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Hopf Bifurcation for a Class of Partial Differential Equation with Delay

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Abstract. We study a kind of delayed reaction-diffusion equation with Dirichlet boundary condition. We show the existence of a sequence of values $\{\tau_{k_n}\}_{n=0,1,2,\dots}$ of the parameter τ such that a Hopf bifurcation occurs when the delay passes through each value $\{\tau_{k_n}\}$. The main techniques used here are some results on nonlinear eigenvalue problems, the analysis of the characteristic equation of the linearized problem, the Liapunov-Schmidt method and the implicit function theorem.

Key Words and Phrases. Partial functional-differential equation, Distributed delay, Periodic solution, Hopf bifurcation, Population dynamics.

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1. Introduction

Reaction-diffusion equations with delay are used as models for some problems in ecology and population dynamic, also for physical and chemical systems. We are specially concerned in the following reaction-diffusion equation with distributed delay incorporated to the control and Dirichlet boundary condition:

$$(1)\qquad \begin{cases} \dfrac{\partial U(t,x)}{\partial t} = \dfrac{\partial^2 U(t,x)}{\partial x^2} + kU(t,x) + \dfrac{k}{\delta}\displaystyle\int_{-\tau}^{-\tau+\delta} g(U(t,x), U(t+s,x))ds, \\ U(t,0) = U(t,\pi) = 0, \quad t \geq 0, \\ U(t,x) = \psi(t,x), \quad (t,x) \in [-\tau,0]\times[0,\pi], \\ \psi \in C([-\tau,0], H_0^1) \end{cases}$$

with k, τ and δ positive constants, $0 < \delta \leq \tau$.

This equation describes the dynamics of a single species population on the interval $[0,\pi]$ and $U(t,x)$ represents the size of the population on (t,x). This boundary condition describes the situation when the environment is surrounded by a totally unfavorable region in which the population density cannot attain positive values. This work is motivated by [1] and [3]. We have generalized [1] and [3], in the sense that we consider distributed delay and the equation in

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[1] is a limit for the equation in this paper when $\delta \to 0$. We consider that kind of control motivated by [5] and [6].

We look for periodic solutions arising by Hopf bifurcation. It is the purpose of this paper to prove the existence of a sequence of Hopf bifurcations which arise from positive equilibrium (spatially nonconstant) as the delay τ varies.

For us $g : \boldsymbol{R}^2 \to \boldsymbol{R}$ is a function of (a,b) and define $h(\xi) := g(\xi,\xi)$, $\gamma(\xi) := g_a(\xi,\xi)$ and $\eta(\xi) := g_b(\xi,\xi)$. The hypothesis on g are:

- $g(a,b)$ is a continuously differentiable function with $g(0,0) = 0$,
- $h(\xi) < 0$ if $\xi > 0$,
- $\dfrac{h(\xi)}{\xi}$ is decreasing for $\xi > 0$,
- $\exists \xi^* > 0$ such that $\dfrac{h(\xi^*)}{\xi^*} = -1$,
- $\eta(\xi) < 0$ if $\xi > 0$,
- $\dfrac{h(\xi)}{\xi} \geq \gamma(\xi)$ for $\xi > 0$,
- $\gamma'(0) = -1$ and there exists $\eta'(0)$,
- $\displaystyle\lim_{(a,b)\to(0,0)} \frac{g(a,b)}{ab} = -1$,
- $\dfrac{g(a,a) - g(b,b)}{a-b} < \dfrac{g(a,a)}{a}$, $a,b > 0$.

We remark that the function $g(a,b) = -ab - \sum_{n=1}^{\infty} \alpha_n (ab)^{2n+1}$, considering, for example, $0 \leq \alpha_n \leq 1/(2n+1)!$, satisfies all the hypothesis above.

First, we need to guarantee the existence of the equilibrium point. The equilibrium solutions of (1) are independent of t but spatially nonconstant. Because the Dirichlet condition, the unique spatially constant solution is the null solution. Within those solutions we look for positive solutions on $(0,\pi)$. The equilibrium solutions of (1) coincide with solutions of the nonlinear boundary value problem

$$\begin{cases} \dfrac{d^2U(x)}{dx^2} + kU(x) + kg(U(x),U(x)) = 0 \\ U(0) = U(\pi) = 0 \end{cases} \tag{2}$$

We summarize in the next lemma results about existence and uniqueness of positive solutions for this problem and its proof follows the one in [7].

