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Large Time Behavior of Small Solutions to Dirichlet Problem for Landau-Ginzburg Type Equations

By

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Abstract. We study the Dirichlet problem for nonlinear dissipative equations with two power (critical and sub-critical) nonlinearities of parabolic type on half lines. Taking the zero boundary conditions into consideration, we present a sufficient condition which gives sharp time asymptotics of small solutions.

Key Words and Phrases. Dissipative nonlinear evolution equation, Large time asymptotics, Landau-Ginzburg equation, Dirichlet problem.

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1. Introduction

We consider the initial-boundary value problem for the nonlinear Landau-Ginzburg type equations with Dirichlet boundary conditions

$$\begin{cases} \mathscr{L}u + \beta|u|^{\sigma}u + \gamma|u|^{\kappa}u = 0, \quad x \in \boldsymbol{R}_+^n,\ t \in \boldsymbol{R}_+, \\ u(0,x) = u_0(x), \quad x \in \boldsymbol{R}_+^n, \\ u(t,x) = 0, \quad x \in \partial\boldsymbol{R}_+^n,\ t \in \boldsymbol{R}_+, \end{cases} \tag{1}$$

where $\mathscr{L} = \partial_t - \alpha\Delta$, $\alpha, \beta, \gamma \in \boldsymbol{C}$, $\operatorname{Re}\alpha > 0$, $0 < \sigma \le \kappa$ and

$$\boldsymbol{R}_+^n = \{x = (x_1, \dots, x_n); x_j > 0, j = 1, \dots, n\}.$$

Equation (1) with $\sigma = 2$, $\gamma = 0$ is known as the complex Landau-Ginzburg equation. The Cauchy problem for the complex Landau-Ginzburg equation

$$\begin{cases} \mathscr{L}u + \beta|u|^{\sigma}u + \gamma|u|^{\kappa}u = 0, \quad x \in \boldsymbol{R}^n,\ t \in \boldsymbol{R}_+, \\ u(0,x) = u_0(x), \quad x \in \boldsymbol{R}^n, \end{cases} \tag{2}$$

was studied extensively by many authors, (see, e.g. [5], [6], [9], [10], [21] and references cited therein). Blow-up in finite time of positive solutions to the Cauchy problem for the heat equation $u_t - \Delta u = u^{1+\sigma}$ was proved in [3], [8], [15], [22]. Large time behavior of positive solutions for nonlinear heat equation (which is a particular case of (2)) $u_t - \Delta u + u^{1+\sigma} = 0$ was studied for any $\sigma > 0$ (see [13] for the super critical case $\sigma > 2/n$, [4], [7], [14] for the critical case

$\sigma = 2/n$ and [1], [2] for the sub critical case $\sigma \in (0, 2/n)$). Global existence in time of small solutions to (2) in the super critical case $\sigma > 2/n$ was also shown in [3]. Large time asymptotic behavior of small solutions to the Cauchy problem (2) was investigated in [10], [11], [12] under the condition

$$\operatorname{Re} \beta((2+\sigma)|\alpha|^2 + \sigma\alpha^2)^{-n/2} > 0, \qquad \gamma = 0.$$

The critical case $\sigma = 2/n$ was studied in paper [11] if the initial data are small in $\boldsymbol{L}^\infty \cap \boldsymbol{L}^{1,a}$, $a \in (0,1)$. In paper [12] it was considered the sub critical case $1 < \sigma < 2/n$, when $2/n - \sigma$ is small.

In this paper we are interested in the global existence and large time behavior of solutions to the initial-boundary value problem (1) in the critical $\sigma = 1/n$ and sub critical $0 < \sigma < 1/n$ cases. Our result below shows that solutions of (1) decay faster than those of the Cauchy problem (2) due to the zero Dirichlet boundary conditions and the critical power $\sigma = 1/n$ for (1) is different from that $\sigma = 2/n$ for the case of the Cauchy problem (2). In what follows we use the Lebesgue space $\boldsymbol{L}^p$ on $\boldsymbol{R}_+^n$ with the norm $\|\phi\|_{\boldsymbol{L}^p} = (\int_{\boldsymbol{R}_+^n} |\phi(x)|^p dx)^{1/p}$, and the weighted Lebesgue space $\boldsymbol{L}^{1,a}$, $a \geq 0$, with norm

$$\|\phi\|_{\boldsymbol{L}^{1,a}} = \|\langle \cdot \rangle^a \phi\|_{\boldsymbol{L}^1}, \qquad \langle x \rangle = \sqrt{1+|x|^2}.$$

By $\boldsymbol{C}\ (\boldsymbol{I}; \boldsymbol{B})$ we denote the space of continuous functions from a time interval $\boldsymbol{I}$ to the Banach space $\boldsymbol{B}$. Different positive constants might be denoted by the same letter C.

We assume that the initial data $u_0 \in \boldsymbol{L}^\infty \cap \boldsymbol{L}^{1,n+1}$ and satisfy conditions

$$\|u_0\|_{\boldsymbol{L}^\infty} + \|u_0\|_{\boldsymbol{L}^{1,n+1}} = \varepsilon, \qquad C\varepsilon \leq |\theta| \leq \varepsilon, \tag{3}$$

where

$$\theta = \int_{\boldsymbol{R}_+^n} \tilde{x} u_0(x) dx, \qquad \tilde{x} = \Pi_{j=1}^n x_j,$$

the angular conditions for $(\arg \alpha, \arg \beta)$

$$-\frac{\pi}{2} < \arg \beta + \frac{\sigma n}{2} \arg \alpha - \frac{\sigma n + 3n}{2} \arctan \frac{\sigma \sin(2 \arg \alpha)}{\sigma + 2 + \sigma \cos(2 \arg \alpha)} < \frac{\pi}{2}, \tag{4}$$

$\arg \alpha \in (-\pi/2, \pi/2)$ and the condition on the order σ

$$0 \leq \frac{1}{n} - \sigma \leq \varepsilon, \tag{5}$$

where $\varepsilon > 0$ is sufficiently small. We denote

$$\eta = \operatorname{Re} \beta\delta(\alpha, \sigma) > 0,$$

where

$$\delta(\alpha,\sigma) = t^{\sigma n}\int_{\boldsymbol{R}_+^n} \tilde{x}|F(t,x)|^{\sigma}F(t,x)dx,$$

$$F(t,x) = (4\pi\alpha^3)^{-n/2}t^{-(3/2)n}\tilde{x}e^{-|x|^2/4\alpha t}.$$

Denote $\tilde{\eta} = -\mathrm{Im}\,\beta t^{\sigma n}\int_{\boldsymbol{R}_+^n}\tilde{x}|F(t,x)|^{\sigma}F(t,x)dx,$

$$\chi_\sigma(t) = g(t) \quad \text{if } \sigma = \frac{1}{n} \qquad \text{and} \qquad \chi_\sigma(t) = 1 + \frac{\sigma|\theta|^\sigma\eta}{1-\sigma n}t^{1-\sigma n} \quad \text{if } \sigma \in \left(0,\frac{1}{n}\right),$$

