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Global Existence and Decay for Kirchhoff Type Wave Equation with Boundary and Localized Dissipations in Exterior Domains*

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Abstract. In this paper we prove the existence of global H^2-solutions and energy decay of small amplitude of the Kirchhoff type wave equation with linear localized dissipation and two types of boundary conditions in an exterior domain. Subsequently, we also consider the same problem with weakly nonlinear dissipation.

Key Words and Phrases. Global existence of H^2-solutions, Energy decay, Exterior domain, Kirchhoff type wave equation.

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1. Introduction

In this paper, we first consider the following Kirchhoff type wave equation with two types of boundary condition:

$$(1.1)\qquad u_{tt} - (1 + \|\nabla u(t)\|^2)\Delta u + a(x)u_t = 0 \qquad \text{in } \Omega \times (0, \infty),$$

$$u = 0 \qquad \text{on } \Gamma_0 \times (0, \infty),$$

$$\frac{\partial u}{\partial \nu} = -g(u_t) \qquad \text{on } \Gamma_1 \times (0, \infty),$$

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x),$$

where Ω is an exterior domain in $\boldsymbol{R}^N$ with smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$.

To state precise assumption on $a(x)$, we present a set of the boundary Γ_0 introduced by Russell [9]:

$$\Gamma(x_0) = \{x \in \Gamma_0 \mid (x - x_0) \cdot \nu(x) > 0\},$$

where $x_0 \in \boldsymbol{R}^N$ and $\nu(x)$ is the outward normal vector at $x \in \partial\Omega$. Note that this set is often used in control or stabilization theory for the wave equation in bounded domains (see Lions [3]). Now, we give assumptions on $a(x)$ and g as follows:

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Hyp.A. There exists $x_0 \in \boldsymbol{R}^N$ and a relatively open set $\omega \subset \bar{\Omega}$ such that

$$\overline{\Gamma(x_0)} \subset \omega \quad \text{and} \quad a(x) \geq \varepsilon_0 > 0 \qquad \text{for } x \in \omega \cup B_L^c \text{ with some } \varepsilon_0,$$

where $B_L = \{x \in \boldsymbol{R}^N \mid |x| \leq L\}$.

Hyp.B. The function g is in $C^1(\boldsymbol{R})$, strictly increasing, $g(0) = 0$ and satisfies $m \leq g'(v) \leq M$ for some positive constants m and M.

Under the assumptions Hyp.A and Hyp.B, we prove the global existence and energy decay of small amplitude solutions for the problem (1.1).

When Ω is a bounded domain in $\boldsymbol{R}^N$ and Γ_0 is star-shaped, Lasiecka and Ong [1] are devoted to the global existence and decay estimates to the initial boundary value problem (1.1) with $a(x) = 0$. In [1], $g(v)$ is assumed to be a function like $g(v) = |v|^r v$, $r \geq 0$. We note that in the case of bounded domain, the boundary dissipation is sufficient for the energy decay at a certain rate, and by use of this estimate, global existence of solution is proved for a small $r > 0$. One of the difficulty in the problem with Neumann boundary dissipation lies in the treatment of tangential derivatives on Γ_1.

On the other hand, when Ω is an exterior domain, Nakao and author [7] presented the results of decay estimate of energy $E(t)$ and related ones for the linear wave equation:

$$(1.2) \qquad \begin{gathered} u_{tt} - \Delta u + a(x) u_t = 0 \qquad \text{in } \Omega \times (0, \infty), \\ u = 0 \qquad \text{on } \Gamma_0 \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = -g(u_t) \qquad \text{on } \Gamma_1 \times (0, \infty), \\ u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x) \end{gathered}$$

under the assumption Hyp.A and Hyp.B. Our result in [7] is an extended result of Lasiecka and Triggiani [2] and Nakao [5]. In [2], Lasiecka and Triggiani treated the case where Ω is a bounded domain and Γ_0 is star shaped. And in [5], Nakao considered the exterior problem (1.2) for the case $\Gamma_1 = \emptyset$.

For the Kirchhoff type quasilinear wave equation it is natural to show the existence of H^2-solutions, which means the solutions in the class $C([0, \infty); H_2) \cap C^1([0, \infty); H_1) \cap C^2([0, \infty); L^2)$ or a little weaker space $L^\infty([0, \infty); H_2) \cap W^{1, \infty} \cdot ([0, \infty); H_1) \cap W^{2, \infty}([0, \infty); L^2)$.

Our first purpose of this paper is to prove the global existence and energy decay of small amplitude H^2-solutions for the Kirchhoff type wave equation (1.1) under the assumptions Hyp.A and Hyp.B. To obtain our aim, we shall derive the a priori estimates

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \leq C(\|u_0\|_{H_1} + \|u_1\|_{L^2})(1 + t)^{-1},$$

and

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le C(\|u_0\|_{H_2} + \|u_1\|_{H_1})(1+t)^{-2}$$

if $\|u_0\|_{H_2} + \|u_1\|_{H_1}$ is small. These a priori estimates are sufficient for the desired global solution. For comparison with earlier works, we mention Yamada [10] and Mochizuki [4]. Yamada [10] devoted himself to the global existence of H^2-solutions of problem (1.1) when $\Omega = \boldsymbol{R}^N$ and $a(x) = \lambda$ is a positive constant. Mochizuki [4] extended his result to the case of localized dissipation near infinity, that is, $a(x) \ge \varepsilon_0 > 0$ for large $|x|$.

Next, we consider the same problem with nonlinear dissipation:

$$(1.3) \qquad u_{tt} - (1 + \|\nabla u(t)\|^2)\Delta u + \rho(x, u_t) = 0 \qquad \text{in } \Omega \times (0, \infty),$$

$$u = 0 \qquad \text{on } \Gamma_0 \times (0, \infty),$$

$$\frac{\partial u}{\partial \nu} = -g(u_t) \qquad \text{on } \Gamma_1 \times (0, \infty),$$

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x),$$

where $\rho(x, v)$ is a nonlinear function such that $\rho_v \ge \varepsilon_0 > 0$ on $\omega \cup B_L^c$ for some relative open set ω and $0 \le \rho_v(x, v) \le k_1$ for $x \in \Omega$, $v \in \boldsymbol{R}$.

When $\rho(x, u_t)$ is nonlinear, generally no result on the decay estimate of the energy is known. But we can expect to derive the second energy decay estimate like $\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le C(1+t)^{-1}$ if (u_0, u_1) is small. Indeed, this estimate is weaker than the case of linear dissipation, but combining this with the L^2-norm bound of u_t, Nakao and author [6] have shown the global existence of H^2-solutions for the Cauchy problem under the assumption $0 < k_0 \le \rho'(v) \le k_1(1 + |v|^\alpha)$, $0 \le \alpha \le 2/(N-2)^+$ ($0 \le \alpha < \infty$ if $N = 1, 2$). Subsequently, in [8], we considered the global existence of H^2-solutions of the exterior problem (1.3) for the case $\Gamma_1 = \emptyset$ by combining the ideas in [6] and [5]. Here, using these ideas, we also discuss in brief the global existence and energy decay of small amplitude solutions for the problem (1.3). We note that due to the boundary condition on Γ_1, our arguments become more complicate.

2. Statement of results

Without loss of generality we may assume $V \equiv \Omega^c \subset B_L = \{x \in \boldsymbol{R}^N \mid |x| \le L\}$ and $\omega \cap \Gamma_1 = \emptyset$. Let us introduce the following solution spaces:

$$X_2(T) \equiv W_{loc}^{2,\infty}([0,T]; L^2(\Omega)) \cap W_{loc}^{1,\infty}([0,T]; H_1(\Omega))$$
$$\cap L_{loc}^{\infty}([0,T]; H_2(\Omega) \cap H_1(\Omega)),$$

$$V_1 = \left\{ (u_0, u_1) \in H_2 \cap H_1^0 \times H_1^0 \mid u_0|_{\Gamma_0} = u_1|_{\Gamma_0} = 0 \right.$$
$$\left. \text{and } \frac{\partial u_0}{\partial \nu} = -g(u_1) \text{ on } \Gamma_1 \right\}.$$

Throughout this paper, we set

$$E(t) = \frac{1}{2}\left[\|u_t(t)\|^2 + \|\nabla u(t)\|^2 + \frac{1}{2}\|\nabla u(t)\|^4 \right],$$

$$E_1(t) = \frac{1}{2}[\|u_{tt}(t)\|^2 + b(t)\|\nabla u_t(t)\|^2], \qquad \text{where } b(t) = (1 + \|\nabla u(t)\|^2).$$

Then our main results are as follows.

