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On the Solvability and the Maximal Regularity of Complete Abstract Differential Equations of Elliptic Type

By

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Abstract. In this paper we give some new results on complete abstract second order differential equations of elliptic type in a Banach space. Existence, uniqueness and maximal regularity of the strict solution are proved under some natural assumptions generalizing previous theorems on the subject.

Key Words and Phrases. Second order abstract differential equation, Boundary condition, Analytic semigroup, Maximal regularity, Elliptic equation, Degenerate parabolic equation.

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1. Introduction and hypotheses

Let us consider the second order abstract differential equation

$$u''(t) + 2Bu'(t) + Au(t) = f(t), \qquad t \in (0,1), \tag{1}$$

together with the boundary conditions

$$\begin{cases} u(0) = u_0, \\ u(1) = u_1. \end{cases} \tag{2}$$

Here f is a continuous X-valued function on $[0,1]$, X being a complex Banach space, u_0, u_1 are given elements of $D(A)$, the domain of A, A and B are two closed linear operators in X.

We seek for a strict solution $u(.)$ to (1), (2), i.e. a function

$$u \in C^2([0,1];X) \cap C^1([0,1];D(B)) \cap C([0,1];D(A)),$$

satisfying (1) and (2).

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Our main goal is to give both an alternative approach with respect to recent results due to El Haial and Labbas [4] and to improve the main result by Favini, Labbas, Tanabe, Yagi [5]. To this end we will assume that

$$(3)\qquad \begin{cases} B^2 - A \text{ is a linear closed densely defined operator in } X \text{ and} \\ \forall\lambda \geqslant 0, \quad \exists(\lambda I + B^2 - A)^{-1} \in L(X): \\ \|(\lambda I + B^2 - A)^{-1}\|_{L(X)} \leqslant C/(1+\lambda), \end{cases}$$

(it is well known that hypothesis (3) implies that $-(B^2 - A)^{1/2}$ is the infinitesimal generator of an analytic semigroup $\{T(t)\}$, $t \geqslant 0$, in X),

$$(4)\qquad D(A) \subseteq D(B^2),$$

$$(5)\qquad \forall y \in D(B) \qquad B(B^2 - A)^{-1}y = (B^2 - A)^{-1}By,$$

$$(6)\qquad A \text{ is boundedly invertible,}$$

$$(7)\qquad D((B^2 - A)^{1/2}) \subseteq D(B),$$

$$(8)\qquad \pm B - (B^2 - A)^{1/2} \text{ generates an analytic semigroup on } X.$$

Remark 1.

1. If X is a Hilbert space, $B, B^2 - A$ are self-adjoint operators, $B^2 - A$ being positive, too, and in addition $D(A) \subseteq D(B^2)$, an easy modification to Heinz Theorem (see [14], p. 44–46) shows that
$$D((B^2 - A)^{1/2}) \subseteq D(B),$$
holds as well, so that (7) is satisfied.
2. In general case of operators defined in Banach spaces (7) implies that for every $\rho > 0$
$$\forall y \in D(B^2 - A) \qquad \|By\| \leqslant C(\rho^{1/2}\|y\| + \rho^{-1/2}\|(B^2 - A)y\|).$$
Conversely if, for some $\gamma \in]0, 1/2[$ and every $\rho \geqslant \rho_0 > 0$, one has
$$\forall y \in D(B^2 - A) \qquad \|By\| \leqslant C(\rho^{\gamma}\|y\| + \rho^{\gamma-1}\|(B^2 - A)y\|),$$
then
$$D((B^2 - A)^{1/2}) \subseteq D(B),$$
(see [13], p. 73–74).
3. Let A_0 and B be two closed linear operators in X commuting in the resolvent sense with $D(A_0) \subseteq D(B^2)$, $D(A_0)$ everywhere dense in X and $B^2 - A_0$ a closed operator. If there exists some $\lambda_0 < 0$ such that
$$\begin{cases} \forall\lambda > 0 \\ \|(A_0 + \lambda_0 I - \lambda I)^{-1}\| \leqslant 1/\lambda, \\ \|(B^2 + \lambda I)^{-1}\| \leqslant 1/\lambda, \end{cases}$$

then for any $s > 0$

$$\begin{cases} \|((s+\lambda_0)I - (\lambda_0 I + A_0) + B^2)^{-1}\| \\ = \|(A_0 - B^2 - sI)^{-1}\| \\ \leqslant 1/s, \end{cases}$$

(see [3], p. 320). Therefore (3) holds for $A = A_0 + \lambda_0 I$.

4. Assumptions (3)$\sim$(4) yield

(9) $$B^2(B^2 - A)^{-1} \in L(X),$$

and

(10) $$A(B^2 - A)^{-1} \in L(X).$$

5. As it is well known, the only assumptions (3), (5) do not imply (8). However, sometimes some conditions easily verifiable, guarantee that assumption (8) is satisfied without asking smallness of B with respect to $(B^2 - A)^{1/2}$. Here we recall the following one from Favini and Triggiani [7], Theorem 1.1, p. 94.

   Let L be a strictly positive self-adjoint operator on the Hilbert space X and let M be another self-adjoint operator on X such that $D(L^{1/2}) \subseteq D(|M|^{1/2})$ where $|M| = (M^2)^{1/2}$. Then $-L \pm iM$ generate analytic semigroups in X.

   On the other hand, if one assumes $D(L) \subseteq D(M)$ $(= D(|M|))$, then by the Corollary in Tanabe [14], p. 45, $D(L^{1/2}) \subseteq D(|M|^{1/2})$ and thus $-L \pm iM$ generate analytic semigroups in X again. Assumption (8) follows if we take $B = iM$ and $(B^2 - A)^{1/2} = L$.

We give an example when all the assumptions (3)$\sim$(8) are satisfied.

Example 2. Take $X = L^2(\boldsymbol{R})$ with the usual inner product

$$\langle f, g\rangle = \int_{\boldsymbol{R}} f(x)\overline{g(x)}dx,$$

and let A and B defined by

$$\begin{cases} D(A) = H^2(\boldsymbol{R}), & Au = au'' - cu, \\ D(B) = H^1(\boldsymbol{R}), & Bu = bu', \end{cases}$$

with $a > b^2$, $b \neq 0$ and $c > 0$. A is stricly negative self-adjoint operator and $B^2 - A$ coincides with

$$D(B^2 - A) = H^2(\boldsymbol{R}), \qquad (B^2 - A)u = (b^2 - a)u'' - cu.$$

Therefore $B^2 - A$ is a strictly positive self-adjoint operator on X. One then knows that $(B^2 - A)^{1/2}$ has the same property as well with

$$D((B^2 - A)^{1/2}) = H^1(\boldsymbol{R}).$$

Now the operator $H = \frac{1}{i}B$, where

$$D(H) = H^1(\boldsymbol{R}), \qquad Hu = \frac{1}{i}bu',$$

is self-adjoint. Then Remark 1, statement 4, applies and

$$-(B^2 - A)^{1/2} \pm iH = \pm B - (B^2 - A)^{1/2},$$

generates an analytic semigroup on X. Thus (8) holds. The assumptions (3) $\sim$ (7) are easily verified together.

We give a direct approach to problem (1)–(2), extending to the case $B \neq 0$, the pioneering work by S. G. Krein [10], pp. 299–270. Moreover, we prove the maximal regularity of the strict solution u, that is

$$u'', Bu', Au \in C^\theta([0,1]; X),$$

provided that $f \in C^\theta([0,1]; X)$ and $f(0), f(1), Au_0$ and Au_1 verify conditions of compatibility with respect to equation (1). Here $C^\theta([0,1]; X)$ denotes the space of all X-valued Hölder continuous functions on $[0,1]$ with exponent θ.

In fact, a representation formula of the solution is found taking into account the basic properties of analytic semigroups.

The plan of the paper is as follows. Section 2 is devoted to the existence and the uniqueness of the strict solution u for (1)–(2). In section 3 we prove the maximal regularity of u. In section 4 we give some examples of application to partial differential equations.

2. Existence and uniqueness of the strict solution

We shall establish the first result as follows.

Theorem 3. *Under assumptions* (3) $\sim$ (8)*, if, in addition* $D(BA) \subset D(B^3)$*, then for all* $f \in C^\theta([0,1]; X)$*,* $0 < \theta < 1$ *and any* $u_0, u_1 \in D(A)$*, problem* (1)–(2) *has a unique strict solution on* $[0,1]$.

For the proof of this Theorem, we need the following Lemmas.