$g(t) = 1 + |\theta|^\sigma\sigma\eta\log(1+t).$

Our purpose in this paper is to prove

Theorem 1.1. *We assume that the conditions* (3)–(5) *are valid. Then there exists a unique solution*

$$u(t,x) \in \boldsymbol{C}([0,\infty);\boldsymbol{L}^\infty \cap \boldsymbol{L}^{1,n+1})$$

of the initial-boundary value problem (1) *satisfying the following large time decay estimate*

$$\|u(t) - |\theta|F(t)\chi_\sigma^{-1/\sigma}(t)e^{i\psi(t)}\|_{\boldsymbol{L}^\infty}$$
$$\leq \begin{cases} C\varepsilon^{1+\sigma}(1+t)^{-n}\chi_\sigma^{-1/\sigma}(t)g^{-1}(t)\log(1+g(t)) & \text{if } \sigma = 1/n, \\ C\varepsilon^{1+\sigma}(1+t)^{-n}\chi_\sigma^{-1/\sigma}(t) & \text{if } \sigma \in (1/n-\varepsilon, 1/n). \end{cases}$$

and ψ has the estimates

$$|\psi(t) - \arg\theta| \leq C\varepsilon^\sigma \qquad \text{for } 0 < t < 1,$$

$$\left|\psi(t) - \arg\theta + |\theta|^\sigma\tilde{\eta}\int_1^t \chi_\sigma^{-1}(\tau)\tau^{-\sigma n}\,d\tau\right|$$
$$\leq \begin{cases} C\varepsilon^\sigma & \text{if } \sigma = 1/n \\ C\varepsilon^\sigma + C\varepsilon^{2\sigma}\int_1^t\chi_\sigma^{-1}(\tau)\tau^{-\sigma n}\,d\tau & \text{if } \sigma \in (1/n-\varepsilon, 1/n) \end{cases} \qquad \text{for } 1 \leq t.$$

It is interesting to compare the results for problems (1) and (2). As we mentioned above the critical value $\sigma = 1/n$ for problem (1) differs from that of the Cauchy problem since the solution of (1) obtains faster time decay rate. Consider the one dimensional case $n = 1$. In the case of problem (1) the points $(\arg\alpha, \arg\beta)$ satisfy condition (4) with $\sigma = 1$, $n = 1$ written as

$$-\frac{\pi}{2} < \arg\beta + \frac{1}{2}\arg\alpha - 2\arctan\frac{\sin(2\arg\alpha)}{3+\cos(2\arg\alpha)} < \frac{\pi}{2}$$

and in the case of problem (2) with $\sigma = 2$, $n = 1$ the points $(\arg\alpha, \arg\beta)$ satisfy

$$-\frac{\pi}{2} < \arg\beta - \arctan\frac{\sin(2\arg\alpha)}{2+\cos(2\arg\alpha)} < \frac{\pi}{2} \tag{6}$$

under the conditions (3), (5). Inequalities (4) and (6) come from the positivity of the values

$$\operatorname{Re} \beta\delta(\alpha, \sigma) \qquad \text{and} \qquad \operatorname{Re} \beta \int_{\boldsymbol{R}^n} |G(x)|^2 G(x) dx$$

respectively, where $G(x) = \mathscr{F}^{-1}(e^{-\alpha|\xi|^2})$. By a direct computation we have

$$\begin{aligned}
&\operatorname{Re} \beta\delta(\alpha, \sigma) \\
&= \frac{|\beta|}{2^{\sigma n/2 - 3n/2} \pi^{\sigma n/2 + n/2} |\alpha|^{\sigma n}} ((\sigma + 2 + \sigma \cos(2 \arg \alpha))^2 + (\sigma \sin(2 \arg \alpha))^2)^{-((\sigma+3)/4)n} \\
&\quad \times \left(2 \int_0^\infty x^{\sigma+2} \exp(-|x|^2) dx \right)^n \\
&\quad \times \cos\left(\arg \beta + \frac{\sigma n}{2} \arg \alpha - \frac{\sigma n + 3n}{2} \arctan \frac{\sigma \sin(2 \arg \alpha)}{\sigma + 2 + \sigma \cos(2 \arg \alpha)} \right).
\end{aligned}$$

If $\operatorname{Im} \alpha = 0$, $\operatorname{Im} \beta = 0$, $\gamma = 0$ namely, if $(\arg \alpha, \arg \beta) = (0,0), (0, \pm\pi)$, then solutions of equations (1) and (2) satisfy the same property. More precisely, if $(\arg \alpha, \arg \beta) = (0,0)$, $(\beta > 0)$, then these problems have solutions globally in time (see [4] for (2) and [17] for (1) with $n = 1$) and if $(\arg \alpha, \arg \beta) = (0, \pm\pi)$, $(\beta < 0)$, then solutions of problems (1) and (2) blow up in finite time (see [3], [8], [15] for (2) and [18], [19], [20] for (1), for review of blow up results, see [16]). Another situation we have in the case of complex coefficients α, β, since there are points $(\arg \alpha, \arg \beta)$ which satisfy (4) but do not (6). This fact implies that the properties of solutions of (1) and (2) in the same point $(\arg \alpha, \arg \beta)$ could be different. For example, when $\arg \alpha = \pi/3$ we find approximately that

$$1/2 \arg \alpha - 2 \arctan \sin(2 \arg \alpha)/(3 + \cos(2 \arg \alpha) \approx -0.1 \qquad \text{and}$$
$$-1/2 \arctan \sin(2 \arg \alpha)/(2 + \cos(2 \arg \alpha)) \approx -0.3.$$

Hence the points $(\pi/3, \arg \beta)$ satisfy condition (4), corresponding to the case of a half-line, if

$$-\frac{\pi}{2} + 0.1 < \arg \beta < \frac{\pi}{2} + 0.1$$

and satisfy (6), corresponding to the case of the line, if

$$-\frac{\pi}{2} + 0.3 < \arg \beta < \frac{\pi}{2} + 0.3$$

respectively.

We organize the rest of our paper as follows. We prove preliminary lemmas in Section 2. In Lemma 2.1 we obtain estimates of the Green operator in the Lebesgue spaces $\boldsymbol{L}^p$, $1 \le p \le \infty$ and $\boldsymbol{L}^{1,a}$. Then in Lemma 2.2 we estimate the Green operator in our basic norm

$$\|\phi\|_{\boldsymbol{X}} = \sup_{t>0}((1+t)^n \|\phi(t)\|_{\boldsymbol{L}^\infty} + (1+t)^{n/2}\|\phi(t)\|_{\boldsymbol{L}^1} + (1+t)^{-1/2}\|\phi(t)\|_{\boldsymbol{L}^{1,n+1}}).$$

Large time behavior of the first moments of the nonlinearity $\beta|u|^\sigma u$ in equation (1) is evaluated in Lemma 2.3. Section 3 is devoted to the proof of Theorem 1.1.

2. Preliminaries

Consider the Dirichlet initial-boundary value problem

$$\begin{cases} \mathscr{L}u = f(t,x), & x \in \boldsymbol{R}^n_+,\ t \in \boldsymbol{R}_+, \\ u(0,x) = u_0(x), & x \in \boldsymbol{R}^n_+, \\ u(t,x) = 0, & x \in \partial\boldsymbol{R}^n_+,\ t \in \boldsymbol{R}_+, \end{cases} \tag{7}$$

where $\mathscr{L} = \partial_t - \alpha\Delta$, Re $\alpha > 0$. We write the solution $u(t,x)$ of the problem (7) by virtue of the Duhamel formula

$$u(t) = \mathscr{G}(t)u_0 + \int_0^t \mathscr{G}(t-\tau)f(\tau)d\tau,$$

where the Green operator $\mathscr{G}(t)$ is given by

$$\mathscr{G}(t)\phi = \int_{\boldsymbol{R}^n_+} G(t,x,y)\phi(y)dy,$$

where the kernel

$$G(t,x,y) = (4\pi\alpha t)^{-n/2}\Pi_{j=1}^n (e^{-(x_j-y_j)^2/4\alpha t} - e^{-(x_j+y_j)^2/4\alpha t}).$$