Theorem 2.1. *Assume the hypotheses Hyp.A and Hyp.B hold. Then for* $K > 0$*, there exists some open set* $S_K \subset V_1$ *including* $(0,0)$ *such that if* $(u_0, u_1) \in S_K$*, the problem* (1.1) *has a solution* $u(t) \in X_2 \equiv X_2(\infty)$ *satisfying*

$$\|u_t(t)\|^2 + \|\nabla u(t)\|^2 \le CI_0^2(1+t)^{-1},$$
$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le C(I_1^2, K)(1+t)^{-2},$$
$$\|\Delta u(t)\|^2 \le C(I_1^2, K)(1+t)^{-1} \qquad \textit{and} \qquad \int_0^t \|\Delta u(s)\|^2 ds \le K^2,$$

where C *is a constant independent of* u, $I_0 = \|u_0\|_{H_1^0} + \|u_1\|_{L^2}$ *and* $I_1 = \|u_0\|_{H^2} + \|u_1\|_{H^2}$.

Next, we turn to the problem (1.3). The assumptions on $\rho(x, v)$ read as follows:

Hyp.C. $\rho(x, v)$ is measurable in $x \in \Omega$ for $v \in \boldsymbol{R}$, almost everywhere differentiable and nondecreasing function in v for a.e. x. Further, there exist $x_0 \in \boldsymbol{R}^N$ and a relatively open set $\omega \subset \bar{\Omega}$ such that

$$\Gamma(x_0) \subset \omega \quad \text{and} \quad \rho_v \ge \varepsilon_0 > 0 \qquad \text{for } x \in \omega \cup B_L^c \text{ with some } \varepsilon_0,$$

where $B_L = \{x \in \boldsymbol{R}^N \mid |x| \le L\}$. Moreover,

$$0 \le \rho_v(x, v) \le k_1 \qquad \text{for } x \in \Omega,\ v \in \boldsymbol{R}.$$

Then we can obtain more or less weaker result for the problem (1.3).

Theorem 2.2. *Suppose the hypotheses Hyp.B and Hyp.C hold. Then for each* $K > 0$*, there exists an open set* $S_K \subset V_1$ *including* $(0,0)$ *such that if* $(u_0, u_1) \in S_K$*, then the problem* (1.3) *admits a unique solution* $u \in X_2 \equiv X_2(\infty)$ *satisfying*

$$\|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le C(I_1^2, K)(1+t)^{-1}, \qquad \int_0^t \|\Delta u(s)\|^2 ds \le K$$

$$\text{and} \qquad \int_0^t \|\nabla u_t(s)\|^2 ds \le K.$$

3. A basic inequality

The existence and uniqueness of local solution $u(t)$ of (1.1) in $X_2(\tilde{T})$ for $(u_0, u_1) \in V_1$ is standard, see Lasiecka and Ong [1]. Here, we derive a basic differential inequality for $u(t)$. First of all, we multiply the problem (1.1) by u_t to obtain usual energy identity:

$$\frac{d}{dt}E(t) + \int_{\Gamma_1} b(t)g(u_t(t))u_t(t)d\Gamma + \int_\Omega a(x)|u_t(t)|^2 dx = 0. \tag{3.1}$$

Let us define the functions $\phi(r), \eta(x) \in C^1(\bar{\Omega})$ and $h(x) = (h_1(x), \dots, h_N(x)) \in W^{1,\infty}(\bar{\Omega})$ as follows:

$$\phi(r) = \begin{cases} \varepsilon_0 & \text{if } 0 \le r \le L, \\ \varepsilon_0 L/r & \text{if } r \ge L, \end{cases}$$

$$\eta(x) = 1 \quad \text{on } \tilde{\omega} \cap \Omega, \qquad \eta = 0 \quad \text{on } \bar{\Omega} \cap \omega^c \qquad \text{and} \qquad \frac{|\nabla \eta|}{\sqrt{\eta}} \in L^\infty(\Omega),$$

$$h \cdot \nu \ge 0, \qquad h = \nu \quad \text{on } \Gamma(x_0) \qquad \text{and} \qquad h(x) = 0 \quad \text{on } \tilde{\omega}^c,$$

where ε_0, L are the positive constants in Hyp.A, and $\tilde{\omega}$ is an open set in $\boldsymbol{R}^N$ such that $\overline{\Gamma(x_0)} \subset \tilde{\omega} \cap \bar{\Omega} \subset \omega$. Then we have the following basic differential inequality for H^2-solution $u(t)$.

Proposition 3.1. *There exists constants $\varepsilon_1 > 0$ and $C > 0$ such that*

$$\begin{aligned} &\frac{d}{dt}X(t) + \varepsilon_1 E(t) + k\int_{\Gamma_1} b(t)g(u_t(t))u_t(t)d\Gamma + k\int_\Omega a(x)|u_t(t)|^2 dx \\ &\quad \le C\int_{\Gamma_1} b(t)\left[\left|\frac{\partial u}{\partial \nu}\right|^2 + |u_t(t)|^2 + \left|\frac{\partial u}{\partial \tau}\right|^2\right] d\Gamma + C\int_{\tilde{\omega}\cap\Omega} b(t)|u(t)|^2 dx \\ &\qquad + C\int_{\Gamma_1} b(t)|u(t)|^2 d\Gamma \qquad \textit{for any large } k, \end{aligned} \tag{3.2}$$

where we set

$$X(t) = \int_\Omega u_t(t)[\phi(|x - x_0|)(x - x_0) - C_0 h] \cdot \nabla u(t)dx + \int_\Omega (\alpha + \eta)u_t(t)u(t)dx$$
$$+ \frac{1}{2}\int_\Omega (\alpha + \eta)a(x)|u(t)|^2 dx + kE(t).$$

Proof. Multiplying the problem (1.1) by $u_t, u, \eta(x)u, \phi(|x - x_0|) \cdot (x - x_0) \cdot \nabla u$, and $h(x) \cdot \nabla u$, respectively, and combining the obtained results, then we have Proposition. In fact, the proof of this is not particularly different with the case of linear wave equation, see [7]. □

Now we consider the following wave equation with some special boundary conditions:

(3.3)
$$u_{tt} - \xi(t)\Delta u = 0 \qquad \text{in } \Omega \times [0, T],$$
$$u = 0 \qquad \text{on } \Gamma_0 \times [0, T],$$
$$u = \frac{\partial u}{\partial \nu} = 0 \qquad \text{on } \Gamma_1 \times [0, T],$$

where $\xi(t)$ is a differentiable and uniformly positive function, say $\xi(t) \geq 1$. As a by-product of (3.2) we can prove a unique continuation property of the problem (3.3) as follows:

Corollary 3.1. *Let* $u \in C^2([0, T]; L^2(\Omega)) \cap C^1([0, T]; H_1^0(\Omega)) \cap C([0, T]; H_1^0(\Omega) \cap H^2(\Omega))$ *be a solution of the problem* (3.3). *If* $u_t(x, t) = 0$ *for* $x \in B_L^c \cup \omega$, $0 \leq t \leq T$ *and* $\sup_{0 \leq t \leq T}\{|\xi'(t)| + |\xi'(t)/\xi(t)|\} \leq \delta$ *with a small* $\delta > 0$, *then there exists* T_0 *such that if* $T > T_0$, $u \equiv 0$ *on* $\Omega \times [0, T]$.