Lemma 4. *Under the hypotheses* (3), (4) *one has*

1. *Assumption* (5) *is equivalent to*

$$(11) \qquad \begin{cases} D(B(B^2 - A)) \subset D((B^2 - A)B) \quad \text{and} \\ \forall z \in D(B(B^2 - A)), \quad B(B^2 - A)z = (B^2 - A)Bz. \end{cases}$$

2. *Assumption* (5) *is equivalent to*

$$\left\{ \begin{array}{l} \forall y \in D(B), \quad (B^2 - A)^{-1/2} y \in D(B) \ \textit{and} \\ B(B^2 - A)^{-1/2} y = (B^2 - A)^{-1/2} By. \end{array} \right. \tag{12}$$

3. *If* (5) *holds, then*

$$\forall y \in D(A), \qquad A(B^2 - A)^{-1/2} y = (B^2 - A)^{-1/2} Ay, \tag{13}$$

and if (5), (7) *hold, then*

$$\forall y \in D(A), \qquad B(B^2 - A)^{1/2} y = (B^2 - A)^{1/2} By. \tag{14}$$

4. *If* (5), (6) *hold, then*

$$\forall y \in X, \qquad (B^2 - A)^{-1/2} A^{-1} y = A^{-1} (B^2 - A)^{-1/2} y, \tag{15}$$

and if (5), (6), (7) *hold, then*

$$\forall y \in D((B^2 - A)^{1/2}), \qquad A^{-1} (B^2 - A)^{1/2} y = (B^2 - A)^{1/2} A^{-1} y. \tag{16}$$

Proof. Assume (3), (4).

1. If (5) holds, then for any $z \in D(B(B^2 - A)) \subset D(B)$ we have

$$Bz = B(B^2 - A)^{-1}(B^2 - A)z = (B^2 - A)^{-1} B(B^2 - A)z \tag{17}$$

so $Bz \in D(B^2 - A)$ and $z \in D((B^2 - A)B)$. Therefore

$$(B^2 - A)Bz = B(B^2 - A)z.$$

Conversely, assume (11) and let $y \in D(B)$. Then

$$(B^2 - A)^{-1} y \in D(B(B^2 - A))$$

and

$$B(B^2 - A)(B^2 - A)^{-1} y = (B^2 - A)B(B^2 - A)^{-1} y$$

which implies

$$(B^2 - A)^{-1} By = B(B^2 - A)^{-1} y.$$

2. If (12) holds, then for any $y \in D(B)$ we have

$$\begin{aligned} B(B^2 - A)^{-1/2}(B^2 - A)^{-1/2} y &= (B^2 - A)^{-1/2} B(B^2 - A)^{-1/2} y \\ &= (B^2 - A)^{-1/2}(B^2 - A)^{-1/2} By, \end{aligned}$$

which implies (5).

Conversely, assume (5). Let $y \in D(B)$, $\lambda \in \rho(-(B^2 - A))$ and set

$$z = (B^2 - A + \lambda I)^{-1} y.$$

Then

$$(B^2 - A)z = y - \lambda z \in D(B),$$

so that $z \in D(B(B^2 - A))$. Now, in virtue of statement 1, we have

$$B(B^2 - A + \lambda I)z = (B^2 - A + \lambda I)Bz$$

and thus

$$(B^2 - A + \lambda I)^{-1} By = B(B^2 - A + \lambda I)^{-1} y.$$

To conclude, let $y \in D(B)$. Then using a suitable curve γ, we can write

$$(B^2 - A)^{-1/2} y = \frac{1}{2\pi i} \int_\gamma (-\lambda)^{-1/2} (B^2 - A + \lambda I)^{-1} y \, d\lambda.$$

Now the integral

$$\int_\gamma B(-\lambda)^{-1/2} (B^2 - A + \lambda I)^{-1} y \, d\lambda,$$

is convergent since

$$\begin{aligned} \|B(-\lambda)^{-1/2}(B^2 - A + \lambda I)^{-1} y\|_X &= |\lambda|^{-1/2} \|(B^2 - A + \lambda I)^{-1} By\| \\ &\leq C \frac{\|By\|_X}{|\lambda|^{3/2}}, \end{aligned}$$

thus $(B^2 - A)^{-1/2} y \in D(B)$ and

$$\begin{aligned} B(B^2 - A)^{-1/2} y &= \frac{1}{2\pi i} \int_\gamma B(-\lambda)^{-1/2} (B^2 - A + \lambda I)^{-1} y \, d\lambda \\ &= \frac{1}{2\pi i} \int_\gamma (-\lambda)^{-1/2} (B^2 - A + \lambda I)^{-1} By \, d\lambda \\ &= (B^2 - A)^{-1/2} By, \end{aligned}$$

from which (12) follows.

3. If (5) holds, then from (12) we get

$$\forall y \in D(B^2), \qquad B^2 (B^2 - A)^{-1/2} y = (B^2 - A)^{-1/2} B^2 y,$$

thus

$$\forall y \in D(A), \qquad A(B^2 - A)^{-1/2}y = (B^2 - A)^{-1/2}Ay.$$

Suppose that (5) and (7) hold. Let $y \in D(A) = D(B^2 - A)$. Then $(B^2 - A)^{1/2}y \in D(B)$ and in virtue of (12) one has

$$B(B^2 - A)^{-1/2}(B^2 - A)^{1/2}y = (B^2 - A)^{-1/2}B(B^2 - A)^{1/2}y,$$

which gives

$$(B^2 - A)^{1/2}By = B(B^2 - A)^{1/2}y.$$

4. It is enough to consider $A^{-1}y \in D(A)$ and apply (13), (14). ■

Lemma 5. *Under assumptions* (3), (4), (5) *and* (7) *one has, for any* $z \in \rho(-B - (B^2 - A)^{1/2})$ *and any* $\lambda \in \rho(B - (B^2 - A)^{1/2})$

1.

$$(zI + B + (B^2 - A)^{1/2})^{-1}(B^2 - A)^{-1/2} = (B^2 - A)^{-1/2}(zI + B + (B^2 - A)^{1/2})^{-1}, \tag{18}$$

$$(\lambda I - B + (B^2 - A)^{1/2})^{-1}(B^2 - A)^{-1/2} = (B^2 - A)^{-1/2}(\lambda I - B + (B^2 - A)^{1/2})^{-1}, \tag{19}$$

2.

$$(zI + B + (B^2 - A)^{1/2})^{-1}(B^2 - A)^{1/2}y = (B^2 - A)^{1/2}(zI + B + (B^2 - A)^{1/2})^{-1}y, \tag{20}$$

$$(\lambda I - B + (B^2 - A)^{1/2})^{-1}(B^2 - A)^{1/2}y = (B^2 - A)^{1/2}(\lambda I - B + (B^2 - A)^{1/2})^{-1}y, \tag{21}$$

where $y \in D((B^2 - A)^{1/2})$.

Proof.

1. Consider $\xi \in X$ and set

$$y = (B^2 - A)^{-1/2}(zI + B + (B^2 - A)^{1/2})^{-1}\xi \in D(B^2 - A) = D(A).$$

Now using (14) we have

$$(B^2 - A)^{1/2}(zI + B + (B^2 - A)^{1/2})y$$
$$= (zI + B + (B^2 - A)^{1/2})(B^2 - A)^{1/2}y,$$

which implies (18). By the same way we obtain (19).

2. Is enough to apply (18), (19) to $\xi \in X$ such that $y = (B^2 - A)^{-1/2}\xi$. ∎

Lemma 6. *Let us assume* (3)∼(7). *Then*

1.

$$\begin{cases} (B - (B^2 - A)^{1/2})(B + (B^2 - A)^{1/2})A^{-1} = I \\ (B + (B^2 - A)^{1/2})(B - (B^2 - A)^{1/2})A^{-1} = I. \end{cases} \tag{22}$$

2. *For any* $y \in D((B^2 - A)^{1/2})$

$$(B - (B^2 - A)^{1/2})A^{-1}(B + (B^2 - A)^{1/2})y = (B - (B^2 - A)^{1/2})(A^{-1}B - BA^{-1})y + y, \tag{23}$$

and

$$(B + (B^2 - A)^{1/2})A^{-1}(B - (B^2 - A)^{1/2})y = (B + (B^2 - A)^{1/2})(A^{-1}B - BA^{-1})y + y. \tag{24}$$

Proof.