We first prepare some preliminary estimates of the Green operator $\mathscr{G}(t)$ in the Lebesgue norms $\|\phi\|_{\boldsymbol{L}^p}$ and $\|\phi\|_{\boldsymbol{L}^{1,a}} = \|\langle\cdot\rangle^a\phi\|_{\boldsymbol{L}^1}$, where $a \ge 0$, $1 \le p \le \infty$. Denote

$$\begin{aligned} F(t,x) &= \partial_{y_1}\partial_{y_2}\ldots\partial_{y_n} G(t,x,y)|_{y=0} \\ &= (4\pi\alpha t)^{-n/2}(4\pi\alpha^3)^{-n/2} t^{-(3/2)n} \tilde{x} e^{-|x|^2/4\alpha t}, \qquad \tilde{x} = \Pi_{j=1}^n x_j \end{aligned}$$

since

$$\begin{aligned} &\partial_{y_1}\partial_{y_2}\ldots\partial_{y_n} G(t,x,y) \\ &\quad = (4\pi\alpha t)^{-n/2}\Pi_{j=1}^n\left(\frac{(x_j-y_j)}{2\alpha t}e^{-(x_j-y_j)^2/4\alpha t} + \frac{(x_j+y_j)}{2\alpha t}e^{-(x_j+y_j)^2/4\alpha t}\right). \end{aligned}$$

Lemma 2.1. *Suppose that* $\phi \in \boldsymbol{L}^p$, *then the estimate*

$$\|\mathscr{G}(t)\phi\|_{\boldsymbol{L}^p} \le C\|\phi\|_{\boldsymbol{L}^p},$$

is true for all $t > 0$, $1 \le p \le \infty$. *Furthermore we assume that* $\phi \in \boldsymbol{L}^{1,n+1}$, *then the estimate*

$$\| |\cdot|^{\omega}(\mathscr{G}(t)\phi - \vartheta F(t))\|_{\boldsymbol{L}^p} \le Ct^{-n+n/2p+(\omega-1)/2}\|\phi\|_{\boldsymbol{L}^{1,n+1}}$$

is valid for all $t > 0$, *where* $1 \le p \le \infty$, $\omega \in [0, n+1]$ *and*

$$\vartheta = \int_{\boldsymbol{R}_+^n} \tilde{x}\phi(x)dx.$$

Proof. Since

$$|G(t,x,y)| \le Ct^{-n/2}e^{-(C/t)|x-y|^2} \tag{8}$$

for all $x, y \in \boldsymbol{R}_+^n$, by the Young inequality we have

$$\begin{aligned}\|\mathscr{G}(t)\phi\|_{\boldsymbol{L}^p} &\le Ct^{-n/2}\left\|\int_{\boldsymbol{R}_+^n} e^{-(C/t)|x-y|^2}\phi(y)dy\right\|_{\boldsymbol{L}^p} \\ &\le Ct^{-n/2}\|e^{-(C/t)|x|^2}\|_{\boldsymbol{L}^1}\|\phi\|_{\boldsymbol{L}^p} \le C\|\phi\|_{\boldsymbol{L}^p}\end{aligned}$$

for all $t > 0$, where $1 \le p \le \infty$. Hence the first estimate of the lemma follows. For the second estimate we write

$$|x|^{\omega}(\mathscr{G}(t)\phi - \vartheta F(t,x)) = \int_{\boldsymbol{R}_+^n} |x|^{\omega}(G(t,x,y) - F(t,x)\tilde{y})\phi(y)dy$$

for any $\omega \in [0, n+1]$. Applying Taylor expansion, we obtain

$$|G(t,x,y) - F(t,x)\tilde{y}| \le Ct^{-n-1/2}|y|^{n+1}(e^{-(C/t)|x-y|^2} + e^{-(C/t)|x|^2}) \tag{9}$$

for all $x, y \in \boldsymbol{R}_+^n$. Hence in the domain $|y| \le |x|/2$

$$\begin{aligned}|x|^{\omega}|G(t,x,y) - F(t,x)\tilde{y}| &\le Ct^{-n-1/2}|y|^{n+1}|x|^{\omega}e^{-(C/t)|x|^2} \\ &\le Ct^{-n+(\omega-1)/2}|y|^{n+1}e^{-(C/t)|x|^2}.\end{aligned}$$

By the Lagrange finite differences Theorem we have

$$|G(t,x,y)| \le Ct^{-n}|y|^n e^{-(C/t)|x-y|^2},$$

Hence in view of (8) we find

$$|G(t,x,y)| \le Ct^{-(n+\nu)/2}|y|^{\nu}e^{-(C/t)|x-y|^2} \tag{10}$$

for all $x, y \in \boldsymbol{R}_+^n$, where $v \in [0, n]$. Taking (10) with $v = n + 1 - \omega$, in the case $\omega \in [1, n+1]$ we get for $|y| \geq |x|/2$

$$
\begin{aligned}
(11) \qquad & |x|^{\omega}|G(t,x,y) - F(t,x)\tilde{y}| \\
& \leq |x|^{\omega}(|G(t,x,y)| + |F(t,x)\tilde{y}|) \\
& \leq Ct^{-n+(\omega-1)/2}|x|^{\omega}|y|^{n+1-\omega}e^{-(C/t)|x-y|^2} + Ct^{-(3/2)n}|x|^{\omega+n}|y|^{n}e^{-(C/t)|x|^2} \\
& \leq Ct^{-n+(\omega-1)/2}|y|^{n+1}(e^{-(C/t)|x-y|^2} + e^{-(C/t)|x|^2}),
\end{aligned}
$$

and in the case $\omega \in [0, 1]$ we write by virtue of (9) and (11) with $\omega = 1$

$$
\begin{aligned}
& |x|^{\omega}|G(t,x,y) - F(t,x)\tilde{y}| \\
& \quad \leq |x|^{\omega}(|G(t,x,y)| + |F(t,x)\tilde{y}|)^{\omega}|G(t,x,y) - F(t,x)\tilde{y}|^{1-\omega} \\
& \quad \leq Ct^{-n\omega}|y|^{(n+1)\omega}t^{-(n+1/2)(1-\omega)}|y|^{(n+1)(1-\omega)} \\
& \qquad \times (e^{-(C/t)|x-y|^2} + e^{-(C/t)|x|^2}) \\
& \quad \leq Ct^{-n+(\omega-1)/2}|y|^{n+1}(e^{-(C/t)|x-y|^2} + e^{-(C/t)|x|^2}),
\end{aligned}
$$

for all $x, y \in \boldsymbol{R}_+^n$, $|y| \geq |x|/2$. Thus we obtain the estimate

$$
|x|^{\omega}|G(t,x,y) - F(t,x)\tilde{y}| \leq Ct^{-n+(\omega-1)/2}|y|^{n+1}(e^{-(C/t)|x-y|^2} + e^{-(C/t)|x|^2})
$$

for all $x, y \in \boldsymbol{R}_+^n$, and for any $\omega \in [0, n+1]$. Applying the above estimate with Young inequality we find

$$
\begin{aligned}
& \| |\cdot|^{\omega}(\mathscr{G}(t)\phi - \vartheta F(t))\|_{\boldsymbol{L}^p} \\
& \quad = \left\| \int_{\boldsymbol{R}_+^n} |x|^{\omega}(G(t,x,y) - F(t,x)\tilde{y})\phi(y)dy \right\|_{\boldsymbol{L}_x^p} \\
& \quad \leq Ct^{-n+(\omega-1)/2}\left\| \int_{\boldsymbol{R}_+^n} (e^{-(C/t)|x-y|^2} + e^{-(C/t)|x|^2})|y|^{n+1}|\phi(y)|dy \right\|_{\boldsymbol{L}_x^p} \\
& \quad \leq Ct^{-n+n/2p+(\omega-1)/2}\|\phi\|_{\boldsymbol{L}^{1,n+1}}.
\end{aligned}
$$

Thus the second estimate of the lemma follows. Lemma 2.1 is proved. ∎

We introduce the function space

$$
\|\phi\|_{\boldsymbol{X}} = \sup_{t>0}((1+t)^n\|\phi(t)\|_{\boldsymbol{L}^\infty} + (1+t)^{n/2}\|\phi(t)\|_{\boldsymbol{L}^1} + (1+t)^{-1/2}\|\phi(t)\|_{\boldsymbol{L}^{1,n+1}}).
$$

Define the function $\tilde{g}(t)$

$$\tilde{g}(t) = 1 + \varsigma \log(1+t)$$

for some $\varsigma > 0$.