Lemma 1. (i) *For $k \leq 1$, there is no positive solution on $(0,\pi)$ of (2).*

(ii) *For $k > 1$, there exists a unique solution $U_k(x) = U(x,k)$ positive on $(0,\pi)$ of (2).*

Proof. (i) Suppose that $k \leq 1$ and exists a positive solution $\tilde{U}(x)$ of (2). Hence $\tilde{U}(x)$ is also a solution of the problem

$$\ddot{y}(x) + ky(x)\left[1 + \frac{g(\tilde{U}(x), \tilde{U}(x))}{\tilde{U}(x)}\right] = 0$$

with $y(0) = y(\pi) = 0$. From the hypothesis, $g(a, a) < 0$ if $a > 0$, so

$$k\left[1 + \frac{g(\tilde{U}(x), \tilde{U}(x))}{\tilde{U}(x)}\right] \leq 1$$

By Sturm Comparison Theorem ([8], page 223), each solution of the equation $\ddot{y}(x) + y(x) = 0$ has at least one zero on $0 < x < \pi$. However the function $y = \sin x$ provides us a contradiction.

To prove the existence of a positive solution we use the global bifurcation results in [10]. We remark that the problem (2) can be considered as a nonlinear Sturm-Liouville problem for second-order ordinary differential equations and the eigenvalues of the operator $-\ddot{U}$ are positive integers, $\lambda = \alpha^2$, $\alpha \in \boldsymbol{N}$. Hence, the first eigenvalue of $-\ddot{U}$ is $\lambda = 1$. Therefore, there exists an unbounded continuum of positive solutions on $(0, \pi)$ bifurcating from $\lambda = 1$. Let $U(x) > 0$ be for $0 < x < \pi$ one of those solutions. If x_0 is a maximum point, then $\ddot{U}(x_0) < 0$ and hence $g(U(x_0), U(x_0))/U(x_0)$ must satisfy

$$-1 < \frac{g(U(x_0), U(x_0))}{U(x_0)} < 0.$$

By assumption, $g(a, a)/a$ is decreasing if $a > 0$ and there exists $a^* \in \boldsymbol{R}$, $a^* > 0$ such that $g(a^*, a^*)/a^* = -1$, and so $U(x_0) < a^*$. Therefore, $U(x) < a^*$ for all $x \in [0, \pi]$, being, this way, bounded by a^*. Since the argument holds for every U at this continuum, this is uniformly bounded and since this continuum is unbounded, it extends for all $k > 1$.

To prove uniqueness, we assume U_1 and U_2 distinct positive solutions of the equation (2). Hence, $U_1 - U_2$ is a solution of the problem

$$\ddot{y} + k\left[1 + \frac{g(U_1, U_1) - g(U_2, U_2)}{U_1 - U_2}\right]y = 0 \tag{3}$$

with boundary condition $y(0) = y(\pi) = 0$. The hypotheses on g ensure that the coefficient $(g(U_1, U_1) - g(U_2, U_2))/(U_1 - U_2)$ is smooth even when $U_1 - U_2$ vanishes. By assumption

$$\frac{g(U_1, U_1) - g(U_2, U_2)}{U_1 - U_2} < \frac{g(U_1, U_1)}{U_1}.$$

So, by Sturm comparison theorem, every solution of the problem

$$\ddot{y} + k\left[1 + \frac{g(U_1, U_1)}{U_1}\right]y = 0, \qquad y(0) = y(\pi) = 0$$

must vanish between 0 and π, that is, between the zeros of the solutions of (3), in particular U_1 should vanish between 0 and π, what is a contradiction. ■

2. Analysis of characteristic equation

Let $k > 1$ and U_k be the positive equilibrium of (1). In order to understand the stability of the equilibrium point and Hopf bifurcation, we need to analyze the characteristic equation. To simplify the notation we denote $g(U_k, U_k) = g(U_k)$. Then the linearization of (1) around U_k is:

$$
(4) \qquad \begin{cases} \dfrac{\partial}{\partial t} V(t,x) = \dfrac{\partial^2}{\partial x^2} V(t,x) + k(1+g_a(U_k))V(t,x) \\ \qquad\qquad + \dfrac{k}{\delta}\displaystyle\int_{-\tau}^{-\tau+\delta} g_b(U_k)V(t+s,x)ds, \quad t>0, \\ V(t,0) = V(t,\pi) = 0, \quad t \geq 0, \\ V(t,x) = \phi(t,x), \quad (t,x) \in [-\tau,0]\times[0,\pi], \end{cases}
$$

where $\phi \in C([-\tau,0],X) = C$, and $X = L^2[0,\pi]$.