1. In virtue of (14), one has

$$(B^2 - A)^{1/2}BA^{-1} = B(B^2 - A)^{1/2}A^{-1},$$

hence

$$(B - (B^2 - A)^{1/2})(B + (B^2 - A)^{1/2})A^{-1}$$
$$= B^2A^{-1} - (B^2 - A)^{1/2}BA^{-1} + B(B^2 - A)^{1/2}A^{-1} - (B^2 - A)A^{-1}$$
$$= I,$$

and also

$$(B + (B^2 - A)^{1/2})(B - (B^2 - A)^{1/2})A^{-1} = I.$$

2. Let $y \in D((B^2 - A)^{1/2})$. Then

$$\begin{aligned}(B - (B^2 - A)^{1/2})A^{-1}(B + (B^2 - A)^{1/2})y \\ = BA^{-1}By - (B^2 - A)^{1/2}A^{-1}By + BA^{-1}(B^2 - A)^{1/2}y \\ - (B^2 - A)^{1/2}A^{-1}(B^2 - A)^{1/2}y.\end{aligned} \tag{25}$$

Now using (16), we get

$$(B^2 - A)^{1/2} A^{-1} (B^2 - A)^{1/2} y = (B^2 - A)^{1/2} (B^2 - A)^{1/2} A^{-1} y \tag{26}$$
$$= B^2 A^{-1} y - y,$$

and

$$BA^{-1} (B^2 - A)^{1/2} y = B(B^2 - A)^{1/2} A^{-1} y \tag{27}$$
$$= (B^2 - A)^{1/2} BA^{-1} y.$$

Using (25), (26) and (27) we obtain

$$(B - (B^2 - A)^{1/2}) A^{-1} (B + (B^2 - A)^{1/2}) y$$
$$= B(A^{-1}B - BA^{-1}) y - (B^2 - A)^{1/2} (A^{-1}B - BA^{-1}) y + y$$
$$= (B - (B^2 - A)^{1/2})(A^{-1}B - BA^{-1}) y + y.$$

Similarly we get

$$(B + (B^2 - A)^{1/2}) A^{-1} (B - (B^2 - A)^{1/2}) y$$
$$= BA^{-1}By + (B^2 - A)^{1/2} A^{-1} By - BA^{-1} (B^2 - A)^{1/2} y$$
$$- (B^2 - A)^{1/2} A^{-1} (B^2 - A)^{1/2} y$$
$$= (B + (B^2 - A)^{1/2})(A^{-1}B - BA^{-1}) y + y. \quad \blacksquare$$

Lemma 7. *Under assumptions* (3)∼(7), $B + (B^2 - A)^{1/2}$ *has a bounded inverse if and only if* $B - (B^2 - A)^{1/2}$ *has a bounded inverse and then*

$$\begin{cases} (B - (B^2 - A)^{1/2})^{-1} = (B + (B^2 - A)^{1/2}) A^{-1}, \\ (B + (B^2 - A)^{1/2})^{-1} = (B - (B^2 - A)^{1/2}) A^{-1}. \end{cases} \tag{28}$$

Proof. Assume that $B + (B^2 - A)^{1/2}$ is boundedly invertible. To prove that $B - (B^2 - A)^{1/2}$ is boundedly invertible it is enough, in virtue of (22), to show that this operator is one-to-one.

So let $x \in D((B^2 - A)^{1/2})$ such that

$$(B - (B^2 - A)^{1/2}) x = 0.$$

Due to (20), we can write

$$(B^2 - A)^{-1} (B^2 - A)^{1/2} (B + (B^2 - A)^{1/2})^{-1} (B^2 - A)^{1/2} x$$
$$= (B^2 - A)^{-1} (B^2 - A)^{1/2} (B^2 - A)^{1/2} (B + (B^2 - A)^{1/2})^{-1} x$$
$$= (B + (B^2 - A)^{1/2})^{-1} x,$$

which implies that $(B+(B^2-A)^{1/2})^{-1}x \in D(A)$ and thus

$$0 = (B-(B^2-A)^{1/2})(B+(B^2-A)^{1/2})A^{-1}A(B+(B^2-A)^{1/2})^{-1}x$$
$$= A(B+(B^2-A)^{1/2})^{-1}x.$$

Since A is one to one, $x = 0$, as desired. ■

Lemma 8. *Under assumptions* (3)∼(7)*, the following assertions are equivalent.*

1. $B+(B^2-A)^{1/2}$ *is boundedly invertible,*
2. $B-(B^2-A)^{1/2}$ *is boundedly invertible,*
3. $\forall y \in D((B^2-A)^{1/2})$ $(B^2-A)^{1/2}(A^{-1}B-BA^{-1})y = 0,$
4. $\forall y \in D(B)$ $(A^{-1}B-BA^{-1})y = 0,$
5. $D(BA) \subset D(B^3).$

Proof. From (22), $B+(B^2-A)^{1/2}$ will be boundedly invertible if and only if for any $y \in D((B^2-A)^{1/2})$

$$(B-(B^2-A)^{1/2})A^{-1}(B+(B^2-A)^{1/2})y = y.$$

Therefore, due to (23), $B+(B^2-A)^{1/2}$ is boundedly invertible if and only if for any $y \in D((B^2-A)^{1/2})$

$$(B-(B^2-A)^{1/2})(A^{-1}B-BA^{-1})y = 0. \tag{29}$$

Similarly $B-(B^2-A)^{1/2}$ is boundedly invertible if and only if for any $y \in D((B^2-A)^{1/2})$

$$(B+(B^2-A)^{1/2})(A^{-1}B-BA^{-1})y = 0. \tag{30}$$

On the other hand by Lemma 7, assertions 1 and 2 are equivalent and, in virtue of (29) and (30), imply

$$(B^2-A)^{1/2}(A^{-1}B-BA^{-1})y = 0, \tag{31}$$

for any $y \in D((B^2-A)^{1/2})$, i.e. assertion 3.

Now assume assertion 3 and let $y \in D(B)$. Then, from (12)

$$(B^2-A)^{-1/2}y \in D(B),$$

and

$$(B^2-A)^{1/2}(A^{-1}B-BA^{-1})(B^2-A)^{-1/2}y = 0,$$

thus

$$(B^2-A)^{1/2}A^{-1}B(B^2-A)^{-1/2}y - (B^2-A)^{1/2}BA^{-1}(B^2-A)^{-1/2}y = 0,$$

so, by (12) and (15), we get

$$(B^2-A)^{1/2}(B^2-A)^{-1/2}A^{-1}By - (B^2-A)^{1/2}(B^2-A)^{-1/2}BA^{-1}y = 0,$$

and

$$(A^{-1}B - BA^{-1})y = 0.$$

We then obtain assertion 4.

Now assume assertion 4. Then in virtue of (7), (29) holds. Hence $B + (B^2 - A)^{1/2}$ is boundedly invertible, that is assertion 1.

To conclude it is enough to prove that assertions 4 and 5 are equivalent. To this end assume assertion 4, then for $y \in D(BA)$ we write

$$By = A^{-1}BAy \in D(A),$$

so $By \in D(A) \subset D(B^2)$ and $y \in D(B^3)$. This gives assertion 5. Conversely if assertion 5 holds then, from (5), we deduce

$$\forall y \in D(B(B^2 - A)), \qquad (B^2 - A)By = B(B^2 - A)y,$$

which implies

$$\forall y \in D(B^3) \cap D(BA), \qquad B^3y - ABy = B^3y - BAy,$$

thus

$$\forall y \in D(BA), \qquad ABy = BAy,$$

from which assertion 4 follows. ■

Proof of Theorem 3. Let us assume (3)∼(8), and $D(BA) \subset D(B^3)$, let $f \in C^\theta([0,1];X)$, where $\theta \in]0,1[$, and $u_0, u_1 \in D(A)$. Our first step consists in finding a particular solution $\bar{u}(.)$ to (1). We introduce $\bar{u}(.)$ by

$$\begin{aligned} \bar{u}(t) = &-\frac{1}{2}\int_0^t V(t-s)(B^2 - A)^{-1/2} f(s)ds \\ &-\frac{1}{2}\int_t^1 U(s-t)(B^2 - A)^{-1/2} f(s)ds, \end{aligned} \tag{32}$$

for $0 \leqslant t \leqslant 1$, where $V(t)$ and $U(t)$ denote the analytic semigroups generated by $-B - (B^2 - A)^{1/2}$ and $B - (B^2 - A)^{1/2}$, respectively.