Lemma 2.2. *Let the function $f(t,x)$ satisfy $\int_{\mathbf{R}_+^n} \bar{y} f(t,x)dx = 0$. Then the following inequality*

$$\left\| \tilde{g}^l(t) \int_0^t \tilde{g}^{-l}(\tau) \mathscr{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{X}} \leq C \|(1+t)f(t)\|_{\mathbf{X}}$$

is valid for $l = 0, 1$, provided that the right-hand side is finite.

Proof. By the estimate $\tilde{g}^{-1}(\tau) \leq C$ and Lemma 2.1 we get

$$\left\| \int_0^t \tilde{g}^{-l}(\tau) \mathscr{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^\infty} + \left\| \int_0^t \tilde{g}^{-l}(\tau) \mathscr{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^{1,n+1}}$$

$$\leq C \|(1+t)f(t,x)\|_{\mathbf{X}} \int_0^4 (1+\tau)^{-1} d\tau \leq C \|(1+t)f(t,x)\|_{\mathbf{X}} \tilde{g}^{-l}(t)$$

for all $0 \leq t \leq 4$. We now consider $t > 4$. Via the condition of the lemma for the function $\tilde{g}(t)$ we have the estimate $(1+t)^{-1/4} \leq C\tilde{g}^{-1}(t)$ and

$$\sup_{\tau \in [\sqrt{t}, t]} \tilde{g}^{-1}(\tau) \leq C(1 + \varsigma \log(1+\sqrt{t}))^{-1}$$

$$\leq C\left(1 + \frac{\varsigma}{2} \log(1+t)\right)^{-1} \leq C\tilde{g}^{-1}(t),$$

hence by virtue of Lemma 2.1 with $\omega = 0$ we obtain

$$\left\| \int_0^t \tilde{g}^{-l}(\tau) \mathscr{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^p}$$

$$\leq C \int_0^{\sqrt{t}} (t-\tau)^{-n+n/2p-1/2} (1+\tau)^{-1/2} d\tau \sup_{\tau>0} (1+\tau)^{-1/2} \|(1+\tau)f(\tau)\|_{\mathbf{L}^{1,n+1}}$$

$$+ C\tilde{g}^{-l}(t) \int_{\sqrt{t}}^{t/2} (t-\tau)^{-n+n/2p-1/2} (1+\tau)^{-1/2} d\tau \sup_{\tau>0} (1+\tau)^{-1/2} \|(1+\tau)f(\tau)\|_{\mathbf{L}^{1,n+1}}$$

$$+ C\tilde{g}^{-l}(t) \int_{t/2}^{t} (1+\tau)^{-n+n/2p-1} d\tau \sup_{\tau>0} (1+\tau)^{n-n/2p} \|(1+\tau)f(\tau)\|_{\mathbf{L}^p}$$

$$\leq Ct^{-n+n/2p} \tilde{g}^{-l}(t) \|(1+t)f\|_{\mathbf{X}}$$

for $1 \leq p \leq \infty$ and using the second estimate of Lemma 2.1 with $\omega = n+1$ we get

$$\left\| \int_0^t \tilde{g}^{-l}(\tau)\mathscr{G}(t-\tau)f(\tau)d\tau \right\|_{\mathbf{L}^{1,n+1}}$$

$$\leq C\int_0^{\sqrt{t}} (1+\tau)^{-1/2}d\tau \sup_{\tau>0}(1+\tau)^{-1/2}\|(1+\tau)f(\tau)\|_{\mathbf{L}^{1,n+1}}$$

$$+ C\tilde{g}^{-l}(t)\int_{\sqrt{t}}^t (1+\tau)^{-1/2}d\tau \sup_{\tau>0}(1+\tau)^{-1/2}\|(1+\tau)f(\tau)\|_{\mathbf{L}^{1,n+1}}$$

$$\leq C(t^{1/4}+\tilde{g}^{-l}(t)t^{1/2})\|(1+t)f\|_{\mathbf{X}} \leq C\tilde{g}^{-l}(t)t^{1/2}\|(1+t)f\|_{\mathbf{X}}$$

for all $t > 4$. Hence the result of the lemma follows. Lemma 2.2 is proved. ∎

Next lemma will be used in the proof of the theorem to evaluate large time behavior of the mean value of the nonlinearity in equation (1). Denote $\eta = \operatorname{Re} \beta\delta(\alpha, \sigma) > 0$,

$$\delta(\alpha,\sigma) = t^{\sigma n}\int_{\mathbf{R}_+^n} \tilde{x}|F(t,x)|^{\sigma}F(t,x)dx,$$

$$F(t,x) = (4\pi\alpha^3)^{-n/2}t^{-(3/2)n}\tilde{x}e^{-|x|^2/4\alpha t},$$

$$\chi_\sigma(t) = g(t) \quad \text{if } \sigma = \frac{1}{n} \qquad \text{and} \qquad \chi_\sigma(t) = 1 + \frac{\sigma|\theta|^{\sigma}\eta}{1-\sigma n}t^{1-\sigma n} \quad \text{if } \sigma \in \left(0, \frac{1}{n}\right),$$

$g(t) = 1 + |\theta|^{\sigma}\sigma\eta \log(1+t)$.

Lemma 2.3. *Assume that $v_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,n+1}$, the norm $\|v_0\|_{\mathbf{L}^\infty} + \|v_0\|_{\mathbf{L}^{1,n+1}} = \varepsilon$ is sufficiently small and $\int_{\mathbf{R}_+^n} \tilde{x}v_0(x)dx = |\int_{\mathbf{R}_+^n} \tilde{x}u_0(x)dx| = |\theta|$. Let function $v(t,x)$ satisfy the estimates*

$$\|v\|_{\mathbf{L}^\infty} \leq C\varepsilon(1+t)^{-n}, \qquad \|v\|_{\mathbf{L}^{1,n}} \leq C\varepsilon \qquad \textit{and}$$

$$\|v(t) - \mathscr{G}(t)v_0\|_{\mathbf{L}^{1,n}} \leq C\varepsilon^{1+\sigma}g^{-l}(t),$$

where $l = 1$ in the critical case $\sigma = 1/n$ and $l = 0$ in the sub critical case $0 < \sigma < 1/n$.

Then the inequality

$$\left| 1 + \frac{\sigma}{|\theta|}\int_0^t d\tau \operatorname{Re} \int_{\mathbf{R}_+^n} \tilde{x}\beta|v|^{\sigma}v(\tau,x)dx - \chi_\sigma(t) \right| \tag{12}$$

$$\leq \begin{cases} C\varepsilon^{\sigma}\log(1+\chi_\sigma(t)) & \textit{if } \sigma = 1/n \\ C(\varepsilon^{\sigma} + \varepsilon^{2\sigma}\chi_\sigma(t)) & \textit{if } \sigma \in (1/n - \varepsilon, 1/n) \end{cases}$$

is valid for all $t > 0$.

Proof. In view of the condition $\|v\|_{L^\infty} + \|v\|_{L^{1,n}} \le C\varepsilon$ we get

$$\left|\frac{\sigma}{|\theta|}\int_0^t d\tau \,\mathrm{Re}\int_{R_+^n} \tilde{x}\beta|v|^\sigma v(\tau,x)dx\right| \le C\varepsilon^\sigma t, \tag{13}$$

hence (12) follows for all $0 < t < 1$.