If we introduce the operator $A(k) : \mathrm{Dom}(A(k)) \subset X \to X$ given by:

$$
A(k) = D^2 + k(1+g_a(U_k)), \qquad D^2 = \frac{\partial^2}{\partial x^2},
$$

where $\mathrm{Dom}(A(k)) = H^2 \cap H_0^1$, then $A(k)$ is infinitesimal generator of a compact C_0-semigroup and we can write (4) as an abstract functional differential equation:

$$
(5) \qquad \begin{cases} \dot{V}(t) = A(k)V(t) + \dfrac{k}{\delta}\displaystyle\int_{-\tau}^{-\tau+\delta} g_b(U_k)V(t+s)ds, \quad t>0, \\ V(t) = \phi(t), \qquad t \in [-\tau,0];\ \phi \in C \end{cases}
$$

with $V(t) = V(t,\cdot)$ and $\phi(t) = \phi(t,\cdot)$. We associate to equation (5) a semigroup called solution semigroup, define by $U(t) : C \to C$,

$$
U(t)\phi = u_t(\phi), \qquad t \geq 0
$$

where $u_t(\phi)$ denotes the solution of (5) subject to the initial condition $u_0 = \phi$.

$\{U(t)\}_{t\geq 0}$ is a C_0-semigroup in C and we define by $A_\tau(k) : \mathrm{Dom}(A_\tau(k)) \subset C \to C$, the infinitesimal generator of the semigroup induced by solutions of the equation (5) with

$$
A_\tau(\phi) = \dot{\phi},
$$

$$\mathrm{Dom}(A_\tau(k)) = \Big\{ \phi \in C \cap C^1 : \phi(0) \in H_0^1 \cap H^2,$$

$$\dot{\phi}(0) = A(k)\phi(0) + \frac{k}{\delta}\int_{-\tau}^{-\tau+\delta} g_b(U_k)\phi(s)ds \Big\}.$$

By the characterization of the point spectrum of $A_\tau(k), \sigma(A_\tau(k))$, we get information about the stability of U_k. Those points of $\boldsymbol{C}$ which belong to the point spectrum of $A_\tau(k)$ are roots of the characteristic equation associated to (5):

$$(6) \qquad \Delta(k,\lambda,\tau)y = 0, \qquad y \in H_0^1 \cap H^2,\ y \neq 0$$

where

$$\Delta(k,\lambda,\tau) = D^2 + k(1+g_a(U_k)) + \frac{k}{\delta}\int_{-\tau}^{-\tau+\delta} g_b(U_k)e^{\lambda s}\,ds - \lambda,$$

or equivalently,

$$\Delta(k,\lambda,\tau) = D^2 + k(1+g_a(U_k)) + kg_b(U_k)e^{-\lambda\tau}\left[\frac{e^{\lambda\delta}-1}{\lambda\delta}\right] - \lambda.$$

We want to find the values of τ for which $\sigma(A_\tau(k))$ contains purely imaginary eigenvalues; they play a key role in the analysis of stability and bifurcation of periodic solutions. The operator $A_\tau(k)$ has a pure imaginary eigenvalue $\lambda = i\omega$, $\omega \neq 0$ for some $\tau > 0$ if and only if the equation

$$(7) \qquad \left[A(k) + kg_b(U_k)e^{-i\omega\tau}\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right) - i\omega\right]y = 0, \qquad 0 \neq y \in H^2 \cap H_0^1,$$

has a solution for some $\omega > 0$, $\tau \in [0, 2\pi)$. If we find a pair (ω,τ) such that (7) has a nonzero solution in $H^2 \cap H_0^1$, then we have $\Delta(k, i\omega, \tau_n) = 0$, for $\tau_n = (\theta + 2n\pi)/\omega$, $n = 0, 1, 2, \ldots$, and $\theta = \omega\tau$, and hence the sequence τ_n is a candidate at which the stability changes and Hopf bifurcation occurs.

