} \\ &+ \hat{\beta}_\varepsilon(x_k, f_1(x_k)) - \hat{\beta}_\varepsilon(y_k, f_2(y_k)). \end{aligned}$$

Taking into account $\xi_\nu' > 0$, and letting $k \to \infty$ together with (2.23) and (2.25), by (2.19), we have

$$\begin{aligned} \alpha(f_1(\tilde{x}) - f_2(\tilde{x})) &\le 2\delta\{r\tilde{x}\xi_\nu'(\tilde{x}) + \mathscr{M}(\xi_\nu'(\tilde{x}), \xi_\nu''(\tilde{x}))\} \\ &\le 2\delta\{\mathscr{H}_\alpha(\tilde{x}, \xi_\nu(\tilde{x}), \xi_\nu'(\tilde{x}), \xi_\nu''(\tilde{x})) + \alpha\xi_\nu(\tilde{x})\} \\ &\le 2\delta\alpha\xi_\nu(\tilde{x}), \end{aligned}$$

which contradicts (2.24). □

3. Estimates on solutions

In this section, we show the $W_{loc}^{2,\infty}$-regularity of viscosity solutions of (1.2). For this purpose, it is enough to obtain the $W_{loc}^{2,\infty}$-estimates on viscosity solutions $u_{\varepsilon,\mu,L,R} \in \mathscr{B}_\zeta$ of (2.6).

We will first show the strict (local) concavity of $u_{\varepsilon,\mu,L,R}$ in Lemma 3.1. Next, we obtain the local bounds of the first derivative in Lemma 3.2. After having observed that the first derivative is positive in Lemma 3.3, we finally derive the estimate on the second derivative in Lemma 3.4.

Since we only need the supersolution property of $u_{\varepsilon,\mu,L,R}$ to show its strict concavity, we consider the following

$$-\alpha u + rxu' + \frac{\mu^2 x^2}{2}u'' + \mathscr{M}^L(u', u'') + \tilde{U}_1^R(u') = 0 \qquad \text{in } (0, \infty). \tag{3.1}$$

Lemma 3.1. *For $T > 0$, there is $C_1(T) > 0$ satisfying the following property: For $T > 0$ and $R \ge C_1(T)$, there is $C_2 = C_2(T, R) > 0$ such that if $w \in \mathscr{B}_\zeta \cap C^2(0, \infty)$ be a supersolution of* (3.1) *for $\mu \in (0, 1]$, $R \ge C_1$ and $L \ge C_2$, then*

$$w''(x) < 0 \qquad \textit{for } x \in (0, T).$$

Proof. Fix $x \in (0, T)$.

Taking $\pi = L \operatorname{sgn} w'(x)$ in $\mathscr{M}^L(w', w'')$ and $c = R$ in $\tilde{U}_1^R(w')$, we have

$$-\alpha w(x) + \{(b - r)L - rT - R\}|w'(x)| + \frac{1}{2}(\mu^2 x^2 + L^2\sigma^2)w''(x) + U_1(R) \le 0.$$

Choosing $C_1 > 0$ so that $\alpha\zeta(T) - U_1(R) < 0$ for $R \ge C_1$, we have

$$\{(b - r)L - rT - R\}|w'(x)| + \frac{1}{2}(\mu^2 x^2 + L^2\sigma^2)w''(x) < 0$$

for $R \ge C_1$. Hence, for $L \ge C_2 := (b - r)^{-1}(rT + R)$, we have

$$w''(x) < 0 \qquad \text{in } (0, T). \quad \square$$

We next show the local estimate of first derivatives:

Lemma 3.2. *Fix $0 < S < T < \infty$. Let $w \in \mathscr{B}_\zeta \cap C^1(0, T + 1)$ satisfy that*

$$w''(x) < 0 \qquad \textit{in } (S/2, T + 1).$$

Then, we have

$$\max_{x \in [S, T]} |w'(x)| \le \frac{2\zeta(T)}{S}.$$

Proof. In view of the concavity, we see that

$$w(x + h) - w(x) \le w'(x)h \qquad \text{for } x \in (S, T) \text{ and } |h| \le \frac{S}{2}.$$

Fix $x \in (S, T)$. Since we may suppose $w'(x) \ne 0$, in view of $w \in \mathscr{B}_\zeta$, taking $h = -2^{-1}S \operatorname{sgn} w'(x)$, we have

$$\frac{S}{2}|w'(x)| \le w(x) \le \zeta(T). \quad \square$$

We shall show the first derivative is indeed positive.