We now consider the case $t \ge 1$. By the second estimate of Lemma 2.1 we find

$$\|\mathscr{G}(t)v_0 - |\theta|F(t)\|_{L^{1,n}} \le Ct^{-1/2}\|v_0\|_{L^{1,n+1}} \le C\varepsilon t^{-1/2}.$$

Hence we find

$$\begin{aligned} &\| |v|^\sigma v - |\theta|^{1+\sigma}|F(t)|^\sigma F(t)\|_{L^{1,n}} \\ &\le C\|v(t) - |\theta|F(t)\|_{L^{1,n}}(\|v\|_{L^\infty}^\sigma + |\theta|^\sigma\|F(t)\|_{L^\infty}^\sigma) \\ &\le C(\|v(t) - \mathscr{G}(t)v_0\|_{L^{1,n}} + \|\mathscr{G}(t)v_0 - |\theta|F(t)\|_{L^{1,n}}) \\ &\quad \times (\|v\|_{L^\infty}^\sigma + |\theta|^\sigma\|F(t)\|_{L^\infty}^\sigma) \\ &\le C\varepsilon^{1+2\sigma}t^{-\sigma n}g^{-l}(t) + C\varepsilon^{1+\sigma}t^{-\sigma n - 1/2} \end{aligned} \tag{14}$$

for all $t \ge 1$, where $l = 1$ if $\sigma = 1/n$ and $l = 0$ if $0 < \sigma < 1/n$, where we have used the inequality $\theta \le \varepsilon$. Since

$$\int_{R_+^n} \tilde{x}|F(t,x)|^\sigma F(t,x)dx = t^{-n\sigma}\delta(\alpha,\sigma) \tag{15}$$

and $\mathrm{Re}\,\beta\delta(\alpha,\sigma) = \eta > 0$ we get

$$\begin{aligned} &\left|\mathrm{Re}\int_{R_+^n} \tilde{x}\beta|v|^\sigma v(t,x)dx - |\theta|^{1+\sigma}t^{-\sigma n}\eta\right| \\ &\le C\| |v|^\sigma v - |\theta|^{1+\sigma}|F(t)|^\sigma F(t)\|_{L^{1,n}} \\ &\le C\varepsilon^{1+2\sigma}t^{-\sigma n}g^{-l}(t) + C\varepsilon^{1+\sigma}t^{-\sigma n-1/2} \end{aligned}$$

for all $t \ge 1$, where $0 < \sigma \le 1/n$. Therefore

$$\begin{aligned} &\left|\frac{\sigma}{|\theta|}\int_1^t d\tau\,\mathrm{Re}\int_{R_+^n} \tilde{x}\beta|v|^\sigma v(\tau,x)dx - |\theta|^\sigma\sigma\eta\log t\right| \\ &\le \int_1^t \frac{C\varepsilon^{2\sigma}\,d\tau}{\tau(1+|\theta|^\sigma\eta\log(1+\tau))} + C\varepsilon^\sigma\int_1^t \tau^{-1-1/2}\,d\tau \\ &\le C\varepsilon^\sigma\log(1+|\theta|^\sigma\eta\log(1+t)) \le C\varepsilon^\sigma\log(1+\chi_\sigma(t)) \end{aligned} \tag{16}$$

for all $t \ge 1$ if $\sigma = 1/n$. Thus in view of (16) and (13) we obtain estimate (12) in the case $\sigma = 1/n$. In the same way as in the proof of (16) we have the inequality

$$\left|\frac{\sigma}{|\theta|}\int_0^t d\tau \,\mathrm{Re}\int_{\boldsymbol{R}_+^n}\tilde{x}\beta|v|^\sigma v(\tau,x)dx - |\theta|^\sigma\frac{\sigma\eta}{1-\sigma n}t^{1-\sigma n}\right|$$

$$\leq C\varepsilon^\sigma + C\varepsilon^{2\sigma}\frac{1}{1-\sigma n}t^{1-\sigma n} \leq C\varepsilon^\sigma + C\varepsilon^{2\sigma}\chi_\sigma(t)$$

for all $t>0$, which implies (12) in the case $1/n-\varepsilon<\sigma<1/n$. Lemma 2.3 is proved. ■

3. Proof of Theorem 1.1

We make a change of the dependent variable $u = ve^{-\varphi(t)+i\psi(t)}$ as in [9], where $\varphi(t)$ and $\psi(t)$ are real valued functions and defined later. Then for the new function v we get the equation

$$\mathscr{L}v + \beta e^{-\sigma\varphi}|v|^\sigma v + \gamma e^{-\kappa\varphi}|v|^\kappa v - (\varphi' - i\psi')v = 0,$$

where $\mathscr{L} = \partial_t - \alpha\Delta$. We take $\varphi(t)$ and $\psi(t)$ satisfy

$$\int_{\boldsymbol{R}_+^n}\tilde{x}(\beta e^{-\sigma\varphi}|v|^\sigma v + \gamma e^{-\kappa\varphi}|v|^\kappa v - (\varphi' - i\psi')v)dx = 0$$

and

$$\int_{\boldsymbol{R}_+^n}\tilde{x}v_0(x)dx = \left|\int_{\boldsymbol{R}_+^n}\tilde{x}u_0(x)dx\right| = |\theta| > 0, \qquad \varphi(0) = 0,$$

where $\tilde{x} = \Pi_{j=1}^n x_j$. Thus we consider the Cauchy problem for the new dependent variables (v,φ)

$$(17)\qquad \begin{cases} \mathscr{L}v = -\beta e^{-\sigma\varphi}\Big(|v|^\sigma - \frac{1}{|\theta|}\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\sigma v\,dx\Big)v \\ -\gamma e^{-\kappa\varphi}\Big(|v|^\kappa - \frac{1}{|\theta|}\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\kappa v\,dx\Big)v, \\ \varphi' = \frac{1}{|\theta|}e^{-\sigma\varphi}\,\mathrm{Re}\,\beta\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\sigma v\,dx + \frac{1}{|\theta|}e^{-\kappa\varphi}\,\mathrm{Re}\,\gamma\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\kappa v\,dx, \\ v(0,x) = v_0(x), \quad \varphi(0) = 0, \end{cases}$$

and

$$\psi'(t) = -\frac{1}{|\theta|}e^{-\sigma\varphi}\,\mathrm{Im}\,\beta\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\sigma v\,dx - \frac{1}{|\theta|}e^{-\kappa\varphi}\,\mathrm{Im}\,\gamma\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\kappa v\,dx.$$

We write (17) as

$$(18)\qquad \begin{cases} \mathscr{L}v = N(v,h), \\ h' = \frac{\sigma}{|\theta|}(\mathrm{Re}\,\beta\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\sigma v\,dx + h^{(\sigma-\kappa)/\sigma}\,\mathrm{Re}\,\gamma\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\kappa v\,dx), \\ v(0,x) = v_0(x), \quad h(0) = 1, \end{cases}$$

where we denote $h = e^{\sigma\varphi(t)}$ and

$$N(v,h) = -\beta h^{-1}\left(|v|^{\sigma} - \frac{1}{|\theta|}\int_{\boldsymbol{R}_+^n} \tilde{x}|v|^{\sigma} v\, dx\right)v - \gamma h^{-\kappa/\sigma}\left(|v|^{\kappa} - \frac{1}{|\theta|}\int_{\boldsymbol{R}_+^n} \tilde{x}|v|^{\kappa} v\, dx\right)v.$$