We next prove that if $0 < k - 1 \ll 1$, there exists a unique pair (ω, θ) which solves:

$$(8) \qquad \left[A(k) + kg_b(U_k)e^{-i\theta}\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right) - i\omega\right]y = 0, \qquad 0 \neq y \in H^2 \cap H_0^1.$$

First, we write X as the direct sum of the kernel of $D^2 + 1$ with the range of this map:

$$L^2[0,\pi] = N(D^2+1) \oplus R(D^2+1)$$

where $N(D^2+1) = \text{span}[\sin(\cdot)]$ and

$$R(D^2+1) = \left\{ y \in L^2[0,\pi]; \langle \sin(\cdot), y \rangle = \int_0^\pi \sin(x) y(x) dx = 0 \right\}$$

by the projection theorem, considering the Dirichlet boundary condition. By projecting the positive equilibrium into $N(D^2+1)$ and $R(D^2+1)$ we can estimate U_k as follows.

Lemma 2. (i) *There exist* $k^* > 1$ *and a continuously differentiable mapping* $k \to (\xi_k, \alpha_k)$ *from* $[1, k^*]$ *to* $(H^2 \cap H_0^1) \cap R(D^2+1) \times \boldsymbol{R}^+$ *such that*

$$U_k = \alpha_k (k-1)[\sin(\cdot) + (k-1)\xi_k], \qquad k \in [1, k^*],$$

$$\alpha_1 = \int_0^\pi \sin^2(x) dx \Big/ \int_0^\pi \sin^3(x) dx$$

and

$$\xi_1 = -(D^2+1)^{-1}[\sin(\cdot)(1 - \alpha_1 \sin(\cdot))], \qquad \langle \sin(\cdot), \xi_1 \rangle = 0, \qquad \textit{with } \xi_1 \in H^2 \cap H_0^1.$$

(ii) *If* (ω, θ, y) *solves* (8) *with* $0 \neq y \in H^2 \cap H_0^1$, *then* $\omega/(k-1)$ *is uniformly bounded for* $k \in [1, k^*]$, *and*

$$\omega \langle y, y \rangle = -k \frac{(\cos(\theta - \omega\delta) - \cos(\theta))}{\omega\delta} \langle g_b(U_k) y, y \rangle.$$

(iii) *If* $z \in H^2 \cap H_0^1$ *and* $\langle \sin(\cdot), z \rangle = 0$ *then* $|\langle (D^2+1)z, z \rangle| \geq 3\|z\|_{L^2}^2$.

Proof. The proofs of (i) and (ii) follow as in [3]. To prove (iii) it is enough to remark that $A(k)$ is self-adjoint and $\lim_{k \to 1} g_b(U_k)/(k-1)$ exists, guaranteed by the assumption on $\eta'(0)$. ■

Now we write the equation (8) as a system of equations. For each $k \in [1, k^*]$, suppose (ω, θ, y) is a solution of (8) with $0 \neq y \in H^2 \cap H_0^1$. Ignoring a scalar factor, y can be represented as

$$y = \beta \sin(\cdot) + (k-1)z, \qquad \langle \sin(\cdot), z \rangle = 0, \qquad \beta \geq 0.$$

We also want that for k close to 1 the following equality is satisfied

$$\|y\|_{L^2}^2 = \beta^2 \|\sin(\cdot)\|_{L^2}^2 + (k-1)^2 \|z\|_{L^2}^2 = \|\sin(\cdot)\|_{L^2}^2.$$

If we call $\omega = (k-1)\rho$ and replace together with the expression to y into (8), then we have:

$$\left[A(k) + kg_b(U_k)e^{-i\theta}\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right) - i\omega\right] y$$

$$= \left[D^2 + k(1+g_a(U_k)) + kg_b(U_k)e^{-i\theta}\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right) - i\omega\right](\beta\sin(\cdot) + (k-1)z)$$

$$= -\beta\sin(\cdot) + (k-1)D^2z + k\beta\sin(\cdot) + k(k-1)z$$

$$+ (\beta\sin(\cdot) + (k-1)z)\left[k(g_a(U_k) + g_b(U_k)e^{-i\theta}\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right) - i\omega\right]$$

$$= (k-1)\left\{\frac{-\beta\sin(\cdot)}{(k-1)} + D^2z + z - z + \frac{k\beta\sin(\cdot)}{(k-1)} + kz + (\beta\sin(\cdot) + (k-1)z)\right.$$

$$\left.\times\left[\frac{k}{(k-1)}\left(g_a(U_k) + g_b(U_k)\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right)e^{-i\theta}\right) - i\rho\right]\right\}$$