Lemma 3.3. *Fix $0 < S < 1 < T < \infty$. Let $w \in \mathscr{B}_\zeta \cap C^2(0, \infty)$ be a supersolution of* (3.1) *such that*

$$w'' < 0 \quad \textit{in } (0, T + 1), \qquad \max_{[S, T+1]} |w'| \le C(S, T)$$

for some $C(S, T) > 0$. Then, there are $C_3 = C_3(S, T) > 0$ and $\mu_0 = \mu_0(S, T) > 0$ satisfying the following property: For $R \ge C_3$, there is $\tau_0 = \tau_0(S, T, R) > 0$ such that if $\mu \in (0, \mu_0]$, $L > 0$ and $R \ge C_3$, then

$$\min_{x \in [S, T]} w' \ge \tau_0.$$

Proof. Since $w'' < 0$ in $(0, T+1)$, it is sufficient to find $\tau_0 > 0$ such that $w'(T) \geq \tau_0$.

Since $\mathscr{M}^L \geq 0$, we have

$$-\alpha\zeta(T+1) - r(T+1)C(S,T) + \frac{\mu^2(T+1)^2}{2} w''(x) + \tilde{U}_1^R(w'(x)) \leq 0$$

for $x \in (S, T+1)$.

Hence, we have

$$w''(x) \leq \frac{2}{\mu^2(T+1)^2}\{\alpha\zeta(T+1) + r(T+1)C(S,T) - \tilde{U}_1^R(w'(x))\}$$

for $x \in (S, T+1)$.

Choose $C_3 > 0$ so that

$$U_1(R) \geq \alpha\zeta(T+1) + r(T+1)C(S,T) + 1 \qquad \text{for } R \geq C_3.$$

Noting $\tilde{U}_1^R(w'(x)) \geq U_1(C_3) - C_3 w'(T)$ for $R \geq C_3$ for $x \in [T, T+1)$, we thus have

$$w''(x) \leq \frac{2}{\mu^2(T+1)^2}\{C_3 w'(T) - 1\} \qquad \text{for } x \in (T, T+1).$$

Integrating the above over $[T, x]$ for $x \in (T, T+1]$, we have

$$w'(x) - w'(T) \leq \frac{2}{\mu^2(T+1)^2}\{C_3 w'(T) - 1\} \qquad \text{for } x \in (T, T+1).$$

Again, integrating the above over $[T, T+1]$, we have

$$0 \leq w(T+1) \leq \zeta(T) + w'(T) + \frac{2}{\mu^2(T+1)^2}\{C_3 w'(T) - 1\}.$$

Thus, we have

$$2 \leq \{\mu^2(T+1)^2 + 2C_3\} w'(T) + \mu^2(T+1)^2 \zeta(T).$$

Setting $\mu_0^2 = 1/\{\zeta(T)(T+1)^2\}$, for $\mu \in (0, \mu_0]$, we have

$$\tau_0 := \frac{1}{\mu_0^2(T+1)^2 + 2C_3} \leq w'(T). \quad \square$$

Finally, we shall derive the local estimate of the second derivative:

Lemma 3.4. *Fix* $0 < S < T < \infty$. *Let* $u_{\varepsilon,\mu,L,R} \in \mathscr{B}_\zeta \cap C^4(0,\infty)$ *be a solution of* (2.6) *such that* $u''_{\varepsilon,\mu,L,R} < 0$ *in* $(0, T+1)$ *and* $u'_{\varepsilon,\mu,L,R} > 0$ *in* $[S/2, T+1]$. *Then, there is* $C_4 = C_4(S,T) > 0$ *such that*

$$\max_{x \in [S,T]} |u''_{\varepsilon,\mu,L,R}(x)| \leq C_4 \qquad \textit{for large } R, L > 0 \textit{ and small } \varepsilon, \mu > 0.$$