We note here that $h^{(\sigma-\kappa)/\sigma} \operatorname{Re}\gamma \int_{\boldsymbol{R}_+^n} \tilde{x}|v|^{\kappa} v\, dx$ decays in time more rapidly than

$$\operatorname{Re}\beta \int_{\boldsymbol{R}_+^n} \tilde{x}|v|^{\sigma} v\, dx \qquad \text{and} \qquad \int_{\boldsymbol{R}_+^n} \tilde{x} N(v,h) dx = 0.$$

We prove a global existence of solutions $(v(t,x), h(t))$ for the Cauchy problem (18) by the successive approximations $(v_m(t,x), h_m(t))$, $m = 1, 2, \ldots,$ defined as follows

$$\begin{cases} \mathscr{L} v_m = N(v_{m-1}, h_{m-1}), \\ \partial_t h_m = \frac{\sigma}{|\theta|}(\operatorname{Re}\beta \int_{\boldsymbol{R}_+^n} \tilde{x}|v_{m-1}|^{\sigma} v_{m-1}\, dx + h_{m-1}^{1-\kappa/\sigma} \operatorname{Re}\gamma \int_{\boldsymbol{R}_+^n} \tilde{x}|v_{m-1}|^{\kappa} v_{m-1}\, dx), \\ v_m(0,x) = v_0(x), \quad h_m(0) = 1, \end{cases}$$

for all $m \geq 2$, where $v_1 = \mathscr{G}(t) v_0$, $h_1(t) = \chi_{\sigma}(t)$,

$$\chi_{\sigma}(t) = g(t) \quad \text{if } \sigma = \frac{1}{n} \qquad \text{and} \qquad \chi_{\sigma}(t) = 1 + \frac{\sigma|\theta|^{\sigma}\eta}{1-\sigma n} t^{1-\sigma n} \quad \text{if } \sigma \in \left(0, \frac{1}{n}\right),$$

$g(t) = 1 + |\theta|^{\sigma}\eta \log(1+t)$. We now prove by induction the following estimates

$$\|v_m\|_{\boldsymbol{X}} \leq C\varepsilon, \qquad \|v_m(t) - \mathscr{G}(t)v_0\|_{\boldsymbol{L}^{1,n}} \leq C\varepsilon^{1+\sigma} g^{-l}(t), \tag{19}$$

$$|h_m - \chi_{\sigma}(t)| \leq \begin{cases} C\varepsilon^{\sigma} \log(1+\chi_{\sigma}(t)) & \text{if } \sigma = 1/n \\ C(\varepsilon^{\sigma} + \varepsilon^{2\sigma}\chi_{\sigma}(t)) & \text{if } \sigma \in (1/n - \varepsilon, 1/n) \end{cases}$$

for all $m \geq 1$, where $l = 1$ in the critical case $\sigma = 1/n$ and $l = 0$ in the sub critical case $1/n - \varepsilon < \sigma < 1/n$, the norm $\|\cdot\|_{\boldsymbol{X}}$ is defined as above by

$$\|\phi\|_{\boldsymbol{X}} = \sup_{t>0}((1+t)^n \|\phi(t)\|_{\boldsymbol{L}^{\infty}} + (1+t)^{n/2}\|\phi(t)\|_{\boldsymbol{L}^1} + (1+t)^{-1/2}\|\phi(t)\|_{\boldsymbol{L}^{1,n+1}}).$$

By virtue of Lemma 2.1 we have

$$\|\mathscr{G}(t)v_0\|_{\boldsymbol{L}^{\infty}} \leq C\varepsilon(1+t)^{-n}, \qquad \|\mathscr{G}(t)v_0\|_{\boldsymbol{L}^1} \leq C\varepsilon(1+t)^{-n/2} \qquad \text{and}$$

$$\|\mathscr{G}(t)v_0\|_{\boldsymbol{L}^{1,n+1}} \leq \|F(t)\|_{\boldsymbol{L}^{1,n+1}} + C\|v_0\|_{\boldsymbol{L}^{1,n+1}} \leq C\varepsilon(1+t)^{1/2},$$

therefore estimates (19) are valid for $m = 1$. We assume that estimates (19) are true with m replaced by $m-1$. The integral equation associated with (18) is written as

$$\begin{cases} v_m(t) = \mathscr{G}(t)v_0 + \int_0^t \mathscr{G}(t-\tau) N(v_{m-1}(\tau), h_{m-1}(\tau)) d\tau, \\ h_m(t) = 1 + \frac{\sigma}{|\theta|}\int_0^t d\tau (\operatorname{Re}\beta \int_{\boldsymbol{R}_+^n} \tilde{x}|v_{m-1}|^{\sigma} v_{m-1}\, dx \\ \qquad\qquad + h_{m-1}^{1-\kappa/\sigma} \operatorname{Re}\gamma \int_{\boldsymbol{R}_+^n} \tilde{x}|v_{m-1}|^{\kappa} v_{m-1}\, dx). \end{cases}$$

Note that in the critical case $\sigma = 1/n$ we have

$$\|N(v_{m-1}(t), h_{m-1}(t))\|_{\boldsymbol{L}^\infty}$$
$$\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\boldsymbol{L}^\infty}^{1+\sigma}\left(1+\frac{1}{|\theta|}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}^{n/(1+n)}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}^{1/(1+n)}\right)$$
$$\leq C\varepsilon^{1+\sigma}(1+t)^{-1-1/\sigma}g^{-1}(t),$$

$$\|N(v_{m-1}(t), h_{m-1}(t))\|_{\boldsymbol{L}^1}$$
$$\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\boldsymbol{L}^\infty}^{\sigma}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}\left(1+\frac{1}{|\theta|}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}^{n/(1+n)}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}^{1/(1+n)}\right)$$
$$\leq C\varepsilon^{1+\sigma}(1+t)^{-1}g^{-1}(t)$$

and

$$\|N(v_{m-1}(t), h_{m-1}(t))\|_{\boldsymbol{L}^{1,n+1}}$$
$$\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\boldsymbol{L}^\infty}^{\sigma}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}\left(1+\frac{1}{|\theta|}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}^{n/(1+n)}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}^{1/(1+n)}\right)$$
$$\leq C\varepsilon^{1+\sigma}(1+t)^{-1/2}g^{-1}(t)$$

for all $t > 0$, provided that $(v_{m-1}(t), h_{m-1}(t))$ satisfies (19). Similarly in the sub critical case $\sigma \in (0, 1/n)$ we obtain

$$\|N(v_{m-1}(t), h_{m-1}(t))\|_{\boldsymbol{L}^\infty}$$
$$\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\boldsymbol{L}^\infty}^{1+\sigma}\left(1+\frac{1}{|\theta|}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}^{n/(1+n)}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}^{1/(1+n)}\right)$$
$$\leq C\varepsilon^{1+\sigma}(1+t)^{-n-\sigma n}\left(1+\frac{n|\theta|^\sigma}{1-\sigma n}t^{1-\sigma n}\right)^{-1}$$
$$\leq C\varepsilon(1-\sigma n)(1+t)^{-1-n} \leq C\varepsilon^{1+\sigma}(1+t)^{-1-n},$$

$$\|N(v_{m-1}(t), h_{m-1}(t))\|_{\boldsymbol{L}^1}$$
$$\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\boldsymbol{L}^\infty}^{\sigma}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}\left(1+\frac{1}{|\theta|}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}^{n/(1+n)}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}^{1/(1+n)}\right)$$
$$\leq C\varepsilon(1-\sigma n)(1+t)^{-1} \leq C\varepsilon^{1+\sigma}(1+t)^{-1}$$