Then $\bar{u}(.)$ is strongly differentiable on $[0,1]$ and due to Lemma 5, we have

$$\begin{aligned} \bar{u}'(t) &= \frac{1}{2}\int_0^t V(t-s)(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2} f(s)ds \\ &\quad + \frac{1}{2}\int_t^1 U(s-t)(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2} f(s)ds \\ &= \frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}\int_0^t V(t-s)f(s)ds \\ &\quad + \frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2}\int_t^1 U(s-t)f(s)ds. \end{aligned}$$

Observe that

$$\begin{aligned}(B^2 - A)^{1/2}\bar{u}(0) &= -\frac{1}{2}\int_0^1 U(s)f(s)ds \\ &= -\frac{1}{2}\int_0^1 U(s)(f(s) - f(0))ds - \frac{1}{2}\int_0^1 U(s)f(0)ds.\end{aligned}$$

We recall that there exists $C > 0$ such that for any $s \in]0,1]$

$$\|(B - (B^2 - A)^{1/2})U(s)\|_X \le \frac{C}{s},$$

therefore, since $f \in C^{\theta}([0,1]; X)$, one has

$$\begin{aligned}&(B^2 - A)\bar{u}(0) \\ &\quad = -\frac{1}{2}(B^2 - A)^{1/2}(B - (B^2 - A)^{1/2})^{-1} \\ &\qquad \times \int_0^1 (B - (B^2 - A)^{1/2})U(s)(f(s) - f(0))ds \\ &\qquad - \frac{1}{2}(B^2 - A)^{1/2}(B - (B^2 - A)^{1/2})^{-1}(U(1) - U(0))f(0),\end{aligned}$$

it follows that

$$\bar{u}(0) \in D(B^2 - A) = D(A). \tag{33}$$

By the same way we obtain

$$\bar{u}(1) \in D(A). \tag{34}$$

Since f is Hölder-continuous, we also deduce that $\bar{u}(.)$ is twice continuously differentiable and

$$\begin{aligned}\bar{u}''(t) = &-\frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}(B + (B^2 - A)^{1/2})\int_0^t V(t-s)f(s)ds \\ &+\frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}f(t) \\ &-\frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2}(B - (B^2 - A)^{1/2})\int_t^1 U(s-t)f(s)ds \\ &-\frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2}f(t)\end{aligned}$$

$$= -\frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2} \int_0^t \frac{\partial V}{\partial s}(t-s)(f(s) - f(t))ds$$

$$- \frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}(I - V(t))f(t)$$

$$+ \frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}f(t)$$

$$- \frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2} \int_t^1 \frac{\partial U}{\partial s}(s-t)(f(s) - f(t))ds$$

$$- \frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2}(U(1-t) - I)f(t)$$

$$- \frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2}f(t)$$

$$= -\frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2} \int_0^t \frac{\partial V}{\partial s}(t-s)(f(s) - f(t))ds$$

$$- \frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2} \int_t^1 \frac{\partial U}{\partial s}(s-t)(f(s) - f(t))ds$$

$$+ \frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2} V(t)f(t)$$

$$- \frac{1}{2}(B - (B^2 - A)^{1/2})(B^2 - A)^{-1/2} U(1-t)f(t).$$

Moreover in virtue of Lemma 5, $\bar{u}(t) \in D(A)$, $A\bar{u}(.) \in C([0,1];X)$ and

$$A\bar{u}(t) = -\frac{1}{2}A(B^2 - A)^{-1/2}\left(\int_0^t V(t-s)f(s)ds + \int_t^1 U(s-t)f(s)ds\right)$$

$$= -\frac{1}{2}A(B^2 - A)^{-1/2}(B + (B^2 - A)^{1/2})^{-1} \int_0^t \frac{\partial V}{\partial s}(t-s)(f(s) - f(t))ds$$

$$- \frac{1}{2}A(B^2 - A)^{-1/2}(B + (B^2 - A)^{1/2})^{-1}(I - V(t))f(t)$$

$$- \frac{1}{2}A(B^2 - A)^{-1/2}(B - (B^2 - A)^{1/2})^{-1} \int_t^1 \frac{\partial U}{\partial s}(s-t)(f(s) - f(t))ds$$

$$- \frac{1}{2}A(B^2 - A)^{-1/2}(B - (B^2 - A)^{1/2})^{-1}(U(1-t) - I)f(t)$$

$$= -\frac{1}{2}A(B^2 - A)^{-1/2}(B + (B^2 - A)^{1/2})^{-1} \int_0^t \frac{\partial V}{\partial s}(t-s)(f(s) - f(t))ds$$

$$
\begin{aligned}
&-\frac{1}{2}A(B^2-A)^{-1/2}(B-(B^2-A)^{1/2})^{-1}\int_t^1 \frac{\partial U}{\partial s}(s-t)(f(s)-f(t))ds \\
&-\frac{1}{2}A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}f(t) \\
&+\frac{1}{2}A(B^2-A)^{-1/2}(B-(B^2-A)^{1/2})^{-1}f(t) \\
&+\frac{1}{2}A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}V(t)f(t) \\
&-\frac{1}{2}A(B^2-A)^{-1/2}(B-(B^2-A)^{1/2})^{-1}U(1-t)f(t).
\end{aligned}
$$

Now

$$
\begin{aligned}
&\frac{1}{2}A(B^2-A)^{-1/2}((B-(B^2-A)^{1/2})^{-1}-(B+(B^2-A)^{1/2})^{-1}) \\
&\quad =\frac{1}{2}A(B^2-A)^{-1/2}((B+(B^2-A)^{1/2})A^{-1}-(B-(B^2-A)^{1/2})A^{-1}) \\
&\quad =A(B^2-A)^{-1/2}(B^2-A)^{1/2}A^{-1} \\
&\quad =I,
\end{aligned}
$$

and thus

$$
\begin{aligned}
A\bar{u}(t)=f(t)&+\frac{1}{2}A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}V(t)f(t) \\
&-\frac{1}{2}A(B^2-A)^{-1/2}(B-(B^2-A)^{1/2})^{-1}U(1-t)f(t) \\
&-\frac{1}{2}A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}\int_0^t \frac{\partial V}{\partial s}(t-s)(f(s)-f(t))ds \\
&-\frac{1}{2}A(B^2-A)^{-1/2}(B-(B^2-A)^{1/2})^{-1}\int_t^1 \frac{\partial U}{\partial s}(s-t)(f(s)-f(t))ds.
\end{aligned}
$$

(notice that $A(B^2-A)^{-1/2}(B\pm(B^2-A)^{1/2})^{-1}\in L(X)$).

Since f is Hölder-continuous and from (8), it is well known that

$$
\int_0^t V(t-s)f(s)ds\in D(B+(B^2-A)^{1/2})=D((B^2-A)^{1/2})
$$

$$
\int_t^1 U(s-t)f(s)ds\in D(-B+(B^2-A)^{1/2})=D((B^2-A)^{1/2}),
$$

then $\bar{u}'(t)\in D(B)$ and

$$
\begin{aligned}
B\bar{u}'(t) &= -\frac{1}{2}B(B^2-A)^{-1/2}V(t)f(t) + \frac{1}{2}B(B^2-A)^{-1/2}U(1-t)f(t) \\
&\quad + \frac{1}{2}B(B+(B^2-A)^{1/2})^{-1} \\
&\quad \times \int_0^t \frac{\partial V}{\partial s}(t-s)(B+(B^2-A)^{1/2})(B^2-A)^{-1/2}(f(s)-f(t))ds \\
&\quad + \frac{1}{2}B(B-(B^2-A)^{1/2})^{-1} \\
&\quad \times \int_t^1 \frac{\partial U}{\partial s}(s-t)(B-(B^2-A)^{1/2})(B^2-A)^{-1/2}(f(s)-f(t))ds.
\end{aligned}
$$

Therefore

$$
\bar{u}''(t) + 2B\bar{u}'(t) + A\bar{u}(t) = f(t) + \frac{1}{2}((i)+(ii)+(iii)+(iv)),
$$

where

$$
\begin{aligned}
(i) &= \{-B(B^2-A)^{-1/2} + I + A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}\}V(t)f(t), \\
(ii) &= \{B(B^2-A)^{-1/2} + I - A(B^2-A)^{-1/2}(B-(B^2-A)^{1/2})^{-1}\}U(1-t)f(t), \\
(iii) &= -\{-B(B^2-A)^{-1/2} + I + A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}\} \\
&\quad \times \int_0^t \frac{\partial V}{\partial s}(t-s)(f(s)-f(t))ds, \\
(iv) &= \{B(B^2-A)^{-1/2} + I - A(B^2-A)^{-1/2}(B-(B^2-A)^{1/2})^{-1}\} \\
&\quad \times \int_t^1 \frac{\partial U}{\partial s}(s-t)(f(s)-f(t))ds.
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1} \\
&\quad = -(B^2-A)^{1/2}(B+(B^2-A)^{1/2})^{-1} \\
&\qquad + B(B^2-A)^{-1/2}B(B+(B^2-A)^{1/2})^{-1} \\
&\quad = -(B^2-A)^{1/2}(B+(B^2-A)^{1/2})^{-1} \\
&\qquad + B(B^2-A)^{-1/2} - B(B+(B^2-A)^{1/2})^{-1} \\
&\quad = B(B^2-A)^{-1/2} - I.
\end{aligned}
$$

Hence $(i) = (iii) = 0$.