Proof. To begin with, we note that $\partial u/\partial \tau = 0$ on Γ_1 because of the fact $u = 0$ on Γ_1. Differentiate the equation (3.3) in t, then $w = u_t$ satisfies the following problem:

(3.4)
$$w_{tt} - \xi(t)\Delta w = \xi'(t)\Delta u\left(= -\frac{\xi'(t)}{\xi(t)}w_t\right) \qquad \text{in } \Omega \times [0, T],$$
$$w = 0 \qquad \text{on } \Gamma_0 \times [0, T],$$
$$w = \frac{\partial w}{\partial \nu} = 0 \qquad \text{on } \Gamma_1 \times [0, T].$$

It is easy to see that the solution $w = u_t$ of the problem (3.4) can be applicable to the inequality (3.2) in place of u and we must add the terms $|\xi'(t)|\,\|w\|\,\|\Delta u\| + |\xi'(t)|^2\|\Delta u\|^2$ on the right hand side of corresponding inequality. Hence, using the assumption on ξ and the boundary conditions, we obtain

$$\begin{aligned}
&\frac{d}{dt}X_1(t) + \varepsilon_1 E_1(t) \\
&\quad \le C|\xi'(t)|(\|\nabla w(t)\|^2 + \|\Delta u(t)\|\,\|w_t(t)\| + \|\Delta u(t)\|\,\|u_t(t)\| \\
&\qquad\qquad + \|\Delta u(t)\|\,\|\nabla w(t)\| + |\xi'(t)|\,\|\Delta u(t)\|^2) \\
&\quad \le C\left(|\xi'(t)|\,\|\nabla w(t)\|^2 + \left|\frac{\xi'(t)}{\xi(t)}\right| \|u_{tt}(t)\|[\|w_t(t)\| + \|u_t(t)\| + \|\nabla w(t)\|]\right) \\
&\qquad + C\left|\frac{\xi'(t)}{\xi(t)}\right|^2 \|w_t(t)\|^2 \\
&\quad \le C_1\delta E_1(t), \qquad 0 \le t \le T,
\end{aligned}$$

where

$$\begin{aligned}
X_1(t) = &\int_\Omega w_t(t)[\phi(|x-x_0|)(x-x_0) - C_0 h]\cdot \nabla w(t)dx \\
&+ \int_\Omega (\alpha+\eta) w_t(t) w(t) dx + kE_1(t)
\end{aligned}$$

and $E_1(t) = [\|w_t(t)\|^2 + \xi(t)\|\nabla w(t)\|^2]/2$. Thus for small $\delta > 0$, we have

$$X_1(T) + \varepsilon_1 \int_0^T E_1(t)dt \le X_1(0) + C_1\delta \int_0^T E_1(t)dt.$$

Using the fact that $X_1(t)$ is equivalent to $E_1(t)$ for large $k > 0$, we get

$$E_1(T) + \tilde{\varepsilon}_1 \int_0^T E_1(t)dt \le C_2 E_1(0) + C_1\delta \int_0^T E_1(t)dt.$$

On the other hand, let $E_1(t^*) = \min_{0\le t\le T} E_1(t)$, $0 \le t^* \le T$, then we have from (3.4) that for $0 \le t \le T$,

$$\begin{aligned}
E_1(t) &= E_1(t^*) + \frac{1}{2}\int_{t^*}^t \xi'(s)\|\nabla w(s)\|^2 ds + \int_{t^*}^t \int_\Omega \xi'(s)\Delta u(s) w_t(s) dx ds \\
&= E_1(t^*) + \frac{1}{2}\int_{t^*}^t \xi'(s)\|\nabla w(s)\|^2 ds + \int_{t^*}^t \frac{\xi'(s)}{\xi(s)}\|w_t(s)\|^2 ds,
\end{aligned}$$

and so

$$E_1(t) \le E_1(t^*) + C_3\delta \int_0^T E_1(s)ds \qquad \text{for all } 0 \le t \le T.$$

We use the last inequality for $t = 0$. Then, if $\delta < \tilde{\varepsilon}_1/C_4$ with some $C_4 = C_1 + C_2C_3 > 0$, we get

$$TE_1(t^*) \le \int_0^T E_1(t)dt \le C_5 E_1(t^*)$$

for some constant $C_5 = C_2/(\tilde{\varepsilon}_1 - C_4\delta) > 0$. Taking $T > C_5 \equiv T_0$, we see $E_1(t^*) = 0$ i.e., $w_t(t^*) = \nabla w(t^*) = 0$. Considering again equation (3.4), we can see that $w(t) = u_t(t) \equiv 0$, and so the original problem (3.3) implies

$$\Delta u = 0 \qquad \text{on } \Omega \times [0, T].$$

Thus we conclude by boundary conditions

$$u(x,t) \equiv 0 \qquad \text{on } \Omega \times [0, T]. \quad \square$$

4. Energy decay

We continue the estimation of the local solutions u in $X_2(\tilde{T})$. Let ε_2, $K > 0$ and we assume for a moment

$$(4.1) \qquad E(t) \le \varepsilon_2 \qquad \text{and} \qquad \|\nabla u_t(t)\| \le K(1+t)^{-1}, \qquad 0 \le t < \tilde{T}.$$

Now, we have to estimate the integrals $\int_{\tilde{\omega}\cap\Omega} b(t)|u(t)|^2 dx + \int_{\Gamma_1} b(t)|u(t)|^2 d\Gamma$ in the right hand side of (3.2).

Proposition 4.1. *Let u be a solution of* (1.1). *In addition to the assumption* (4.1), *we assume that*

$$(4.2) \qquad K\|\nabla u(t)\| \le \delta$$

with small $\delta > 0$. Then there exists $T_0 > 0$ such that if $T > T_0$, the solution u satisfies

$$(4.3) \quad \int_t^{t+T}\int_{\Omega_L} |u(s)|^2 dxds + \int_t^{t+T}\int_{\Gamma_1} |u(s)|^2 d\Gamma ds$$
$$\le C_0\left\{\int_t^{t+T}\int_\Omega a(x)|u_t(s)|^2 dxds + \int_t^{t+T}\int_{\Gamma_1}\left[\left|\frac{\partial u}{\partial \nu}\right|^2 + |u_t(s)|^2\right] d\Gamma ds\right\}$$
$$+ \varepsilon\int_t^{t+T} E(s)ds, \qquad 0 \le t < \tilde{T} - T,$$

where $C_0 = C_0(T, K, \varepsilon_2)$ is a constant independent of u.

Proof. We can prove Proposition using contradiction method as in [7] and [1]. Assume that (4.3) does not hold, there would exist sequences of numbers $\{t_n\}$ and solutions $\{u_n\}$, respectively, such that

$$\int_{t_n}^{t_n+T}\int_{\Omega_L}|u_n(t)|^2dxdt+\int_{t_n}^{t_n+T}\int_{\Gamma_1}|u_n(t)|^2d\Gamma dt$$
$$\geq n\left\{\int_{t_n}^{t_n+T}\left[\int_{\Omega}a(x)|u_{nt}(t)|^2dx+\int_{\Gamma_1}\left(\left|\frac{\partial u_n}{\partial \nu}(t)\right|^2+|u_{nt}(t)|^2\right)d\Gamma\right]dt\right\}$$
$$+\varepsilon\int_{t_n}^{t_n+T}E_n(t)dt,$$

where $E_n(t)$ is defined by $E(t)$ with u replaced by u_n.

Set $\lambda_n^2=\int_{t_n}^{t_n+T}\int_{\Omega_L}|u_n(t)|^2dxdt+\int_{t_n}^{t_n+T}\int_{\Gamma_1}|u_n(t)|^2d\Gamma dt$ and $v_n(t)=u_n(t+t_n)/\lambda_n$. Then we can derive

$$\int_0^T\int_{\Omega}a(x)|v_{nt}(t)|^2dxdt+\int_0^T\int_{\Gamma_1}\left(\left|\frac{\partial v_n}{\partial \nu}(t)\right|^2+|v_{nt}(t)|^2\right)d\Gamma dt\to 0\qquad\text{as } n\to\infty,$$
$$\int_0^T\int_{\Omega}(|v_{nt}(t)|^2+|\nabla v_n(t)|^2)dx\leq\frac{2}{\lambda_n^2}\int_{t_n}^{t_n+T}E_n(t)dt\leq\frac{2}{\varepsilon}<\infty,$$
$$\int_0^T\int_{\Omega_L}|v_n(t)|^2dxdt+\int_0^T\int_{\Gamma_1}|v_n(t)|^2d\Gamma dt=1.$$