and

$$\|N(v_{m-1}(t), h_{m-1}(t))\|_{\boldsymbol{L}^{1,n+1}}$$
$$\leq Ch_{m-1}^{-1}(t)\|v_{m-1}(t)\|_{\boldsymbol{L}^\infty}^{\sigma}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}\left(1+\frac{1}{|\theta|}\|v_{m-1}(t)\|_{\boldsymbol{L}^{1,n+1}}^{n/(1+n)}\|v_{m-1}(t)\|_{\boldsymbol{L}^1}^{1/(1+n)}\right)$$
$$\leq C\varepsilon(1-\sigma n)(1+t)^{-1/2} \leq C\varepsilon^{1+\sigma}(1+t)^{-1/2}$$

for all $t > 0$. This yields the estimate

$$\|(1+t)g^l(t)N(v_{m-1}(t), h_{m-1}(t))\|_{\boldsymbol{X}} \le C\varepsilon^{1+\sigma} \tag{20}$$

if we suppose that $1/n - \sigma \le \varepsilon$. Since $N(v_{m-1}(\tau), h_{m-1}(\tau))$ satisfies condition $\int_{\boldsymbol{R}^n_+} \tilde{x} N(v_{m-1}(\tau), h_{m-1}(\tau)) dx = 0$, we get via Lemma 2.2

$$\left\| g^l(t) \int_0^t g^{-l}(\tau) \mathscr{G}(t-\tau) N(v_{m-1}(\tau), h_{m-1}(\tau)) d\tau \right\|_{\boldsymbol{X}} \le C\varepsilon^{1+\sigma}$$

hence it follows that

$$\|v_m\|_{\boldsymbol{X}} \le C\varepsilon, \qquad \|v_m(t) - \mathscr{G}(t)v_0\|_{\boldsymbol{L}^{1,n}} \le C\varepsilon^{1+\sigma} g^{-l}(t).$$

By virtue of Lemma 2.3 we find that

$$|h_m(t) - \chi_\sigma(t)| \le \begin{cases} C\varepsilon^\sigma \log(1+\chi_\sigma(t)) & \text{if } \sigma = 1/n \\ C(\varepsilon^\sigma + \varepsilon^{2\sigma}\chi_\sigma(t)) & \text{if } \sigma \in (1/n - \varepsilon, 1/n) \end{cases}$$

for all $t > 0$. Thus by induction we see that estimates (19) are valid for all $m \ge 1$. In the same way by induction we can prove that

$$\|v_m - v_{m-1}\|_{\boldsymbol{X}} \le \frac{1}{4}\|v_{m-1} - v_{m-2}\|_{\boldsymbol{X}},$$

$$\sup_{t>0} \chi_\sigma^{-1}(t)|h_m(t) - h_{m-1}(t)| \le \frac{1}{4} \sup_{t>0} \chi_\sigma^{-1}(t)|h_{m-1}(t) - h_{m-2}(t)| + \frac{1}{4}\|v_{m-1} - v_{m-2}\|_{\boldsymbol{X}}$$

for all $m > 2$. Therefore taking limits $\lim_{m\to\infty} v_m(t,x) = v(t,x)$ and $\lim_{m\to\infty} h_m(t) = h(t)$ we obtain a unique solution $v(t,x) \in \boldsymbol{X}$, $h(t) \in \boldsymbol{C}(0,\infty)$ satisfying the estimates

$$|h(t) - \chi_\sigma(t)| \le \begin{cases} C\varepsilon^\sigma \log(1+\chi_\sigma(t)) & \text{if } \sigma = 1/n \\ C(\varepsilon^\sigma + \varepsilon^{2\sigma}\chi_\sigma(t)) & \text{if } \sigma \in (1/n - \varepsilon, 1/n), \end{cases} \tag{21}$$

$$\|v\|_{\boldsymbol{X}} \le C\varepsilon, \qquad \|x(v(t) - \mathscr{G}(t)v_0)\|_{\boldsymbol{L}^1} \le C\varepsilon^{1+\sigma} g^{-l}(t),$$

and integral equations

$$\begin{cases} v(t) = \mathscr{G}(t)v_0 + \int_0^t \mathscr{G}(t-\tau)N(v(\tau), h(\tau))d\tau, \\ h(t) = 1 + \frac{\sigma}{|\theta|}\int_0^t d\tau(\operatorname{Re}\beta \int_{\boldsymbol{R}^n_+} \tilde{x}|v|^\sigma v\, dx + h^{1-\kappa/\sigma} \operatorname{Re}\gamma \int_{\boldsymbol{R}^n_+} \tilde{x}|v|^\kappa v\, dx), \end{cases}$$

$$\psi(t) = \arg \int_{\boldsymbol{R}^n_+} \tilde{x} u_0\, dx - \frac{1}{|\theta|}\int_0^t d\tau \left(e^{-\sigma\varphi} \operatorname{Im}\beta \int_{\boldsymbol{R}^n_+} \tilde{x}|v|^\sigma v\, dx - e^{-\kappa\varphi} \operatorname{Im}\gamma \int_{\boldsymbol{R}^n_+} \tilde{x}|v|^\kappa v\, dx \right).$$

By (20) replaced $v_{m-1}(t), h_{m-1}(t)$ by $v(t), h(t)$ and Lemma 2.2, we see that

$$\|v(t) - \mathscr{G}(t)v_0\|_{\boldsymbol{L}^\infty} \le \begin{cases} C\varepsilon^{1+\sigma}(1+t)^{-n}g^{-1}(t) & \text{if } \sigma = 1/n \\ C\varepsilon^{1+\sigma}(1+t)^{-n} & \text{if } \sigma \in (1/n - \varepsilon, 1/n). \end{cases} \tag{22}$$

We have with $h = e^{\sigma\varphi}$

$$\begin{aligned} &\left|\psi(t) - \arg\int_{\boldsymbol{R}^n_+} \tilde{x}u_0\,dx + |\theta|^\sigma \tilde{\eta}\int_1^t h^{-1}(\tau)\tau^{-\sigma n}\,d\tau\right| \\ &\le \left| -\frac{1}{|\theta|}\int_0^t d\tau e^{-\sigma\varphi}\,\mathrm{Im}\,\beta\int_{\boldsymbol{R}^n_+}\tilde{x}|v|^\sigma v\,dx + |\theta|^\sigma\tilde{\eta}\int_1^t h^{-1}(\tau)\tau^{-\sigma n}\,d\tau\right| \\ &\quad + \frac{1}{|\theta|}\left|\int_0^t d\tau e^{-\kappa\varphi}\,\mathrm{Im}\,\gamma\int_{\boldsymbol{R}^n_+}\tilde{x}|v|^\kappa v\,dx\right| \\ &\le \left|\frac{1}{|\theta|}\int_0^1 d\tau e^{-\sigma\varphi}\,\mathrm{Im}\,\beta\int_{\boldsymbol{R}^n_+}\tilde{x}|v|^\sigma v\,dx\right| \\ &\quad + \left|\frac{1}{|\theta|}\int_1^t d\tau e^{-\sigma\varphi}\left(\mathrm{Im}\,\beta\int_{\boldsymbol{R}^n_+}\tilde{x}|v|^\sigma v\,dx - |\theta|^\sigma\theta\tau^{-\sigma n}\tilde{\eta}\right)\right| \\ &\quad + C\frac{1}{|\theta|}\int_0^t d\tau e^{-\kappa\varphi}\|v(\tau)\|^\kappa_{\boldsymbol{L}^\infty}\|v(\tau)\|_{\boldsymbol{L}^n} \end{aligned} \tag{23}$$

where $\tilde{\eta} = -\mathrm{Im}\,\beta\delta(\alpha,\sigma)$ and $\delta(\alpha,\sigma)$ is defined in (15). By the time decay property of the solution v and $e^{-\kappa\varphi} = h^{-\kappa/\sigma}$