Analogously it is readily seen that $(ii) = (iv) = 0$. We have proved that $\bar{u}(t)$ is the unique strict solution of (1) satisfying the boundary conditions

$$\bar{u}(0) = -\frac{1}{2}\int_0^1 U(s)(B^2 - A)^{-1/2} f(s)ds,$$

$$\bar{u}(1) = -\frac{1}{2}\int_0^1 V(1-s)(B^2 - A)^{-1/2} f(s)ds.$$

To conclude our proof, let us now consider the homogeneous problem

$$v''(t) + 2Bv'(t) + Av(t) = 0, \qquad t \in [0,1], \tag{35}$$

with non-homogeneous boundary conditions

$$v(0) = x_0, \qquad v(1) = x_1, \tag{36}$$

where $x_0, x_1 \in D(A)$. We have the lemma as follows.

Lemma 9. *Assume* (3)$\sim$(8) *and* $D(BA) \subset D(B^3)$. *If* $x_0, x_1 \in D(A)$, *then problem* (35)–(36) *has a unique strict solution.*

Proof. It suffices to show that under the indicated assumptions, problem (35)–(36) has one strict solution. To accomplish this, we in fact furnish an explicit solution to it, precisely

$$v(t) = V(t)\xi_0 + U(1-t)\xi_1, \tag{37}$$

where

$$\begin{cases} Z = e^{-2(B^2-A)^{1/2}} \\ \xi_0 = (I - Z)^{-1}(x_0 - U(1)x_1) \\ \xi_1 = (I - Z)^{-1}(x_1 - V(1)x_0). \end{cases}$$

Notice that since the imaginary axis is contained in the resolvent set

$$\rho(-(B^2 - A)^{1/2}),$$

$I - Z$ has a bounded inverse (see Lunardi [11], p. 60)

$$(I - Z)^{-1} = \frac{1}{2\pi i}\int_{\gamma_\#} \frac{e^{2z}}{1 - e^{2z}}(zI + (B^2 - A)^{1/2})^{-1} dz + I,$$

where $\gamma_\# = \gamma_1 - \gamma_2$ is a suitable curve in the complex plane (see Lunardi [11], p. 59). On the other hand, since $x_0, x_1 \in D(A)$ and in virtue of assumption (4), there exists $\eta \in X$ such that

$$(zI + (B^2 - A)^{1/2})^{-1}(x_0 - U(1)x_1)$$
$$= (zI + (B^2 - A)^{1/2})^{-1}(B^2 - A)^{-1}\eta$$
$$= (B^2 - A)^{-1}(zI + (B^2 - A)^{1/2})^{-1}\eta \in D(A),$$

therefore $\xi_0 = (I - Z)^{-1}(x_0 - U(1)x_1) \in D(A)$. Similarly $\xi_1 \in D(A)$.

From (20) of Lemma 5 it follows that for $z \in \rho(-B - (B^2 - A)^{1/2})$ and $y \in D((B^2 - A)^{1/2})$ that

(38)
$$\begin{aligned}
&(zI + B + (B^2 - A)^{1/2})^{-1}By \\
&\quad = (zI + B + (B^2 - A)^{1/2})^{-1} \\
&\qquad \times ((zI + B + (B^2 - A)^{1/2})y - zy - (B^2 - A)^{1/2}y) \\
&\quad = y - z(zI + B + (B^2 - A)^{1/2})^{-1}y \\
&\qquad - (B^2 - A)^{1/2}(zI + B + (B^2 - A)^{1/2})^{-1}y \\
&\quad = ((zI + B + (B^2 - A)^{1/2}) - zI - (B^2 - A)^{1/2}) \\
&\qquad \times (zI + B + (B^2 - A)^{1/2})^{-1}y \\
&\quad = B(zI + B + (B^2 - A)^{1/2})^{-1}y.
\end{aligned}$$

Analogously

(39)
$$(\lambda I - B + (B^2 - A)^{1/2})^{-1}By = B(\lambda I - B + (B^2 - A)^{1/2})^{-1}y$$

for $\lambda \in \rho(B - (B^2 - A)^{1/2})$ and $y \in D((B^2 - A)^{1/2})$. Hence, again using the second part of Lemma 5 one gets

$$(zI + B + (B^2 - A)^{1/2})^{-1}(\lambda - B + (B^2 - A)^{1/2})y$$
$$= (\lambda - B + (B^2 - A)^{1/2})(zI + B + (B^2 - A)^{1/2})^{-1}y,$$

which yields

(40)
$$\begin{aligned}
&(\lambda - B + (B^2 - A)^{1/2})^{-1}(zI + B + (B^2 - A)^{1/2})^{-1} \\
&\quad = (zI + B + (B^2 - A)^{1/2})^{-1}(\lambda - B + (B^2 - A)^{1/2})^{-1}.
\end{aligned}$$

It follows from (40) that $U(t)$ and $V(t)$ commute, and

$$\frac{d}{dt}(U(t)V(t)) = -2U(t)(B^2 - A)^{1/2}V(t) = -2(B^2 - A)^{1/2}U(t)V(t),$$

which implies

$$U(t)V(t) = V(t)U(t) = e^{-2t(B^2-A)^{1/2}},$$

in particular

$$U(1)V(1) = V(1)U(1) = Z.$$

Since

$$U(1)(I-Z)^{-1} = (I-Z)^{-1}U(1),$$

on $D(A) = D(B^2 - A)$ (see Lemma 5), we have

$$\begin{aligned} v(0) &= \xi_0 + U(1)\xi_1 \\ &= (I-Z)^{-1}(x_0 - U(1)x_1) + U(1)(I-Z)^{-1}(x_1 - V(1)x_0) \\ &= (I-Z)^{-1}(x_0 - U(1)x_1) + (I-Z)^{-1}U(1)(x_1 - V(1)x_0) \\ &= x_0. \end{aligned}$$

Analogousely

$$v(1) = x_1.$$

We also have that $v(.)$ is strongly differentiable for $t \in [0,1]$ and

$$v'(t) = -V(t)(B + (B^2-A)^{1/2})\xi_0 - U(1-t)(B - (B^2-A)^{1/2})\xi_1.$$

Recall that since $\xi_0, \xi_1 \in D(A)$, then

$$(B + (B^2-A)^{1/2})(B - (B^2-A)^{1/2})\xi_i = A\xi_i, \qquad i \in \{0,1\}.$$

By virtue of (38) and (39) one has for $y \in D((B^2-A)^{1/2})$

$$BV(t)y = V(t)By, \qquad BU(1-t)y = U(1-t)By.$$

Therefore

$$\begin{aligned} 2Bv'(t) = &-V(t)2B(B - (B^2-A)^{1/2})^{-1}A\xi_0 \\ &- U(1-t)2B(B + (B^2-A)^{1/2})^{-1}A\xi_1; \end{aligned} \tag{41}$$

moreover, Lemma 5 guarantees that v is two times differentiable and

$$\begin{aligned} v''(t) = &V(t)(2B^2 - A + 2B(B^2-A)^{1/2})\xi_0 \\ &+ U(1-t)(2B^2 - A - 2B(B^2-A)^{1/2})\xi_1. \end{aligned} \tag{42}$$

Commutativity of the involved operators yields that for $t \in [0,1]$, $v(t) \in D(A)$ and

$$Av(t) = V(t)A\xi_0 + U(1-t)A\xi_1. \tag{43}$$

Summing (41), (42) and (43) we get by Lemmas 7 and 8

$$\begin{aligned} & v''(t) + 2Bv'(t) + Av(t) \\ & \quad = V(t)[2B(B + (B^2 - A)^{1/2}) - 2B(B - (B^2 - A)^{1/2})^{-1}A]\xi_0 \\ & \qquad + U(1-t)[2B(B - (B^2 - A)^{1/2}) - 2B(B + (B^2 - A)^{1/2})^{-1}A]\xi_1 \\ & \quad = 0, \end{aligned}$$

for any $t \in [0,1]$. ■

To conclude the proof of Theorem 3, note that due to (33), (34)

$$u_0 - \bar{u}(0) \in D(A), \qquad u_1 - \bar{u}(1) \in D(A).$$

Now, let $\bar{\bar{u}}$ the strict solution of problem (35)–(36) with

$$x_0 = u_0 - \bar{u}(0), \qquad x_1 = u_1 - \bar{u}(1),$$

then it is a simple matter to recognize that

$$u(.) = \bar{u}(.) + \bar{\bar{u}}(.),$$

is the unique solution to problem (1)–(2).