Thus there exists a subsequence of $v_n(t)$ (same symbol) such that $v_n(t)$ converges weakly to a function $v\in L^2(0,T;L^2_{loc}(\Omega))$ such that $v_t\in L^2(0,T;L^2(\Omega))\cap L^2(0,T;L^2(\Gamma_1))$, $\nabla v\in L^2(0,T;L^2(\Omega))$. By Rellich's lemma, we may assume that $v_n(t)$ converges to $v(t)$ strongly in $L^2(\Omega_L\times[0,T])\cap L^2(\Gamma_1\times[0,T])$. On the other hand, taking into account the assumption (4.1) and Ascoli-Arzela Theorem, there exists a subsequence $u_n(t)$ (same symbol) such that

$$\lim_{n\to\infty}\|\nabla u_n(t)\|\to\lambda(t)\qquad\text{in } C([0,T]).$$

Hence the limit function $v(t)$ satisfies

$$\int_0^T\int_{\Omega_L}|v(t)|^2dxdt+\int_0^T\int_{\Gamma_1}|v(t)|^2d\Gamma dt=1\tag{4.4}$$

and

$$v_{tt}(t)-(1+\lambda(t)^2)\Delta v(t)=0\qquad\text{on } \Omega\times[0,T],\tag{4.5}$$
$$v=0\qquad\text{on } \Gamma_0\times(0,\infty),$$
$$v_t=0,\qquad\frac{\partial v}{\partial \nu}=0\qquad\text{on } \Gamma_1\times(0,\infty).$$

Since $a(x)\geq\varepsilon_0>0$ on $B_L^c\cup\omega$ and $\int_0^T\int_{\Omega}a(x)|v_{nt}(s)|^2dxds\to 0$ as $n\to\infty$,

$$v_t(x,t)=0\qquad\text{on } (B_L^c\cup\omega)\times[0,T].$$

We also note that the assumptions (4.1)–(4.2) imply

$$\frac{1}{1+\lambda(t)^2}\frac{d}{dt}|1+\lambda(t)^2| \le \frac{d}{dt}|1+\lambda(t)^2| \le \lim_{n\to\infty}\left|\frac{d}{dt}(1+\|\nabla u_n(t)\|^2)\right|$$
$$= 2\lim_{n\to\infty}|(\nabla u_n(t), \nabla u_{nt}(t))|$$
$$\le 2K\|\nabla u_n(t)\| \le 2\delta.$$

Applying Corollary 3.1, we see that if $T > T_0$,

(4.6) $$v_t(x,t) = 0 \qquad \text{on } \Omega\times[0,T].$$

The fact (4.6) means that $v(x,t) = v(x)$ is independent of t. Thus by (4.5), we have $-\Delta v = 0$ on $\Omega\times[0,T]$. Since $\nabla v \in L^2(\Omega)$ and $v|_{\Gamma_0} = \partial u/\partial v|_{\Gamma_1} = 0$, we conclude $v(x) \equiv 0$ in Ω. This contradicts to (4.4). □

Now, we take T with $T_0 \le T < 2T_0$ and a natural number $n \in N$ with $nT \le t < (n+1)T$. Applying Proposition 4.1 on the interval $[t-(i+1)T, t-iT]$, $i = 0,1,\dots,n-1$, then for all $i = 0,1,\dots,n-1$, we have n-inequalities $\int_{t-(i+1)T}^{t-iT}\int_{\Omega_L}|u(s)|^2dxds + \int_{t-(i+1)T}^{t-iT}\int_{\Gamma_1}|u(s)|^2d\Gamma ds$. Combining them, we have the following inequality independent on T in the interval of integration, which is an essential tool in proving our argument.

Proposition 4.2. *Let $u(t)$ be a solution of* (1.1). *Under the assumptions* (4.1) *and* (4.2), *we have*

(4.7) $$\int_0^t\int_{\Omega_L}|u(s)|^2dxds + \int_0^t\int_{\Gamma_1}|u(s)|^2d\Gamma ds$$
$$\le C_\varepsilon\left\{\int_0^t\int_\Omega a(x)|u_t(s)|^2dxds + \int_0^t\int_{\Gamma_1}\left\{\left|\frac{\partial u}{\partial v}(s)\right|^2 + |u_t(s)|^2\right\}d\Gamma ds\right\}$$
$$+ \varepsilon\int_0^t E(s)ds,$$

for any $t \ge T_0$ and $0 < \varepsilon \ll 1$, where C_ε is a constant independent of t and u.

Applying the inequality in Proposition 3.1 along with the ones in Proposition 4.1 and Proposition 4.2, respectively, we have the followings.

Proposition 4.3. *Let u be a solution of the problem* (1.1). *Then for large $k > 0$ and $T > T_0$ fixed, if $0 < t+T < \tilde{T}$, we have*

(4.8) $$X(t+T) - X(t) + \frac{\varepsilon_1}{2}\int_t^{t+T}E(s)ds$$
$$\le C_\varepsilon\int_t^{t+T}\int_{\Gamma_1}b(s)\left[\left|\frac{\partial u}{\partial v}\right|^2 + |u_t(s)|^2 + \left|\frac{\partial u}{\partial \tau}\right|^2\right]d\Gamma ds$$

and

$$X(t) + \varepsilon_1 \int_0^t E(s)ds \le C \int_0^t \int_{\Gamma_1} b(s) \left[\left| \frac{\partial u}{\partial \nu} \right|^2 + |u_t(s)|^2 + \left| \frac{\partial u}{\partial \tau} \right|^2 \right] d\Gamma ds + CX(0), \qquad t \ge 0 \tag{4.9}$$

with some constant $C > 0$ independent of t and u.

The second inequality (4.9) is proved by using (4.7) and the following two inequalities:

$$m \int_0^t \int_{\Gamma_1} |u_t(s)|^2 d\Gamma ds \le \int_0^t \int_{\Gamma_1} g(u_t(s))u_t(s) d\Gamma ds \le CE(0) \tag{4.10}$$

and

$$\int_0^t \int_{\Omega} a(x)|u_t(s)|^2 dx ds \le CE(0). \tag{4.11}$$

Now, to control the tangential derivatives in the inequality (4.9), we use Lasiecka and Ong's result [1]. But they estimated the size of tangential derivative in bounded domain, so we have to restrict our problem to the ones in bounded domain. For this reason as in [7], we take a function $\psi(x) \in C^1(\bar{\Omega})$ such that $\psi(x) = 1$ in a neighborhood of Γ_1 and $\psi(x) = 0$ for $|x| \ge L$ and $w(x,t) = \psi(x)u$. Then we have

$$w_{tt} - b(t)\Delta w = -a(x)w_t - 2b(t)\nabla\psi(x) \cdot \nabla u - b(t)\Delta\psi u$$
$$\equiv f(x,t) \qquad \text{in } \Omega_L \times [0, \infty),$$
$$w|_{\Gamma_0} = w|_{\partial B_L} = 0$$

and we can use the following Lasiecka and Ong's result in bounded domain by the condition of ψ.

Proposition 4.4 (Lasiecka and Ong [1]). *We fix $T > 0$. Let $t \ge T > 0$ and let u be a solution in Theorem* 2.1. *Then for any constant ε_0 with $0 < \varepsilon_0 < 1/2$ the following trace estimate holds*

$$\int_{t-3T/4}^{t-T/4} \int_{\Gamma_1} \left| \frac{\partial u}{\partial \tau} \right|^2 d\Gamma ds \le C \int_{t-T}^{t} \left[\int_{\Gamma_1} \left(\left| \frac{\partial u}{\partial \nu} \right|^2 + |u_t(s)|^2 \right) d\Gamma + \|f(s)\|^2_{H^{-1/2+\varepsilon_0}(\Omega_L)} \right] ds + C_{\varepsilon_0} \|w\|^2_{H^{1/2+\varepsilon_0}(Q_{L,T,t})}, \tag{4.12}$$

where we set $Q_{L,T,t} = \Omega_L \times [t-T, t]$. *The constants* C, C_{ε_0} *in the above are independent of* t *and* u.

Remark. In [1], Lasiecka and Ong treated the size of the tangential derivative in $\Gamma_1 \times [\delta, t-\delta]$, $\delta > 0$, we apply their result for $t \geq T > 0$, $t - T$ and $T/4$ in place of 0 and δ, respectively.