Proof. In view of Lemmas 3.2 and 3.3, we may suppose that $\tilde{U}_1^R = \tilde{U}_1$ for large $R > 0$. We shall write u for $u_{\varepsilon,\mu,L,R}$ for simplicity.

Following [15], we introduce the mapping $I : (0,\infty) \to (0,\infty)$ defined by

$$I(p) := (U_1')^{-1} \quad \text{for } p > 0; \qquad \tilde{U}_1(p) = U_1(I(p)) - pI(p).$$

We shall use the cut off function $\eta \in C(\boldsymbol{R})$ (as in [15]) such that

$$\begin{cases} (1) & 0 \le \eta \le \hat{C} \quad \text{in } \boldsymbol{R}, \\ (2) & \eta(x) = 0 \quad \text{for } x \notin (S/2, T+1), \\ (3) & \eta(x) \ge 1 \quad \text{for } x \in [S,T], \\ (4) & |\eta'(x)| \le \hat{C}\eta^{1/3}(x) \quad \text{for } x \in (S/2, T+1), \\ (5) & \eta|_{(S/2,T+1)} \in C^2(S/2, T+1). \end{cases}$$

In fact, setting $\eta_0(x) = -\Lambda\{(x - S/2)(x - T - 1)\}^{3/2}$ for $x \in (S/2, T+1)$ with large $\Lambda > 0$, we take η by the extension of η_0 by 0 outside of $[S/2, T+1]$.

Setting $W(x) = \eta^2|u''|^2 + \lambda|u'|^2$, we choose $\hat{x} \in (S/2, T+1)$ such that $\max_{x\in[0,T+2]} W(x) = W(\hat{x}) > 0$.

We may suppose that $(b - r)u'(\hat{x}) < -\sigma^2 Lu''(\hat{x})$ because otherwise, we immediately obtain the desired estimate by Lemma 3.2.

Now, we differentiate (2.6) to get

$$(r - \alpha - 2\gamma_0)u' + (\mu^2 + r)xu'' + \frac{\mu^2x^2}{2}u''' + \gamma_0\left|\frac{u'}{u''}\right|u''' - u''I + (u' - U_2')\beta_\varepsilon' = 0$$

near $\hat{x}$. Hence, setting $\rho = 2^{-1}\mu^2\hat{x}^2 + \gamma_0|u'(\hat{x})/u''(\hat{x})|^2$, we have

$$\rho u''' \ge -(\mu^2 + r)\hat{x}u'' + u''I - (u' - U_2')\beta_\varepsilon' \qquad \text{at } \hat{x}. \tag{3.2}$$

Furthermore, we differentiate (2.6) twice to get

$$\begin{aligned} 0 = (2r + \mu^2 - \alpha - 2\gamma_0)u'' + (\mu^2 + r)xu''' + \frac{\mu^2x^2}{2}u'''' + 2\gamma_0\frac{u'u'''}{u''} + \gamma_0\frac{|u'|^2u''''}{|u''|^2} \\ - 2\gamma_0\frac{|u'|^2|u'''|^2}{(u'')^3} - u'''I - |u''|^2I' + |u' - U_2'|^2\beta_\varepsilon'' + (u'' - U_2'')\beta_\varepsilon'. \end{aligned}$$

Thus, we verify that

$$\begin{aligned} \rho u'''' \le C|u''| - (\mu^2 + r)\hat{x}u''' - 2\gamma_0\frac{u'u'''}{u''} + 2\gamma_0\frac{|u'|^2|u'''|^2}{(u'')^3} \\ + u'''I + |u''|^2I' - (u'' - U_2'')\beta_\varepsilon' \qquad \text{at } \hat{x}. \end{aligned} \tag{3.3}$$