3. Maximal regularity of the strict solution

In this section we will prove the following maximal regularity theorem.

Theorem 10. *Under assumptions* (3) ∼ (8)*, if, in addition* $D(BA) \subset D(B^3)$*, then for all* $f \in C^\theta([0,1]; X)$*,* $0 < \theta < 1$ *and any* $u_0, u_1 \in D(A)$ *satisfying*

$$f(i), Au_i \in D_{-(B^2-A)}(\theta/2, \infty) = (D(A), X)_{1-\theta/2,\infty}, \qquad i = 0, 1,$$

the unique strict solution u *to problem* (1)–(2) *has the maximal regularity property*: $u'', Bu', Au \in C^\theta([0,1]; X)$.

Here, $D_{-(B^2-A)}(\theta/2, \infty)$ is the well known real interpolation space

$$(D(B^2 - A), X)_{1-\theta/2,\infty},$$

characterized by

$$\begin{aligned} & D_{-(B^2-A)}(\theta/2; +\infty) \\ & \qquad = \left\{ \varphi \in X : \sup_{r>0} r^{\theta/2} \|(B^2 - A)(rI + B^2 - A)^{-1}\varphi\|_X < \infty \right\}. \end{aligned}$$

Proof. We recall that

$$\begin{aligned} u(t) &= V(t)\xi_0 + U(1-t)\xi_1 \\ &\quad - \frac{1}{2}(B^2-A)^{-1/2}\left(\int_0^t V(t-s)f(s)ds + \int_t^1 U(s-t)f(s)ds\right) \end{aligned}$$

where

$$\begin{aligned} \xi_0 &= (I-Z)^{-1}(u_0 - U(1)u_1) \\ &\quad + \frac{1}{2}(I-Z)^{-1}(B^2-A)^{-1/2}\left(\int_0^1 U(s)f(s)ds - U(1)\int_0^1 V(1-s)f(s)ds\right), \end{aligned}$$

$$\begin{aligned} \xi_1 &= (I-Z)^{-1}(u_1 - V(1)u_0) \\ &\quad + \frac{1}{2}(I-Z)^{-1}(B^2-A)^{-1/2}\left(\int_0^1 V(1-s)f(s)ds - V(1)\int_0^1 U(s)f(s)ds\right). \end{aligned}$$

(See section 2).

One writes

$$\begin{aligned} A &= (B + (B^2-A)^{1/2})(B - (B^2-A)^{1/2}) \\ &= (B - (B^2-A)^{1/2})(B + (B^2-A)^{1/2}), \end{aligned}$$

and

$$\begin{aligned} Au(t) &= (V(t)A\xi_0 + U(1-t)A\xi_1) \\ &\quad + \left(-\frac{1}{2}A(B^2-A)^{-1/2}\left(\int_0^t V(t-s)f(s)ds + \int_t^1 U(s-t)f(s)ds\right)\right) \\ &= (I) + (II). \end{aligned}$$

Now

$$\begin{aligned} (II) &= -\frac{1}{2}(B(B^2-A)^{-1/2} - I)(B + (B^2-A)^{1/2})\int_0^t V(t-s)f(s)ds \\ &\quad - \frac{1}{2}(B(B^2-A)^{-1/2} + I)(B - (B^2-A)^{1/2})\int_t^1 U(s-t)f(s)ds \\ &= -\frac{1}{2}(B(B^2-A)^{-1/2} - I)\int_0^t \frac{\partial V}{\partial s}(t-s)(f(s)-f(t))ds \\ &\quad - \frac{1}{2}(B(B^2-A)^{-1/2} + I)\int_t^1 \frac{\partial U}{\partial s}(s-t)(f(s)-f(t))ds \end{aligned}$$

$$-\frac{1}{2}(B(B^2-A)^{-1/2}-I)f(t)+\frac{1}{2}(B(B^2-A)^{-1/2}-I)V(t)f(t)$$

$$+\frac{1}{2}(B(B^2-A)^{-1/2}+I)f(t)-\frac{1}{2}(B(B^2-A)^{-1/2}+I)U(1-t)f(t)$$

$$=-\frac{1}{2}(B(B^2-A)^{-1/2}-I)\int_0^t\frac{\partial V}{\partial s}(t-s)(f(s)-f(t))ds$$

$$-\frac{1}{2}(B(B^2-A)^{-1/2}+I)\int_t^1\frac{\partial U}{\partial s}(s-t)(f(s)-f(t))ds$$

$$+f(t)+\frac{1}{2}(B(B^2-A)^{-1/2}-I)V(t)(f(t)-f(0))$$

$$+\frac{1}{2}(B(B^2-A)^{-1/2}-I)V(t)f(0)$$

$$-\frac{1}{2}(B(B^2-A)^{-1/2}+I)U(1-t)(f(t)-f(1))$$

$$-\frac{1}{2}(B(B^2-A)^{-1/2}+I)U(1-t)f(1).$$

Therefore, $(II)\in C^{\theta}([0,1];X)$ provided that $f\in C^{\theta}([0,1];X)$ and

$$f(0)\in D_{-B-(B^2-A)^{1/2}}(\theta,\infty)=D_{-(B^2-A)^{1/2}}(\theta,\infty)=D_{-(B^2-A)}(\theta/2,\infty)$$

$$f(1)\in D_{B-(B^2-A)^{1/2}}(\theta,\infty)=D_{-(B^2-A)^{1/2}}(\theta,\infty)=D_{-(B^2-A)}(\theta/2,\infty),$$

(see [2], Proposition 1.3 and Theorem 1.4, pp. 360–361).

We turn to (I). One has

$$(I)=(I-Z)^{-1}V(t)Au_0-(I-Z)^{-1}U(1)V(t)Au_1$$

$$+(I-Z)^{-1}U(1-t)Au_1-(I-Z)^{-1}V(1)U(1-t)Au_0$$

$$+\frac{1}{2}(I-Z)^{-1}V(t)A(B^2-A)^{-1/2}\int_0^1U(s)f(s)ds$$

$$-\frac{1}{2}(I-Z)^{-1}V(t)U(1)A(B^2-A)^{-1/2}\int_0^1V(1-s)f(s)ds$$

$$+\frac{1}{2}(I-Z)^{-1}U(1-t)A(B^2-A)^{-1/2}\int_0^1V(1-s)f(s)ds$$

$$-\frac{1}{2}(I-Z)^{-1}V(1)U(1-t)A(B^2-A)^{-1/2}\int_0^1U(s)f(s)ds$$

$$=(I_1)+(I_2)+(I_3)+(I_4)+(I_5)+(I_6).$$

Since $Au_0, Au_1 \in D_{-(B^2-A)}(\theta/2, \infty)$, then $(I_1), (I_2) \in C^\theta([0,1]; X)$. Write

$$\begin{aligned}
&A(B^2-A)^{-1/2}\int_0^1 U(s)f(s)ds \\
&\quad = (B(B^2-A)^{-1/2}+I)(B-(B^2-A)^{1/2})\int_0^1 U(s)f(s)ds \\
&\quad = (B(B^2-A)^{-1/2}+I)(B-(B^2-A)^{1/2})\int_0^1 U(s)(f(s)-f(0))ds \\
&\qquad + (B(B^2-A)^{-1/2}+I)(B-(B^2-A)^{1/2})\int_0^1 U(s)f(0)ds \\
&\quad = (B(B^2-A)^{-1/2}+I)\int_0^1 (B-(B^2-A)^{1/2})U(s)(f(s)-f(0))ds \\
&\qquad + (B(B^2-A)^{-1/2}+I)(U(1)-I)f(0).
\end{aligned}$$

Now, it is known that

$$\int_0^1 (B-(B^2-A)^{1/2})U(s)(f(s)-f(0))ds \in D_{B-(B^2-A)^{1/2}}(\theta, \infty),$$

see [2], Theorem 1.4, p. 361. Thus

$$V(t)\int_0^1 (B-(B^2-A)^{1/2})U(s)(f(s)-f(0))ds \in C^\theta([0,1]; X).$$

On the other hand, the assumption on $f(0)$ implies $V(t)f(0) \in C^\theta([0,1]; X)$. Then $(I_3) \in C^\theta([0,1]; X)$.