It is easy to see that as in [7]

$$(4.13)\quad \int_{t-T}^{t} \|f(s)\|^2_{H^{-1/2+\varepsilon_0}(\Omega_L)} ds \leq C \int_{t-T}^{t} \|a(x)u_t(s)\|^2_{L^2(\Omega_L)} ds + C_{\varepsilon_0} \|u\|^2_{H^{1/2+\varepsilon_0}(Q_{L,T,t})},$$

$$\|w\|^2_{H^{1/2+\varepsilon_0}(Q_{L,T})} \leq C\|u\|^2_{H^{1/2+\varepsilon_0}(Q_{L,T,t})}.$$

Reiterating the inequality (4.12) to $t = iT/2$, $i = 2, 3, \ldots, 2n$ with $nT \leq t < (n+1)T$, summing up the resulted inequalities and then combining with (4.13), we have

$$(4.14)\quad \int_{T/4}^{t-T/4} \int_{\Gamma_1} \left|\frac{\partial u}{\partial \tau}\right|^2 d\Gamma ds \leq C \int_0^t \left[\int_{\Gamma_1} \left(\left|\frac{\partial u}{\partial \nu}\right|^2 + |u_t(s)|^2 \right) d\Gamma + \|a(x)u_t(s)\|^2_{L^2(\Omega_L)} \right] ds + C_{\varepsilon_0} \|u\|^2_{H^{1/2+\varepsilon_0}(Q_{L,t})}.$$

Using the interpolation theory, Proposition 4.2 and the boundary condition on Γ_1, we get

$$(4.15)\quad C\|u\|^2_{H^{1/2+\varepsilon_0}(Q_{L,t})} \leq \frac{\varepsilon_1}{4} \int_0^t (\|u(s)\|^2_{H^1(\Omega_L)} + \|u_t(s)\|^2) ds + C \int_0^t \int_{\Omega_L} |u(s)|^2 dx ds \leq CE(0) + \frac{\varepsilon_1}{2} \int_0^t E(s) ds.$$

Applying Proposition 4.3 on $[T/4, t - T/4]$ in place of $[t, t+T]$, and then combining the estimates (4.14), (4.15) and the inequality $E(t) \leq E(0)$ for $t \geq 0$, we have

$$(4.16)\quad X(t - T/4) + \frac{\varepsilon_1}{2} \int_0^t E(s) ds \leq C(X(0) + E(0)).$$

Using the equivalence relation of $X(t)$ with $E(t) + \|u(t)\|^2$ and the fact $\|u(t)\|^2 \leq \int_{t-T/4}^t \|u_t(s)\|^2 ds + \|u(t - T/4)\|^2$, we can get $X(t) \leq CX(t - T/4)$ for $t \geq T/4$. Thus we have

(4.17) $$X(t) + \tilde{\varepsilon}_1 \int_0^t E(s)ds \le CX(0) \qquad \text{for some } \tilde{\varepsilon}_1 > 0$$

or

(4.18) $$\sup_{0 \le t < \infty} X(t) + \tilde{\varepsilon}_1 \int_0^\infty E(s)ds \le CX(0).$$

By (3.1), we have

(4.19) $$\frac{d}{dt}[(1+t)E(t)] = E(t) + (1+t)\frac{d}{dt}E(t) \le E(t).$$

Integrating (4.19) over $(0, t)$, we obtain

$$(1+t)E(t) \le \int_0^t E(s)ds + E(0) \le CX(0)$$

and so we can conclude

(4.20) $$E(t) \le CX(0)(1+t)^{-1}.$$

5. Estimates for the second order derivatives and proof of Theorem 2.1

To end the proof of Theorem 2.1 we remain to derive some estimates for the second order derivatives of a local solution $u(t)$ satisfying (4.1). Note that $w = u_t$ satisfies the following linear equation:

(5.1) $$w_{tt} - b(t)\Delta w - b'(t)\Delta u + a(x)w_t = 0 \qquad \text{in } \Omega \times (0, \infty),$$

$$w = 0 \qquad \text{on } \Gamma_0 \times (0, \infty),$$

$$\frac{\partial w}{\partial \nu} = -g'(u_t)w_t \qquad \text{on } \Gamma_1 \times (0, \infty),$$

$$w(0) = u_1, \qquad w_t(0) = b(0)\Delta u_0 - a(x)u_1 \qquad \text{in } \Omega.$$

First of all, we have the following energy identity

(5.2) $$\frac{d}{dt}E_1(t) + \int_{\Gamma_1} b(t)g'(u_t(t))|w_t(t)|^2 d\Gamma + \int_\Omega a(x)|w_t(t)|^2 dx$$
$$= \frac{1}{2}\int_\Omega b'(t)|\nabla w(t)|^2 dx + \int_\Omega b'(t)\Delta u(t)w_t(t)dx.$$

Applying the usual multiplier method as section 3, we get

$$(5.3)\quad \frac{d}{dt}X_1(t)+\varepsilon_1E_1(t)+k\int_{\Gamma_1}b(t)g'(w(t))|w_t(t)|^2d\Gamma+\int_{\Omega}a(x)|w_t(t)|^2dx$$
$$\le C\int_{\Gamma_1}b(t)\left[\left|\frac{\partial w}{\partial \nu}\right|^2+|w_t(s)|^2+\left|\frac{\partial w}{\partial \tau}\right|^2\right]d\Gamma$$
$$+C\int_{\Omega}|b'(t)|(|w_t(t)|^2+|\nabla w(t)|^2+|\Delta u(t)|^2+|w(t)|^2$$
$$+|b'(t)|\,|\Delta u(t)|^2)dx$$
$$+C\int_{\tilde\omega\cap\Omega}b(t)|w(t)|^2dx+C\int_{\Gamma_1}b(t)|w(t)|^2d\Gamma,$$

where

$$X_1(t)=\int_{\Omega}w_t(t)[\phi(|x-x_0|)(x-x_0)-C_0h]\cdot\nabla w(t)dx$$
$$+\int_{\Omega}(\alpha+\eta)w_t(t)w(t)dx+\frac{1}{2}\int_{\Omega}(\alpha+\eta)a(x)|w(t)|^2dx+kE_1(t).$$

Also we treat integral $\int_{\tilde\omega\cap\Omega}|w(t)|^2dx+\int_{\Gamma_1}|w(t)|^2d\Gamma$ using unique continuation principle as section 4.

Proposition 5.1. *Let u be a solution of* (1.1). *Then under the assumptions* (4.1) *and* (4.2), *there exists $T_0>0$ such that if $T>T_0$, $w=u_t$ satisfies*

$$(5.4)\quad \int_t^{t+T}\int_{\Omega_L}|w(s)|^2dxds+\int_t^{t+T}\int_{\Gamma_1}|w(s)|^2d\Gamma ds$$
$$\le C\left\{\int_t^{t+T}\int_{\Omega}a(x)|w_t(s)|^2dxds+\int_t^{t+T}\int_{\Gamma_1}\left[\left|\frac{\partial w}{\partial\nu}\right|^2+|w_t(s)|^2\right]d\Gamma ds\right\}$$
$$+\int_t^{t+T}\int_{\Omega}|b'(s)|^2|\Delta u(s)|^2dxds+\varepsilon\int_t^{t+T}E_1(s)ds,$$

where $C=C(T,K,\varepsilon_2)$ is a constant independent of u.

Reiterating the argument obtaining (4.8) and (4.9), we can derive for sufficiently small $\varepsilon>0$

$$(5.5)\quad X_1(t+T) - X_1(t) + \varepsilon_1 \int_t^{t+T} E_1(s)ds$$
$$\leq C\int_t^{t+T}\int_{\Gamma_1} b(s)\left[\left|\frac{\partial w}{\partial \nu}\right|^2 + |w_t(s)|^2 + \left|\frac{\partial w}{\partial \tau}\right|^2\right] d\Gamma ds$$
$$+ C\int_t^{t+T}\int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|^2 + |w(s)|^2)dxds$$
$$+ C\int_t^{t+T}\int_\Omega |b'(s)|^2|\Delta u(s)|^2 dxds$$

and

$$(5.6)\quad X_1(t) + \varepsilon_1 \int_0^t E_1(s)ds$$
$$\leq CX_1(0) + C\int_0^t\int_{\Gamma_1} b(s)\left[\left|\frac{\partial w}{\partial \nu}\right|^2 + |w_t(s)|^2 + \left|\frac{\partial w}{\partial \tau}\right|^2\right] d\Gamma ds$$
$$+ C\int_0^t\int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|^2 + |w(s)|^2)dxds$$
$$+ C\int_0^t\int_\Omega |b'(s)|^2|\Delta u(s)|^2 dxds.$$

Applying Proposition 4.4 to $w = u_t$ in place of u and reiterating the argument obtaining (4.17), we get the following inequality

$$(5.7)\quad X_1(t) + \varepsilon_1 \int_0^t E_1(s)ds$$
$$\leq CX_1(0) + C\int_0^t\int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|^2$$
$$+ |w(s)|^2)dxds + C\int_0^t\int_\Omega |b'(s)|^2|\Delta u(s)|^2 dxds.$$

To estimate the integrals in the right hand side of (5.7), we assume for a moment

$$(5.8)\quad E(t) \leq \varepsilon_2, \qquad E_1(t) \leq K^2(1+t)^{-2}, \qquad \|\Delta u(t)\|^2 \leq K^2(1+t)^{-1}$$
$$\text{and} \qquad \int_0^t \|\Delta u(s)\|^2 ds \leq K^2.$$

Using the fact that

$$E(t) \le CX(0)(1+t)^{-1} \tag{5.9}$$

and the preassumption (5.8), we have the following Proposition.