We calculate the first and second derivatives of W:

$$W' = 2\eta\eta'|u''|^2 + 2\eta^2 u''u''' + 2\lambda u'u'',$$

$$\begin{aligned} W'' &= 2(\eta\eta'' + |\eta'|^2)|u''|^2 + 8\eta\eta' u''u''' + 2\eta^2(|u'''|^2 + u''u'''') + 2\lambda(u'u''' + |u''|^2) \\ &\geq \eta^2|u'''|^2 + \lambda|u''|^2 + 2\eta^2 u''u'''' + 2\lambda u'u''', \end{aligned}$$

where $\lambda > 1$ is a fixed number.

Since $0 \geq \rho W''(\hat{x})$, by (3.2) and (3.3), we have

$$\begin{aligned} (3.4) \quad & \rho(\eta^2|u'''|^2 + \lambda|u''|^2) \\ &\leq -2\eta^2 u''\rho u'''' - 2\lambda u'\rho u''' \\ &\leq C\eta^2(|u''|^2 + |u''u'''| + |u'''| - u''u'''I - (u'')^3 I') \\ &\quad - 2\lambda u'u''I + 2\{\eta^2(|u''|^2 - u''U_2'') + \lambda(|u'|^2 - u'U_2')\}\beta_\varepsilon' \qquad \text{at } \hat{x}. \end{aligned}$$

Since we may suppose that $W \geq \eta^2|U_2''|^2 + \lambda|U_2'|^2$, we can neglect the last term of (3.14). In view of Lemma 3.2, we find $\bar{C} > 0$ such that

$$\sup_{x \in (S/2, T+1)} |u'(x)| \leq \bar{C}.$$

Noting $W'(\hat{x}) = 0$, by (3.4), we have

$$\gamma_0 \frac{|u'u'''|^2}{|u''|^2} + \eta^2(u'')I' \leq C\eta^2|u''|^2 + C\eta^2|u'u'''| + C\eta|\eta'|\,|u''|^2 + 2\eta\eta'|u''|^2 I.$$

Hence, we have

$$\frac{|u'u'''|^2}{|u''|^2} + \eta^2(u'')^3 I' \leq C\eta^{4/3}|u''|^2 + C\eta^{4/3} I.$$

In view of Lemma 3.2, we can find $\delta > 0$ such that

$$I'(u') \leq -\delta \qquad \text{and} \qquad I(u') \leq \frac{1}{\delta}.$$

Furthermore, recalling the property (4) of η, we have

$$\delta\eta^2|u''|^3 \leq C(1 + \eta^{4/3}|u''|^2) \qquad \text{at } \hat{x},$$

which implies $\eta|u''|(\hat{x}) \leq C$. □

Since, for fixed $0 < S < T < \infty$, $u_{\varepsilon,\mu,L,R}$ coincide with the unique C^2 solution $u \in C^2(S,T) \cap C[S,T]$ of (2.6) in (S,T) under $u(S) = u_{\varepsilon,\mu,L,R}(S)$ and $u(T) = u_{\varepsilon,\mu,L,R}(T)$, we have the following

Theorem 3.5. *Assume that* (2.1), (2.2), (2.4), (2.5) *hold. Then, there is a unique viscosity solution* $v \in \mathscr{B}_\zeta$ *of* (1.2).

Moreover, we have

$$v \in \bigcap_{0<S<T<\infty} W^{2,\infty}(S,T). \tag{3.5}$$

Proof. In view of Lemmas 3.1, 3.2, 3.3 and 3.4, it is easy to find a function $v \in C[0,\infty)$ as a local uniform limit of $u_{\varepsilon,\mu,L,R} \in \mathscr{B}_\zeta$, as $\varepsilon, \mu \to 0$ and $L, R \to \infty$, (by taking a subsequence if necessary) satisfying (3.5). In order to verify that v is a viscosity solution of (1.2), we refer to [15] since it is rather standard.