Concerning (I_4), one writes

$$\begin{aligned}
&A(B^2-A)^{-1/2}\int_0^1 V(1-s)f(s)ds \\
&\quad = (B(B^2-A)^{-1/2}-I)(B+(B^2-A)^{1/2})\int_0^1 V(1-s)(f(s)-f(1))ds \\
&\qquad + (B(B^2-A)^{-1/2}-I)(B+(B^2-A)^{1/2})\int_0^1 V(1-s)f(1)ds \\
&\quad = (B(B^2-A)^{-1/2}-I)\int_0^1 (B+(B^2-A)^{1/2})V(1-s)(f(s)-f(1))ds \\
&\qquad + (B(B^2-A)^{-1/2}-I)(I-V(1))f(1).
\end{aligned}$$

The same arguments used below imply $(I_4) \in C^\theta([0,1]; X)$.

As well (I_5) and (I_6) are handled analogously by changing $V(t)$ to $U(1-t)$. Therefore, under the preceding assumptions, $Au(.) \in C^\theta([0,1];X)$. On the other hand

$$\begin{aligned} Bu'(t) &= -V(t)B(B+(B^2-A)^{1/2})\xi_0 - U(1-t)B(B-(B^2-A)^{1/2})\xi_1 \\ &\quad + \frac{1}{2}B(B^2-A)^{-1/2}\Big\{(B+(B^2-A)^{1/2}) \\ &\qquad \cdot \int_0^t V(t-s)(f(s)-f(t))ds + (I-V(t))f(t)\Big\} \\ &\quad + \frac{1}{2}B(B^2-A)^{-1/2}\Big\{(B-(B^2-A)^{1/2}) \\ &\qquad \cdot \int_t^1 U(s-t)(f(s)-f(t))ds + (U(1-t)-I)f(t)\Big\} \\ &= (J_1)+(J_2)+(J_3)+(J_4). \end{aligned}$$

The previous arguments apply to (J_3) and (J_4). Moreover, since $\xi_0, \xi_1 \in D(A)$, we get

$$B(B+(B^2-A)^{1/2})A^{-1}A\xi_0 = B(B-(B^2-A)^{1/2})^{-1}A\xi_0$$

$$B(B-(B^2-A)^{1/2})A^{-1}A\xi_1 = B(B+(B^2-A)^{1/2})^{-1}A\xi_1.$$

But we already know that $V(.)A\xi_0 \in C^\theta([0,1];X)$, $U(1-.)A\xi_1 \in C^\theta([0,1];X)$. ■

4. Examples

Example 1 (Periodic boundary conditions).

Take $X = L^2(0,1)$ and let us introduce an operator $T: D(T) \subset X \to X$ defined by

$$\begin{cases} D(T) = \{f \in H^1(0,1) : f(0) = f(1)\} \\ Tf = if'. \end{cases}$$

It is well known that T is self-adjoint and its spectrum is $\sigma(T) = 2\pi\boldsymbol{Z}$, (see [12], p. 75). So that T^2, where

$$\begin{cases} D(T^2) = \{f \in H^2(0,1) : f(0) = f(1), f'(0) = f'(1)\} \\ T^2 f = -f'', \end{cases}$$

is positive self-adjoint. We then introduce $B = -iT$ and A defined by

$$\begin{cases} D(A) = D(T^2) \\ Af = (-2T^2 - aI)f = 2f'' - af, \end{cases}$$

(where $a > 0$). Then $B^2 - A = T^2 + aI$ with domain $D(T^2)$ is positive self-adjoint. Therefore $D(T)$ coincides with the complex interpolation space $[X, D(T^2)]_{1/2}$ (see [15], p. 143), and $(T^2 + aI)^{1/2}$ is positive self-adjoint. We have from Remark 1, statement 5, that $\pm B - (B^2 - A)^{1/2}$ generates an analytic semigroup in X.

It follows that we can solve the boundary-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) + 2\dfrac{\partial^2 u}{\partial x \partial t}(x,t) + 2\dfrac{\partial^2 u}{\partial x^2}(x,t) - au(x,t) \\ \qquad = f(x,t), \quad (x,t) \in (0,1) \times (0,1), \\ u(x,0) = u_0(x), \quad 0 < x < 1, \\ u(x,1) = u_1(x), \quad 0 < x < 1, \\ u(0,t) = u(1,t), \quad 0 < t < 1, \\ \dfrac{\partial u}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(1,t), \quad 0 < t < 1, \\ \dfrac{\partial u}{\partial t}(0,t) = \dfrac{\partial u}{\partial t}(1,t), \quad 0 < t < 1, \end{cases}$$

with $u_0, u_1 \in D(A) = D(T^2)$, provided that $f \in C^\theta([0,1]; L^2(0,1))$.

Example 2 (Degenerate parabolic operators).

Let $a \in C^1([0,1])$ be a real valued function which is strictly positive on $(0,1)$, $a(0) = a(1) = 0$. Let us define the differential operator

$$Tu = \frac{d}{dx}\left(a\frac{du}{dx}\right), \qquad u \in D(T),$$

where

$$D(T) = \{u \in L^2(0,1) : \\ u \text{ is locally absolutely continuous in } (0,1) \text{ and } au' \in H_0^1(0,1)\}.$$

Then it shown (see [1], Lemma 2.7 and Theorem 2.8), that T is self-adjoint and generates an analytic semigroup with angle $\pi/2$ and bounded in $L^2(0,1)$. Let

$$\begin{cases} D(B) = D(T) \\ B = iT, \end{cases}$$

and

$$\begin{cases} D(A) = D(T^2) = \{u \in L^2(0,1) : au' \in H_0^1(0,1) \text{ and } a(au')'' \in H_0^1(0,1)\} \\ A = -\alpha T^2 - cI \end{cases}$$

where $\alpha > 1$ and $c > 0$

Since

$$B^2 - A = (\alpha - 1)T^2 + cI,$$

$B^2 - A$ is a positive self-adjoint operator. Then $(B^2 - A)^{1/2}$ is positive with domain $D(T)$ and thus by Remark 1 (statement 5) $\pm B - (B^2 - A)^{1/2}$ generates an analytic semigroup in X, with domain $D(T)$. Hence we can handle the boundary-value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) + 2i\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial^2 u}{\partial x \partial t}(x,t)\right) \\ \qquad - \alpha\dfrac{\partial}{\partial x}\left(a(x)\dfrac{\partial^2}{\partial x^2}\left(a(x)\dfrac{\partial u}{\partial x}(x,t)\right)\right) - cu(x,t) \\ \qquad = f(x,t), \quad (x,t) \in (0,1) \times (0,1), \\ u(x,0) = u_0(x), \quad 0 < x < 1, \\ u(x,1) = u_1(x), \quad 0 < x < 1, \\ \left(a\dfrac{\partial u}{\partial x}\right)(0,t) = \left(a\dfrac{\partial u}{\partial x}\right)(1,t) = 0, \quad 0 < t < 1, \\ \left(a\dfrac{\partial^2}{\partial x^2}\left(a\dfrac{\partial u}{\partial x}\right)\right)(0,t) = \left(a\dfrac{\partial^2}{\partial x^2}\left(a\dfrac{\partial u}{\partial x}\right)\right)(1,t) = 0, \quad 0 < t < 1, \\ \left(a\dfrac{\partial^2 u}{\partial x \partial t}\right)(0,t) = \left(a\dfrac{\partial^2 u}{\partial x \partial t}\right)(1,t) = 0, \quad 0 < t < 1, \end{cases}$$

with $u_0, u_1 \in D(A)$ provided that $f \in C^\theta([0,1]; L^2(0,1))$.

Example 3.

To begin with, we recall that if M is a non-negative self-adjoint operator in the Hilbert space X, then M has the imaginary power $M^{it} \in L(X)$ and $\|M^{it}\| \leq C$ for $|t| \leq \varepsilon$, where ε, C are suitable positive constants (see [15], p. 143). Therefore the complex interpolation space $[X, D(M^n)]_{m/n}$ coincides with $D(M^m)$ for all $m, n \in \mathbf{N}$, $m < n$ (see [15], p. 103).

Take B a strictly positive self-adjoint operator in X and let $A = -B^3$. Then

$$B^2 - A = B^2 + B^3 = B^2 + (B^2)^{3/2},$$

is strictly positive self-adjoint, and

$$D((B^2 - A)^{1/2}) = [X, D(B^3)]_{1/2}$$
$$= D(B^{3/2}).$$

$-(B^2 - A)^{1/2}$ generates of course an analytic semigroup. Moreover there exists a constant $C > 0$ such that

$$\|Bu\| \le C\|u\|^{1/3}\|B^{3/2}u\|^{2/3}$$
$$\le C\|u\|^{1/3}\|B^{3/2}(B^2 - A)^{-1/2}(B^2 - A)^{1/2}u\|^{2/3},$$

for $u \in D(B^{3/2})$, so that $\pm B$ is bounded with respect to $-(B^2 - A)^{1/2}$ and has a $(B^2 - A)^{1/2}$ bound equal to 0. It follows that $\pm B - (B^2 - A)^{1/2}$ generates an analytic semigroup in X. In this case $D(A) \subsetneq D(B^2)$, however all our assumptions are satisfied.