Proposition 5.2. *Under the assumption* (5.8)*, we have*

$$\begin{aligned} &\int_0^{\tilde{T}} \int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|^2 + |w(s)|^2 + |b'(s)|\,|\Delta u(s)|^2) dxds \\ &\quad \le C[K^3 E(0)^{1/2} + KE(0)^{1/2} X(0) + K^4 E(0)] \\ &\quad \equiv q(K, E(0)), \end{aligned} \tag{5.10}$$

where $\lim_{E(0)\to 0} q(K, E(0)) = 0$.

Using this result, we derive the following result.

Proposition 5.3. *Under the assumption* (5.8)*, we can deduce*

$$X_1(t) + C\int_0^{\tilde{T}} E_1(s)ds \le CX_1(0) + q(K, E(0)), \tag{5.11}$$

$$\begin{aligned} &k\int_0^{\tilde{T}} \int_\Omega a(x)|w_t(s)|^2 dxds + k\int_0^{\tilde{T}} \int_{\Gamma_1} b(s)|w_t(s)|^2 d\Gamma ds \\ &\quad \le CX_1(0) + q(K, E(0)), \end{aligned} \tag{5.12}$$

$$\begin{aligned} \int_0^{\tilde{T}} \|w(t)\|^2 dt &\le \frac{1}{\varepsilon_0} E(0) + CX_1(0) + q(K, E(0)) \\ &\equiv q_0(K, E_1(0), E(0)). \end{aligned} \tag{5.13}$$

Proof. Using (5.7), (5.10) and (5.2), we easily obtain (5.11) and (5.12). Next, from (5.12), the assumption on g and boundary condition, we have

$$k\int_0^t \int_{\Gamma_1} \left(|w_t(s)|^2 + \left|\frac{\partial w}{\partial \nu}\right|^2 \right) d\Gamma ds \le CX_1(0) + q(K, E(0)). \tag{5.14}$$

Thus (5.11), (5.12) and (5.14) imply

$$\begin{aligned} \int_0^{\tilde{T}} \|w(t)\|^2 dt &= \int_0^{\tilde{T}} \int_{B_L^c} |w(t)|^2 dxdt + \int_0^{\tilde{T}} \int_{\Omega_L} |w(t)|^2 dxdt \\ &\le \frac{1}{\varepsilon_0} E(0) + CX_1(0) + q(K, E(0)) \equiv q_0(K, E_1(0), E(0)). \quad \square \end{aligned}$$

Note that $\lim_{E(0), E_1(0)\to 0} q_0(K, E_1(0), E(0)) = 0$.

Next we shall show

$$\int_0^{\tilde{T}} (1+t)E_1(t)dt < \infty. \tag{5.15}$$

Now, the equation (5.2) yields

$$\begin{aligned}\frac{d}{dt}[(1+t)E_1(t)] &= E_1(t) + (1+t)\frac{d}{dt}E_1(t)\\ &\le E_1(t) - (1+t)\left[b(t)\int_{\Gamma_1} g'(u_t(t))|w_t(t)|^2 d\Gamma + \int_\Omega a(x)|w_t(t)|^2 dx\right]\\ &\quad + C(1+t)\int_\Omega |b'(t)|[|\Delta u(t)|^2 + |w_t(t)|^2 + |\nabla w(t)|^2]dx.\end{aligned}$$

Integrating the above inequality, (5.11) implies

$$\begin{aligned}&(1+t)E_1(t) + \int_0^t (1+s)\left[b(s)\int_{\Gamma_1} g'(u_t(s))|w_t(s)|^2 d\Gamma + \int_\Omega a(x)|w_t(s)|^2 dx\right]ds\\ &\quad\le CX_1(0) + CE_1(0) + q(K, E(0))\\ &\qquad + C\int_0^t (1+s)\int_\Omega |b'(s)|[|\Delta u(s)|^2 + |w_t(s)|^2 + |\nabla w(s)|^2]dxds.\end{aligned} \tag{5.16}$$

Substitute $\sqrt{1+s}w$ in replace of w in Proposition 5.1, we can get from (5.16),

$$\begin{aligned}&\int_t^{t+T}\int_{\Omega_L}(1+s)|w(s)|^2 dxds\\ &\quad\le C_\varepsilon \int_t^{t+T}(1+s)\left\{\int_\Omega a(x)|w_t(s)|^2 dx + \int_{\Gamma_1}\left[|w_t(s)|^2 + \left|\frac{\partial w}{\partial \nu}\right|^2\right]d\Gamma\right\}ds\\ &\qquad + C_\varepsilon\int_t^{t+T}\int_\Omega (1+s)|b'(s)|^2|\Delta u(s)|^2 dxds + \varepsilon\int_t^{t+T}(1+s)E_1(s)ds\\ &\quad\le CX_1(0) + CE_1(0) + q(K, E(0)) + \varepsilon\int_0^t (1+s)E_1(s)ds\\ &\qquad + C\int_0^t (1+s)\int_\Omega |b'(s)|[|\Delta u(s)|^2 + |w_t(s)|^2 + |\nabla w(s)|^2]dxds\\ &\qquad + C\int_0^t (1+s)\int_\Omega |b'(s)|^2|\Delta u(s)|^2 dxds.\end{aligned} \tag{5.17}$$

We multiply (5.5) by $(1+t+T)$, we can show for any $t \ge 0$,

$$(5.18)\quad (1+t)X_1(t)+\varepsilon_1\int_0^t E_1(s)ds$$

$$\le (1+T)\sup_{0\le s\le T} X_1(s)+\sum_{i=1}^n TX_1(t-iT)$$

$$+C\int_0^t(1+s)\int_{\Gamma_1} b(s)\left[\left|\frac{\partial w}{\partial \nu}\right|^2+|w_t(s)|^2+\left|\frac{\partial w}{\partial \tau}\right|^2\right]d\Gamma ds$$

$$+C\int_0^t(1+s)\int_\Omega |b'(s)|(|w_t(s)|^2+|\nabla w(s)|^2+|\Delta u(s)|^2+|w(s)|^2)dxds$$

$$+C\int_0^t(1+s)\int_\Omega |b'(s)|^2|\Delta u(s)|^2dxds.$$

We note that

$$\{\sqrt{1+t}w\}_{tt}-b(t)\Delta\{\sqrt{1+t}w\}=-\frac{1}{4}(1+t)^{-3/2}w+(1+t)^{-1/2}w_t$$
$$-\sqrt{1+t}a(x)w_t+\sqrt{1+t}b'(t)\Delta u(t).$$

So we can apply (4.14) to $\sqrt{1+s}w$ in place of u, then we have for $t\ge T$,

$$(5.19)\quad C\int_{T/4}^{t-T/4}(1+s)\int_{\Gamma_1}\left|\frac{\partial w}{\partial \tau}\right|^2 d\Gamma ds$$

$$\le C\int_0^t(1+s)\left\{\int_{\Gamma_1}\left[\left|\frac{\partial w}{\partial \nu}\right|^2+|w_t(s)|^2\right]d\Gamma+\int_{\Omega_L}a(x)|w_t(s)|^2dx\right\}ds$$

$$+C\int_0^t(1+s)\int_{\Omega_L}|w(s)|^2dxds+C\int_0^t\int_{\Omega_L}(|w(s)|^2+|w_t(s)|^2)dxds$$

$$+C\int_0^t(1+s)\int_\Omega |b'(s)|^2|\Delta u(s)|^2ds+\frac{\varepsilon_1}{4}\int_0^t(1+s)E_1(s)ds.$$