$$\begin{aligned} &\int_0^t e^{-\kappa\varphi}\|v(\tau)\|^\kappa_{\boldsymbol{L}^\infty}\|v(\tau)\|_{\boldsymbol{L}^n}d\tau \\ &\le C\int_0^t (|h^{-\kappa/\sigma}(\tau) - \chi_\sigma^{-\kappa/\sigma}(\tau)| + \chi_\sigma^{-\kappa/\sigma}(\tau))\|v(\tau)\|^\kappa_{\boldsymbol{L}^\infty}\|v(\tau)\|_{\boldsymbol{L}^{1,n}}d\tau \\ &\le C\int_0^t \chi_\sigma^{-\kappa/\sigma}(\tau)\|v(\tau)\|^\kappa_{\boldsymbol{L}^\infty}\|v(\tau)\|_{\boldsymbol{L}^{1,n}}d\tau \\ &\le \begin{cases} C\varepsilon^{1+\kappa}\int_0^t(\log(1+\tau))^{-\kappa/\sigma}(1+\tau)^{-n\kappa}d\tau \le C\varepsilon^{1+\kappa} & \text{if } \sigma < \kappa = 1/n, \\ C\varepsilon^{1+\kappa}\int_0^t(1+\tau)^{-1-n(\kappa-\sigma)}d\tau \le C\varepsilon^{1+\kappa} & \text{if } \sigma < \kappa < 1/n. \end{cases} \end{aligned} \tag{24}$$

It is easy to see that

$$\left|\frac{1}{|\theta|}\int_0^1 d\tau e^{-\sigma\varphi}\,\mathrm{Im}\,\beta\int_{\boldsymbol{R}_+^n}\tilde{x}|v|^\sigma v\,dx\right| \le C\varepsilon^\sigma, \tag{25}$$

from which it follows that

$$\left|\psi(t) - \arg\int_{\boldsymbol{R}_+^n}\tilde{x}u_0\,dx\right| \le C\varepsilon^\sigma \qquad \text{for } 0 < t < 1. \tag{26}$$

By the estimate (14) and the identity (15) we get

$$e^{-\sigma\varphi}\left|\mathrm{Im}\int_{\boldsymbol{R}_+^n}\tilde{x}\beta|v|^\sigma v(t,x)dx - |\theta|^\sigma\theta t^{-\sigma n}\tilde{\eta}\right|$$
$$\le C(\varepsilon^{1+2\sigma}t^{-\sigma n}g^{-l}(t) + \varepsilon^{1+\sigma}t^{-\sigma n-1/2})h^{-1}(t).$$

for all $t \ge 1$, where $l = 1$ if $\sigma = 1/n$ and $l = 0$ if $0 < \sigma < 1/n$. Therefore the second term of the right hand side of (23) is estimated from above by

$$C\varepsilon^{2\sigma}\int_1^t \tau^{-\sigma n}g^{-l}(\tau)h^{-1}(\tau)d\tau.$$

We also have by (21)

$$\left|\int_1^t (h^{-1}(\tau) - \chi_\sigma^{-1}(\tau))\tau^{-\sigma n}\,d\tau\right| \tag{27}$$
$$\le C\int_1^t \chi_\sigma^{-2}(\tau)|h(\tau) - \chi_\sigma(\tau)|\tau^{-\sigma n}\,d\tau$$
$$\le \begin{cases} C\varepsilon^\sigma\int_1^t \chi_\sigma^{-2}(\tau)(\log\chi_\sigma(\tau))\tau^{-1}\,d\tau \le C & \text{if } \sigma = 1/n, \\ C\int_1^t \chi_\sigma^{-2}(\tau)(\varepsilon^\sigma + \varepsilon^{2\sigma}\chi_\sigma(\tau))\tau^{-\sigma n}\,d\tau & \\ \le C\varepsilon^\sigma + C\varepsilon^{2\sigma}\int_1^t \chi_\sigma^{-1}(\tau)\tau^{-\sigma n}\,d\tau & \text{if } \sigma \in (1/n - \varepsilon, 1/n), \end{cases}$$

Hence by (23)–(27) we find that for $t \ge 1$

$$\left|\psi(t) - \arg\int_{\boldsymbol{R}_+^n}\tilde{x}u_0\,dx + |\theta|^\sigma\tilde{\eta}\int_1^t \chi_\sigma^{-1}(\tau)\tau^{-\sigma n}\,d\tau\right| \tag{28}$$
$$\le \begin{cases} C\varepsilon^\sigma & \text{if } \sigma = 1/n, \\ \le C\varepsilon^\sigma + C\varepsilon^{2\sigma}\int_1^t \chi_\sigma^{-1}(\tau)\tau^{-\sigma n}\,d\tau & \text{if } \sigma \in (1/n - \varepsilon, 1/n). \end{cases}$$

This estimate and (26) imply the second estimate of Theorem 1.1. Then via formulas $u(t,x) = e^{-\varphi(t)+i\psi(t)}v(t,x) = h^{-1/\sigma}(t)e^{i\psi(t)}v(t,x)$ we obtain

$$(29)\qquad \|u(t)-|\theta|F(t)\chi_\sigma^{-1/\sigma}(t)e^{i\psi(t)}\|_{\boldsymbol{L}^\infty}$$
$$=\|v(t)h^{-1/\sigma}(t)-\theta F(t)\chi_\sigma^{-1/\sigma}(t)\|_{\boldsymbol{L}^\infty}$$
$$\le\|(v(t)-(\mathscr{G}(t)v_0))\chi_\sigma^{-1/\sigma}(t)\|_{\boldsymbol{L}^\infty}$$
$$+\|(\mathscr{G}(t)v_0-\theta F(t))\chi_\sigma^{-1/\sigma}(t)\|_{\boldsymbol{L}^\infty}+\|v(t)(h^{-1/\sigma}(t)-\chi_\sigma^{-1/\sigma}(t))\|_{\boldsymbol{L}^\infty}$$

By (22) the first term of the right hand side of (29) is estimated as follows

$$\|(v(t)-(\mathscr{G}(t)v_0))\chi_\sigma^{-1/\sigma}(t)\|_{\boldsymbol{L}^\infty}$$
$$\le\begin{cases} C\varepsilon^{1+\sigma}(1+t)^{-n}g^{-1-1/\sigma}(t) & \text{if }\ \sigma=1/n\\ C\varepsilon^{1+\sigma}(1+t)^{-n}\chi_\sigma^{-1/\sigma}(t) & \text{if }\ \sigma\in(1/n-\varepsilon,1/n).\end{cases}$$

and by Lemma 2.1

$$\|(\mathscr{G}(t)v_0-|\theta|F(t))\chi_\sigma^{-1/\sigma}(t)\|_{\boldsymbol{L}^\infty}\le Ct^{-n-1/2}\|v_0\|_{\boldsymbol{L}^{1,n+1}}.$$

We also have by (21)

$$\|v(t,x)(h^{-1/\sigma}(t)-\chi_\sigma^{-1/\sigma}(t))\|_{\boldsymbol{L}^\infty}$$
$$\le C\varepsilon(1+t)^{-n}\chi_\sigma^{-1-1/\sigma}(t)|h(t)-\chi_\sigma(t)|.$$

Therefore it follows that with (29)

$$\|u(t)-|\theta|F(t)\chi_\sigma^{-1/\sigma}(t)e^{i\psi(t)}\|_{\boldsymbol{L}^\infty}$$
$$\le\begin{cases} C\varepsilon^{1+\sigma}(1+t)^{-n}g^{-n-1}(t)\log(1+g(t)) & \text{if }\ \sigma=1/n,\\ C\varepsilon^{1+\sigma}(1+t)^{-n}\chi_\sigma^{-1/\sigma}(t) & \text{if }\ \sigma\in(1/n-\varepsilon,1/n)\end{cases}$$

for large time t. This completes the proof of Theorem 1.1.

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