As an example, take A, B defined in $X = L^2(\Omega)$ by

$$D(B) = H_0^1(\Omega) \cap H^2(\Omega), \qquad B = -\Delta,$$
$$D(A) = \{u \in H^6(\Omega); u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = \Delta^2 u_{|\partial\Omega} = 0\}, \qquad A = \Delta^3,$$

where Ω is a bounded domain in $\boldsymbol{R}^q$, $q \ge 1$, with a smooth boundary $\partial\Omega$.

One can then handles the boundary problem in a cylinder

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - 2\Delta \dfrac{\partial u}{\partial t}(x,t) + \Delta^3 u(x,t) = f(x,t), & (x,t) \in \Omega \times (0,1), \\ u(x,0) = u_0(x),\ u(x,1) = u_1(x), & x \in \Omega, \\ u(\sigma,t) = \Delta u(\sigma,t) = \Delta^2 u(\sigma,t) = 0, & (\sigma,t) \in \partial\Omega \times (0,1), \\ \dfrac{\partial u}{\partial t}(\sigma,t) = 0, & (\sigma,t) \in \partial\Omega \times (0,1), \end{cases}$$

provided that $f \in C^\theta([0,1]; L^2(\Omega))$; $u_0, u_1 \in D(A)$. Maximal regularity of solutions is correspondingly treated.

Example 4.

Let K be the infinitesimal generator of an analytic semigroup of angle π in the complex Banach space X, i.e., K is closed linear, densely define and for each $\varepsilon \in]0, \pi/2[$ there exists an $M_\varepsilon \ge 1$ such that

$$\|(\lambda I - K)^{-1}\| \le \frac{M_\varepsilon}{|\lambda|},$$

for all $\lambda \in \Sigma_{\pi-\varepsilon}$, where

$$\Sigma_{\pi-\varepsilon} = \{z \in \boldsymbol{C}^*; |\arg z| < \pi - \varepsilon\}.$$

Suppose that $0 \in \rho(K)$ too. Then for all $n \in \boldsymbol{N}^*$, $-K^{2^n}$ also generates an analytic semigroup of angle π. Indeed it suffices to prove this statement for $n = 1$. Let $\lambda = re^{i\theta}$ where $r > 0$, $\theta \in [0, \pi[\cup]\pi, 2\pi]$. One writes

$$\lambda I + K^2 = re^{i\theta} I + K^2 = (\sqrt{r} e^{i(\pi+\theta)/2} - K)(\sqrt{r} e^{i(3\pi+\theta)/2} - K).$$

Now

$$\theta \in [0, \pi[\Rightarrow (\pi + \theta)/2 \in [\pi/2, \pi[\qquad \text{and} \qquad (3\pi + \theta)/2 \in [3\pi/2, 2\pi[,$$

$$\theta \in]\pi, 2\pi] \Rightarrow (\pi + \theta)/2 \in]\pi, 3\pi/2] \qquad \text{and} \qquad (3\pi + \theta)/2 \in [2\pi, 5\pi/2[,$$

and thus the assumption yields easily the conclusion. Take $B = -K$, $A = -K^{2^r}$ so that $B^2 - A = K^2 + K^{2^r}$. Let $r \geq 2$ and assume $-K$ to admit bounded imaginary powers. It follows that $B^2 - A$ has bounded imaginary powers. Therefore

$$D((B^2 - A)^{1/2}) = [X, D(B^2 - A)]_{1/2} = [X, D(K^{2^r})]_{1/2} = D(K^{2^{r-1}}),$$

(see [15], p. 151). $-(B^2 - A)^{1/2}$ generates an analytic semigroup in X. Moreover, moment's inequality yields that $\pm B$ is $(B^2 - A)^{1/2}$ bounded with bound equal to 0 (see [8], p. 65, whose proof is readily extended to arbitrary C_0-semigroup). Then $\pm B - (B^2 - A)^{1/2}$ generates an analytic semigroup and all our results work.

Concerning this example, take $X = L^p(\Omega)$, $1 < p < \infty$, when Ω is a bounded domain in $\boldsymbol{R}^n$, $n \geqslant 1$, with a smooth boundary $\partial\Omega$. If $B = -\Delta$ with

$$D(B) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

and $A = -\Delta^4$ with

$$D(A) = \{u \in W^{8,p}(\Omega) : \Delta^j u_{|\partial\Omega} = 0 \text{ for } j = 0, 1, 2, 3\},$$

then our conditions apply.

Take also $B = -\Delta + kI$ with

$$D(B) = \left\{ u \in W^{2,p}(\Omega) : \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \right\},$$

and k strictly positive. Consider $A = -\Delta^4$ with

$$D(A) = \left\{ u \in W^{8,p}(\Omega) : \frac{\partial \Delta^j u}{\partial \nu} = 0 \text{ on } \partial\Omega \text{ for } j = 0, 1, 2, 3 \right\}.$$

Then we are in the situation described above, again.

Example 5.

Let B be the generator of a bounded analytic semigroup (therefore B is of negative type) and suppose $0 \in \rho(B)$. Observe that

$$(B^2)^{1/2} = -B.$$

Take $A = bB^2$, when $b < 0$. Then

$$B^2 - A = (1 - b)B^2,$$

yields

$$(B^2 - A)^{1/2} = -(1 - b)^{1/2} B,$$

and thus

$$\pm B - (B^2 - A)^{1/2} = \pm B + (1 - b)^{1/2} B = ((1 - b)^{1/2} \pm 1)B$$

generates an analytic semigroup. Since all other assumptions (3)$\sim$(7) hold together with $D(BA) \subset D(B^3)$, our results work.

As an example, if $X = L^p(\Omega)$, $1 < p < \infty$, Ω is a bounded domain in $\boldsymbol{R}^n$, $n \geqslant 1$, with a smooth boundary $\partial\Omega$, $B = \Delta$ with

$$D(B) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

and $A = b\Delta^2$, $b < 0$, with

$$D(A) = \{u \in W^4(\Omega) : u_{|\partial\Omega} = \Delta u_{|\partial\Omega} = 0\},$$

we can handle the boundary value problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) + 2\Delta \dfrac{\partial u}{\partial t}(x,t) + b\Delta^2 u(x,t) = f(x,t), \quad (x,t) \in \Omega \times (0,1), \\ u(x,0) = u_0(x), \quad x \in \Omega, \\ u(x,1) = u_1(x), \quad x \in \Omega, \\ u(x,t) = \Delta u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,1), \\ \dfrac{\partial u}{\partial t}(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,1), \end{cases}$$

provided that $f \in C^\theta([0,1]; L^p(\Omega))$; $u_0, u_1 \in D(A)$.

Maximal regularity is obtained using Theorem 10.

Example 6 (Degenerate parabolic equations, continued).

Refering to Example 2, take

$$B = i(T - bI), \qquad D(B) = D(T),$$

when b is a positive real number, and let A be the operator defined by

$$\begin{cases} D(A) = D(T^2) \\ A = -\alpha T^2 - cI, \end{cases}$$

when $\alpha > 1$, $c > 0$. Then $D(A) = D(B^2)$ and

$$B^2 - A = (\alpha - 1)T^2 + 2bT + (c - b^2)I$$

is self-adjoint.

Notice that T is a negative operator so that $-(-T)^{1/2}$ generates a bounded analytic semigroup in $X = L^2(0,1)$, with

$$\|e^{-t(-T)^{1/2}}\|_{L(X)} \leqslant 1, \qquad t \geqslant 0.$$

Take $u \in D(A)$ and evaluate

$$\begin{aligned}\langle (B^2 - A)u, u\rangle &= (\alpha - 1)\|Tu\|^2 - 2b\|(-T)^{1/2}u\|^2 + (c - b^2)\|u\|^2 \\ &\geqslant (\alpha - 1)\|Tu\|^2 - 4b\|Tu\|\,\|u\| + (c - b^2)\|u\|^2 \\ &\geqslant (\alpha - 1 - 2b\varepsilon)\|Tu\|^2 + (c - b^2 - 2b/\varepsilon)\|u\|^2\end{aligned}$$

for all $\varepsilon > 0$. (See [8], Theorem 9.9, p. 65). Take $\varepsilon = (\alpha - 1)/2b$.

To conclude, we see that if

$$c > b^2 \frac{\alpha + 3}{\alpha - 1}$$

then $B^2 - A$ is a strictly positive operator. Since all assumptions (4)$\sim$(8) hold (see Remark 1.4), our results apply as well.

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