From the equivalence relation of $X_1(t)$ and $E_1(t)+\|w(t)\|^2$, if $nT\le t<(n+1)T$, we see

$$(5.20)\quad \sum_{i=1}^n TX_1(t-iT)\le\sum_{i=0}^{n-1}TX_1(iT)\le C\sum_{i=0}^{n-1}T(E_1(iT)+E(iT))$$

$$\le C\int_0^t(E_1(s)+E(s))ds\le C(X(0)+X_1(0)+q(K,E(0))).$$

Combining the estimates (5.16)–(5.20) and considering again the argument obtaining (4.17), we have for $t\ge T$,

$$
\begin{aligned}
(5.21)\quad & (1+t)X_1(t) + \tilde{\varepsilon}_1 \int_0^t (1+s)E_1(s)ds \\
&\le C(X_1(0) + E_1(0) + X(0)) + q(K, E(0)) \\
&\quad + C\int_0^t (1+s)\int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|^2 + |w(s)|^2)dxds \\
&\quad + C\int_0^t (1+s)\int_\Omega |b'(s)|^2|\Delta u(s)|^2 dxds \qquad \text{for all } t > 0.
\end{aligned}
$$

Now, assumption (5.8) and the result (5.13) imply

$$
\begin{aligned}
(5.22)\quad & \int_0^t (1+s)\int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|^2 + |w(s)|^2)dxds \\
&\quad + C\int_0^t (1+s)\int_\Omega |b'(s)|^2|\Delta u(s)|^2 dxds \\
&\le C[K^3E(0)^{1/2} + K^4E(0) + KE(0)^{1/2}q_0(K, E_1(0), E(0))].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
(5.23)\quad & (1+t)X_1(t) + \tilde{\varepsilon}_1 \int_0^t (1+s)E_1(s)ds \\
&\le C[X_1(0) + E_1(0) + X(0) + q(K, E(0)) \\
&\quad + KE(0)^{1/2}q_0(K, E_1(0), E(0))]
\end{aligned}
$$

for all $t > 0$.

Now, we can obtain the decay of $E_1(t)$, which is the final step of a priori estimates.

Proposition 5.3. *We have under the assumption* (5.8)

$$
\begin{aligned}
(5.24)\quad E_1(t) &\le C[X_1(0) + E_1(0) + X(0) + K^4X(0)^{1/2} + K^4X(0) \\
&\quad + q(K, E(0)) + KE(0)^{1/2}q_0(K, E_1(0), E(0))](1+t)^{-2}.
\end{aligned}
$$

Proof. By (5.2) we have

$$
\begin{aligned}
(5.25)\quad \frac{d}{dt}[(1+t)^2E_1(t)] &= (1+t)E_1(t) + (1+t)^2\frac{d}{dt}E_1(t) \\
&\le (1+t)E_1(t) \\
&\quad + C(1+t)^2\int_\Omega |b'(s)|(|\nabla w(s)|^2 + |\Delta u(s)|\,|w_t(s)|)dx.
\end{aligned}
$$

The integrability of $(1+t)E_1(t)$ is already proved (see (5.23)). Thus from (4.17), (5.8) and (5.11), we have

$$\begin{aligned}(5.26)\qquad &\int_0^{\tilde{T}}(1+t)^2\int_\Omega |b'(s)|(|\nabla w(s)|^2+|\Delta u(s)|\,|w_t(s)|)dxdt\\ &\le CK^2X(0)^{1/2}[X_1(0)+q(K,E(0))]^{1/2}+CK^4X(0)^{1/2}\\ &\le C\{K^4X(0)+X_1(0)+q(K,E(0))+K^4X(0)^{1/2}\}.\end{aligned}$$

Thus, integrating (5.25), then (5.23) and (5.26) imply (5.24). □

Completion of the proof of Theorem 2.1

Set

$$\begin{aligned}Q(K,E_1(0),X(0)) = C\{&X(0)+E_1(0)+X_1(0)+K^4X(0)^{1/2}+K^4X(0)\\ &+q(K,E(0))+(KE(0)^{1/2}+1)q_0(K,E_1(0),E(0))\}\end{aligned}$$

and

$$S_K \equiv \{(u_0,u_1)\in V_1\cap(H_2\times H_1)\,|\,Q(K,E_1(0),X(0))<K^2\}.$$

Note that if $K > C(\|u_0\|_{H_2}+\|u_1\|_{H_1})$ and $\|u_0\|_{H_1}+\|u_1\|$ is sufficiently small, then $(u_0,u_1)\in S_K$. Assume that $(u_0,u_1)\in S_K$, then we get

$$\|u_{tt}(t)\|^2+\|\nabla u_t(t)\|^2 \le Q(1+t)^{-2} < K^2(1+t)^{-2}$$

and

$$\begin{aligned}\|\Delta u(t)\|^2 &\le \|(1+\|\nabla u(t)\|^2)\Delta u(t)\|^2\\ &\le [\|u_{tt}(t)\|^2+\|a(x)u_t(t)\|^2]\\ &\le Q(1+t)^{-2}+CX(0)(1+t)^{-1}\\ &< K^2(1+t)^{-1}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\int_0^{\tilde{T}}\|\Delta u(t)\|^2dt &\le \int_0^{\tilde{T}}\|(1+\|\nabla u(t)\|^2)\Delta u(t)\|^2dt\\ &\le \int_0^{\tilde{T}}[\|u_{tt}(t)\|^2+\|a(x)u_t(t)\|^2]dt\\ &\le \int_0^{\tilde{T}}E_1(t)dt + C\int_0^{\tilde{T}}\|u_t(t)\|^2dt\\ &\le C(X_1(0)+E_1(0)+q(K,E(0))+Cq_0(K,E_1(0),E(0))\\ &\le Q < K^2.\end{aligned}$$

Finally, we note that

$$\|u(t)\| \leq \|u_0\| + \int_0^t \|u_t(s)\| ds < \infty, \qquad 0 < t < \infty.$$

These estimates show that the assumption (5.8) is true as long as this solution exists and the local solution $u(t)$ can be continued, in fact, on the half interval $[0, \infty)$. This completes the proof of Theorem 2.1.

6. Proof of Theorem 2.2

In this section, we consider the problem (1.3). The existence and uniqueness of local solution $u(t)$ of (1.3) in $X_2(\tilde{T})$ for $(u_0, u_1) \in V_1$ is standard, see Lasiecka and Ong [1]. Here, we derive a basic differential inequality for $u(t)$. First, we have the following energy identity

$$\frac{d}{dt}E(t) + \int_{\Gamma_1} b(t)g(u_t(t))u_t(t)d\Gamma + \int_\Omega \rho(x, u_t(t))u_t(t)dx = 0. \tag{6.1}$$

Now, we differentiate the problem (1.3) with respect to t to linearize it, then $w = u_t$ satisfies the following equation;

$$\begin{aligned} &w_{tt} - b(t)\Delta w - b'(t)\Delta u + \rho_v(x, w)w_t = 0 \qquad \text{in } \Omega \times (0, \infty), \\ &w = 0 \qquad \text{on } \Gamma_0 \times (0, \infty), \\ &\frac{\partial w}{\partial v} = -g'(u_t)w_t \qquad \text{on } \Gamma_1 \times (0, \infty), \\ &w(0) = u_1, \qquad w_t(0) = b(0)\Delta u_0 - \rho(x, u_1) = w_1 \qquad \text{in } \Omega. \end{aligned} \tag{6.2}$$

Moreover, the following energy identity holds:

$$\begin{aligned} &\frac{d}{dt}E_1(t) + \int_{\Gamma_1} b(t)g'(u_t(t))|w_t(t)|^2 d\Gamma + \int_\Omega \rho_v(x, w(t))|w_t(t)|^2 dx \\ &= \frac{1}{2}\int_\Omega b'(t)|\nabla w(t)|^2 dx + \int_\Omega b'(t)\Delta u(t)w_t(t)dx. \end{aligned} \tag{6.3}$$

Note that since $\rho_v \leq k_1 < \infty$ for all $t \geq 0$ and $\rho_v \geq \varepsilon_0$ on $B_L^c \cup \omega$, it is not different particularly up to the estimates (5.7), we have only to replace $a(x)$ by ρ_v. Thus we consider the argument proving (5.7) repeatedly, then

$$\begin{aligned} &X_1(t) + \varepsilon_1 \int_0^t E_1(s)ds \\ &\leq CX_1(0) + C\int_0^t \int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|\,|w(s)|)dxds \\ &\quad + C\int_0^t \int_\Omega |b'(s)|^2 |\Delta u(s)|^2 dxds, \end{aligned} \tag{6.4}$$

where

$$X_1(t) = \int_\Omega w_t(t)[\phi(|x-x_0|)(x-x_0) - C_0 h]\cdot \nabla w(t)dx$$
$$+ \int_\Omega (\alpha+\eta)w_t(t)w(t)dx + \int_\Omega \int_0^w (\alpha+\eta)\rho_v(x,\xi)\xi\, d\xi dx + kE_1(t).$$

Now, consider the integrals in the right hand side of (6.4). Let $\varepsilon_2 > 0$, $K > 0$ and we assume for a moment

$$E(t) \le \varepsilon_2, \qquad \|u_{tt}(t)\|^2 + \|\nabla u_t(t)\|^2 \le K(1+t)^{-1}, \tag{6.5}$$
$$\int_0^t \|\Delta u(s)\|^2 ds \le K \qquad \text{and} \qquad \int_0^t \|\nabla u_t(s)\|^2 ds \le K.$$

Then we have the following Proposition.