The uniqueness follows from the same argument as in the proof of Theorem 2.4. □

4. Optimal policies

In this section we present a synthesis of optimal policies. Let us consider the stochastic differential equation

$$dX^*(t) = 1_{\{t \le \tau^*\}}[\{rX^*(t) - C^*(t)\}dt + \pi^*(t)\{(b-r)dt + \sigma\, dW(t)\}] \tag{4.1}$$

under $X^*(0) = x$, where

$$C^*(t) = (U_1')^{-1}(v'(X^*(t))), \tag{4.2}$$

$$\pi^*(t) = -\frac{(b-r)}{\sigma^2}\frac{v'(X^*(t))}{\sigma^2 v''(X^*(t))}, \tag{4.3}$$

$$\tau^* = \inf\{t : v(X^*(t)) = U_2(X^*(t))\}. \tag{4.4}$$

Here, v denotes the function in Theorem 3.5.

Theorem 4.1. *Assume that* (2.1), (2.2), (2.4), (2.5) *hold. Assume also that* $\lim_{x\to\infty} U_2(x) = \infty$. *Then, there exists a unique solution* $X^*(t)$ *of* (4.1), *and the optimal policy in* $\mathscr{A}^0 \times \mathscr{S}$ *is given by* (4.2), (4.3) *and* (4.4).

Proof. Fix any $(C,\pi,\tau) \in \mathscr{A}^0 \times \mathscr{S}$ and $L, R \ge 1$. We let $X_{C,\pi,\tau,L,R,x}(\cdot)$ be the solution of (2.7) for $(C_R, \pi_L) \in \mathscr{A}^{L,R}$ defined by $C_R(t) = C(t)1_{\{C(t)\le R\}}$ and $\pi_L(t) = \pi(t)1_{\{|\pi(t)|\le L\}}$. In view of (2.9) with $\alpha' = \alpha$, we have

$$E\left[e^{-\alpha(\theta^0\wedge\tau)}u_{\varepsilon,\mu,L,R}(X_{C,\pi,L,R,x}(\theta^0\wedge\tau)) + \int_0^{\theta^0\wedge\tau} e^{-\alpha s}U_1(C_R(s))ds\right] \le u_{\varepsilon,\mu,L,R}(x),$$

where $u_{\varepsilon,\mu,L,R}$ is the solution of (2.6). Since

$$u_{\varepsilon,\mu,L,R}(X_{C,\pi,L,R,x}(t\wedge\tau)) \to v(X_{C,\pi,x}(t\wedge\tau)) \qquad \text{in probability}$$

as $(\varepsilon, \mu, L, R) \to (0, 0, \infty, \infty)$, using Fatou's lemma, we have

$$E\left[e^{-\alpha(\theta^0 \wedge \tau)} v(X_{C,\pi,x}(\theta^0 \wedge \tau)) + \int_0^{\theta^0 \wedge \tau} e^{-\alpha s} U_1(C_R(s)) ds\right] \le v(x), \tag{4.5}$$

from which it turns out that

$$J(C, \pi, \tau) \le v(x).$$

Furthermore, denoting $\theta^* = \theta^0_{C^*,\pi^*,x}$, by (4.4), we have

$$e^{-\alpha(\theta^* \wedge \tau^*)} v(X^*(\theta \wedge \tau^*)) = e^{-\alpha(\theta^* \wedge \tau^*)} U_2(X^*(\theta^* \wedge \tau^*)).$$

By Theorem 3.5 and (1.2), we note that $v \in C^2(O)$, where $O = \{x > 0 \mid v(x) > U_2(x)\}$. Therefore, applying Ito's formula to (4.1), we obtain

$$J(C^*, \pi^*, \tau^*) = v(x).$$

In order to complete the proof, we shall show the existence of a unique solution $X^*(\cdot)$ of (4.1) with $X^*(0) = x$.