$$= (k-1)\left\{(D^2+1)z + (\beta\sin(\cdot) + (k-1)z)\right.$$

$$\left.\times\left[\frac{k}{(k-1)}\left[g_a(U_k) + g_b(U_k)e^{-i\theta}\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right)\right] - i\rho + 1\right]\right\}$$

$$= (k-1)h_1(z,\beta,\rho,\theta,k).$$

Therefore, (8) is equivalent to the system:

$$\text{(9)}\qquad \begin{cases} h_1(z,\beta,\rho,\theta,k) = 0, \\ h_2(z) = \mathrm{Re}(\langle\sin(\cdot), z\rangle) = 0, \\ h_3(z) = \mathrm{Im}(\langle\sin(\cdot), z\rangle) = 0, \\ h_4(z,\beta,k) = (\beta^2-1)\|\sin(\cdot)\|_{L^2}^2 + (k-1)^2\|z\|_{L^2}^2 = 0. \end{cases}$$

Let us calculate $\lim_{k\to 1}\omega/(k-1)$, that is, ρ_1:

$$\frac{\omega}{(k-1)}\langle y, y\rangle = -\frac{k}{(k-1)}\left(\frac{\cos(\theta-\omega\delta)-\cos(\theta)}{\omega\delta}\right)\langle g_b(U_k)y, y\rangle$$

$$= -k\left(\frac{\cos(\theta-\omega\delta)-\cos(\theta)}{\omega\delta}\right)\left\langle\frac{g_b(U_k)}{U_k}\frac{U_k}{k-1}y, y\right\rangle.$$

Considering $y = \beta\sin(\cdot) + (k-1)z$, when $k\to 1$ we have

$$\lim_{k\to 1}\frac{\omega}{(k-1)}\langle\beta\sin(\cdot), \beta\sin(\cdot)\rangle = -\sin(\theta)\lim_{k\to 1}\frac{g_b(U_k)}{U_k}\langle\alpha_1\beta\sin^2(\cdot), \beta\sin(\cdot)\rangle.$$

Therefore, $\lim_{k\to 1}\omega/(k-1) = -\sin(\theta)\lim_{k\to 1}g_b(U_k)/U_k$. We also can check that (note that $\lim_{k\to 1}(e^{i\omega\delta}-1)/(i\omega\delta) = 1$)

$$\lim_{k\to 1}\frac{1}{(k-1)}\left[g_a(U_k)+g_b(U_k)e^{-i\theta}\left(\frac{e^{i\omega\delta}-1}{i\omega\delta}\right)\right]=-\alpha_1\left(1-\lim_{k\to 1}\frac{g_b(U_k)}{U_k}e^{-i\theta}\right)\sin(\cdot)$$

Theorem 3. *If* $0<k^*-1\ll 1$, *then there is a continuously differentiable mapping from* $[1,k^*]$ *to* $H^2\cap H_0^1\times \boldsymbol{R}^3$, $k\to(z_k,\beta_k,\rho_k,\theta_k)$ *such that* $z_1=(1+i\lim_{k\to 1} g_b(U_k)/U_k)\xi_1$, $\beta_1=1$, $\theta_1=\pi/2$, $\rho_1=-\lim_{k\to 1} g_b(U_k)/U_k$ *and* $(z_k,\beta_k,\rho_k,\theta_k)$ *solves* (9) *for* $k\in[1,k^*]$, *with* ξ_1 *given in Lemma* 2. *In addition, if* $k\in[1,k^*]$ *and* $(z^k,\beta^k,\rho^k,\theta^k)$ *solves* (9) *for* $k\in[1,k^*]$, *with* $\rho^k>0$ *and* $\theta\in[0,2\pi)$, *then* $(z^k,\beta^k,\rho^k,\theta^k)=(z_k,\beta_k,\rho_k,\theta_k)$.

Proof. To prove this theorem we follow as in [3]. ■

As an immediate consequence we have:

Corollary 4. *If* $0<k^*-1\ll 1$, *then for each* $k\in(1,k^*)$, *the eigenvalue problem*

$$\Delta(k,i\omega,\tau)y=0,\qquad 0\neq y\in H^2\cap H_0^1$$

has a solution (ω,τ,y), *or equivalently,* $i\omega\in\sigma(A_\tau(k))$ *if and only if*

$$\omega=\omega_k=(k-1)\rho_k,$$

$$\tau=\tau_{k_n}=\frac{\theta_k+2n\pi}{\omega_k},\qquad n=0,1,2,\dots$$

$$y=cy_k,\qquad y_k=\beta_k\sin(\cdot)+(k-1)z_k$$

Here, c is any non-zero constant, and $z_k,\beta_k,\rho_k,\theta_k$ *are defined as in Theorem* 3.