Proposition 6.1. *Under the assumption* (6.5), *we have*

$$\int_0^{\tilde T}\int_\Omega |b'(s)|(|w_t(s)|^2 + |\nabla w(s)|^2 + |\Delta u(s)|\,|w(s)| + |b'(s)|\,|\Delta u(s)|^2)dxds \tag{6.6}$$
$$\le C[K^{3/2}E(0)^{1/2} + KE(0) + K^2E(0)] \equiv q(K,E(0)),$$

$$X_1(t) + \int_0^{\tilde T} E_1(s)ds \le CX_1(0) + CE_1(0) + q(K,E(0)), \tag{6.7}$$

$$k\int_0^{\tilde T}\int_\Omega \rho_v(x,w(t))|w_t(s)|^2 dxds + k\int_0^{\tilde T}\int_{\Gamma_1} b(s)|w_t(s)|^2 d\Gamma ds \tag{6.8}$$
$$\le CX_1(0) + CE_1(0) + q(K,E(0)),$$

$$\int_0^{\tilde T}\|w(t)\|^2 dt \le \frac{1}{\varepsilon_0}E(0) + CX_1(0) + CE_1(0) + q(K,E(0)) \tag{6.9}$$
$$\equiv q_0(K,E_1(0),E(0)).$$

As section 5, to complete the proof of Theorem 2.2, we set

$$Q(K,E_1(0),X(0)) = C\{X(0) + E_1(0) + X_1(0) + KE(0)^2 + q(K,E(0)) + k_1 q_0(K,E_1(0),E(0))\}$$

and

$$S_K \equiv \{(u_0,u_1) \in V_1 \cap (H_2 \times H_1) \mid Q(K,E_1(0),X(0)) < K\}.$$

Assume that $(u_0,u_1) \in S_K$. A usual computation gives

$$\int_\Omega b'(t)\Delta u(t)w_t(t)dx = -\frac{1}{4}\frac{d}{dt}[b'(t)]^2 + b'(t)\int_\Omega |\nabla w(t)|^2 dx$$
$$- b'(t)\int_{\Gamma_1} g(u_t(t))w_t(t)d\Gamma.$$

Combining the above estimates, (6.3) and the condition on g, we have

$$\frac{d}{dt}\tilde{E}_1(t) + \int_{\Gamma_1} b(t)g'(u_t(t))|w_t(t)|^2 d\Gamma + \int_\Omega \rho_v(x, w(t))|w_t(t)|^2 dx$$
$$= \frac{3}{2}b'(t)\int_\Omega |\nabla w(t)|^2 dx - b'(t)\int_{\Gamma_1} g(u_t(t))w_t(t)d\Gamma$$
$$\le \frac{3}{2}b'(t)\int_\Omega |\nabla w(t)|^2 dx + C_\varepsilon |b'(t)|^2 \int_{\Gamma_1} |u_t(t)|^2 d\Gamma + \varepsilon \int_{\Gamma_1} |w_t(t)|^2 d\Gamma,$$

where $\tilde{E}_1(t) = E_1(t) + [b'(t)]^2/4$. Thus the condition on g yields

$$\frac{d}{dt}\tilde{E}_1(t) + \int_{\Gamma_1} b(t)g'(u_t(t))|w_t(t)|^2 d\Gamma + \int_\Omega \rho_v(x, w(t))|w_t(t)|^2 dx \tag{6.10}$$
$$\le C\left[|b'(t)|\int_\Omega |\nabla w(t)|^2 dx + |b'(t)|^2 \int_{\Gamma_1} |u_t(t)|^2 d\Gamma\right].$$

Hence, from (6.10) we have

$$\frac{d}{dt}[(1+t)\tilde{E}_1(t)] = \tilde{E}_1(t) + (1+t)\frac{d}{dt}\tilde{E}_1(t) \tag{6.11}$$
$$\le \tilde{E}_1(t) + C(1+t)\left[|b'(t)|\int_\Omega |\nabla w(t)|^2 dx + |b'(t)|^2 \int_{\Gamma_1} |u_t(t)|^2 d\Gamma\right].$$

Now, the estimate (6.7) and assumption (6.5) imply

$$\int_0^t \tilde{E}_1(s)ds = \int_0^t \left[E_1(s) + \frac{1}{4}[b'(s)]^2\right] ds \tag{6.12}$$
$$\le CX_1(0) + CE_1(0) + q(K, E(0)) + CKE(0).$$

Now, we have to derive the integrability of the second term of right hand side of (6.11). Note that from the boundary condition, we have

$$|b'(t)| = |2(\nabla u(t), \nabla u_t(t))| = 2\left|-(\Delta u(t), u_t(t)) + \int_{\Gamma_1} \frac{\partial u}{\partial v}u_t(t)d\Gamma\right| \tag{6.13}$$
$$\le 2\|\Delta u(t)\|\,\|u_t(t)\| + 2\left|\int_{\Gamma_1} g(u_t(t))u_t(t)d\Gamma\right|$$
$$\le 2\|\Delta u(t)\|\,\|u_t(t)\| + 2M\int_{\Gamma_1} |u_t(t)|^2 d\Gamma.$$

From the above estimate (6.13), (6.1) and the assumption (6.5), we have

$$\begin{aligned}(6.14)\qquad &\int_0^t (1+s)\int_\Omega |b'(s)|\,|\nabla w(s)|^2 dxds \\ &\le C\int_0^t (1+s)\left[\|\Delta u(s)\|\,\|u_t(s)\| + \int_{\Gamma_1} |u_t(s)|^2 d\Gamma\right]\|\nabla w(s)\|^2 ds \\ &\le C[K^{3/2}q_0(K, E_1(0), E(0))^{1/2} + KE(0)].\end{aligned}$$

Similarly, (6.1) and assumption (6.5) imply

$$\begin{aligned}(6.15)\qquad &\int_0^t (1+s)|b'(s)|^2 \int_{\Gamma_1} |u_t(s)|^2 d\Gamma ds \\ &\le \int_0^t (1+s)\|\nabla u(s)\|^2\|\nabla u_t(s)\|^2 \int_{\Gamma_1} |u_t(s)|^2 d\Gamma ds \\ &\le KE(0)^2.\end{aligned}$$

Using the above facts (6.14), (6.15) and (6.12), we get

$$\begin{aligned}(1+t)E_1(t) \le (1+t)\tilde{E}_1(t) &\le C[X_1(0) + E_1(0) + q(K, E(0)) + KE(0)^2 \\ &\quad + K^{3/2}q_0(K, E_1(0), E(0))^{1/2}] \\ &\le Q < K,\end{aligned}$$

where we have used the fact $\tilde{E}_1(0) = E_1(0) + [b'(0)]^2/4 \le E_1(0) + CKE(0) \le Cq(K, E(0))$. Moreover, we get

$$\begin{aligned}\int_0^{\tilde{T}} \|\Delta u(s)\|^2 ds &\le \int_0^{\tilde{T}} [\|u_{tt}(t)\|^2 + \|\rho(x, u_t(t))\|^2]dt \\ &\le \int_0^{\tilde{T}} E_1(t)dt + k_1\int_0^{\tilde{T}} \|u_t(t)\|^2 dt \\ &\le C[X_1(0) + E_1(0) + q(K, E(0))] + k_1 q_0(K, E_1(0), E(0)) \\ &\le Q < K.\end{aligned}$$

Also, we can obtain

$$\int_0^{\tilde{T}} \|\nabla w(s)\|^2 ds \le \int_0^{\tilde{T}} E_1(s)ds \le Q < K.$$

Finally, $\|u(t)\|$ is estimated as usual. Thus as section 5, we conclude that the local solution $u(t)$ can be continued on the half interval $[0, \infty)$. This completes the proof of Theorem 2.2.

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