By (1.2), we have

$$-\alpha v + rxv' - \frac{(b-r)^2 |v'|^2}{2\sigma^2 v''} + \tilde{U}_1(v') = 0 \qquad \text{in } O.$$

Hence, we observe that

$$v'' = \frac{(b-r)^2 |v'|^2}{2\sigma^2\{-\alpha v + rxv' + \tilde{U}_1(v')\}}. \tag{4.6}$$

Thus, in view of Lemma 3.4, for $x \in \partial O \backslash \{\infty\}$, there exists $\lim_{y \in O \to x} v''(y) < 0$. We shall denote by $\bar{v}''$ the continuous extension of v'' on $\bar{O} \cap (0, \infty)$.

Consider the equation

$$dz(t) = \left[r - e^{-z(t)} g_1(e^{z(t)}) + e^{-z(t)}(b-r) g_2(e^{z(t)}) - \frac{e^{-2z(t)}\sigma^2}{2} g_2(e^{z(t)})^2\right] dt \tag{4.7}$$
$$+ e^{-z(t)} g_2(e^{z(t)}) dW(t),$$

under $z(0) = \log x$, where

$$g_1(x) = -(U_1')^{-1}(v'(x)),$$

$$g_2(x) = -\frac{(b-r)v'(x)}{\sigma^2 \bar{v}''(x)}.$$

Since $g_i \in C(0, \infty)$ for $i = 1, 2$, we know that for Theorem 2.3 in Chapter IV of [11], (4.7) has a weak solution up to an explosion time $\chi \in \mathscr{S}$.

Setting $Z(t) = e^{z(t)}$, we easily verify that

$$dZ(t) = \{rZ(t) - g_1(Z(t))\}dt + g_2(Z(t))\{(b - r)dt + \sigma \, dW(t)\} \tag{4.8}$$

with $Z(0) = x$.

By the same calculation as for (4.5), we have

$$E[e^{-\alpha\chi} U_2(Z(\chi)) \,|\, \chi < \infty] \le E[e^{-\alpha\chi} v(Z(\chi))] \le v(x).$$

Thus, we have $P(\chi < \infty) = 0$ since $\lim_{x\to\infty} v(x) \ge \lim_{x\to\infty} U_2(x) = \infty$.

To prove the exsitence of a unique strong solution of (4.8), it is sufficient to show that the pathwise uniqueness holds.

Let $Z_i(t)$ $(i = 1, 2)$ be two solutions of (4.8) satisfying $Z_1(0) = Z_2(0) = x$, and let $\tau_n^i = \inf\{t > 0 \,|\, Z_i(t) \le 1/n$ or $Z_i(t) \ge n\}$ for $n = 1, 2, \ldots$ It is clear that $\lim_{n\to\infty} \tau_n^i = \infty$.

By (4.6), we note that v'/v'' is continuously differentiable in O. This yields the local Lipschitz continuity of g_2. Hence, Ito's formula implies that

$$E[|Z_1(t_n) - Z_2(t_n)|^2] \le C_n \int_0^{t_n} E[|Z_1(s) - Z_2(s)|^2] ds, \tag{4.9}$$

for some constant C_n depending on n, where $t_n = t \wedge \tau_n^1 \wedge \tau_n^2$. Therefore, we conclude that $Z_1(t) = Z_2(t)$ for all $t > 0$.

Finally, we define $X^*(t) = Z(t \wedge \tilde{\tau})$, where $\tilde{\tau} = \inf\{t > 0 \,|\, v(Z(t)) = U_2(Z(t))\}$. Then, it is easy to see that $\tilde{\tau} = \tau^*$ and X^* satisfies (4.1).

To prove the uniqueness, assuming that there are two solutions $X_i^*(\cdot)$ $(i = 1, 2)$ of (4.1), as for (4.9), we can derive that $X_1^*(t \wedge \tau_1^* \wedge \tau_2^*) = X_2^*(t \wedge \tau_1^* \wedge \tau_2^*)$, where τ_i^* denote τ^* of (4.4) for X_i. Thus, we obtain that $\tau_1^* = \tau_2^*$ and $X_1^* = X_2^*$. □

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