3. Stability of equilibrium point

By using the results from Huang [16], we stablish sufficient conditions for the asymptotic stability of U_k.

Theorem 5. *Let* $k>1$ *and* $M(k)=\max_{x\in[0,\pi]}\{-g_b(U_k(x))\}$. *If* $(kM(k)/\delta)\int_{-\tau}^{-\tau+\delta}s\,ds>-\pi/2$, *then the positive equilibrium is locally asymptotically stable.*

In [16], Huang describes the localization of the roots of the equation $\Delta(\lambda)=\lambda+a+\int_0^r e^{-\lambda s}\,d\eta(s)$ using the following lemmas:

Lemma 6 (Huang, [16]). *For any* $\theta\in(0,\pi)$, *introduce*

$$D_\theta=\left\{x+iy: x\in\boldsymbol{R},\, y\geq-\frac{\sin\theta-\theta\cos\theta}{\theta\sin\theta}x\right\},$$

a half of the complex plane $\boldsymbol{C}$. *Then for any* $z_i \in D_\theta$, $\alpha_i \geq 0$, $i = 1, 2$, *we have* $\alpha_1 z_1 + \alpha_2 z_2 \in D_\theta$. *Furthermore* $z_1 + z_2 = 0$ *if and only if* $z_1, z_2 \in \partial D_\theta$ *and* $z_1 = -z_2$.

Lemma 7 (Huang, [16]). *For each* $\theta \in (0, \pi)$, *let* $W_\theta : [0, \infty) \to \boldsymbol{C}$ *be given by*

$$\begin{aligned} W_\theta(v) &= -\theta \cos\theta/\sin\theta + iv + \theta e^{-iv}/\sin\theta \\ &= -\theta(\cos\theta - \cos v)/\sin\theta + i(v - \theta \sin v/\sin\theta), \end{aligned}$$

then $W_\theta([0, \infty)) \in D_\theta$.

Before we prove Theorem 5, we establish the following lemma:

Lemma 8. *All eigenvalues of* $A(k)$ *are real and non positive.*

Proof. Since $A(k)$ is a self-adjoint operator, all eigenvalues of $A(k)$ are real. Suppose that $A(k)$ has some eigenvalue $\lambda > 0$ and let y be the corresponding eigenfunction. We have

$$D^2 y + [k(1 + g_a(U_k)) - \lambda] y = 0.$$

However $D^2 U_k + k(1 + g(U_k)/U_k) U_k = 0$ and, by assumption, close to the zero, $g(\xi, \xi)/\xi \geq g_a(\xi)$, so, for k sufficiently close to 1, we have

$$k\left(1 + \frac{g(U_k)}{U_k}\right) > k(1 + g_a(U_k)) - \lambda.$$

By Sturm comparison theorem U_k has at least one zero between 0 and π, which contradicts the fact that U_k is positive in this interval. Therefore $A(k)$ has only eigenvalues non positive. ■

As a corollary:

Corollary 9. *For all* $\psi \in H^2 \cap H_0^1$,

$$\int_0^\pi (A(k)\psi)\bar{\psi}\, dx \leq 0. \tag{10}$$

Proof. Since $C_0^2 = \{y \in C^2((0, \pi)) \cap C([0, \pi]), y(0) = y(\pi) = 0\}$ is dense in $L^2[0, \pi]$, it is enough consider $\psi \in C_0^2$. Since $A(k)$ is self-adjoint, the collection $\{\psi\}$ of all eigenfunctions with $\int_0^\pi \psi^2\, dx = 1$ form an orthonormal basis of C_0^2 ([21], page 374). Inequality (10) follows from (8) and Parseval's equation. ■

Proof of Theorem 5. Recall that the eigenvalue problem consists in finding $\psi \in H^2 \cap H_0^1$, $\psi \neq 0$, and $\lambda \in \boldsymbol{C}$, so that

$$\varDelta(k,\lambda,\tau)\psi := \left[D^2 + k(1+g_a(U_k)) + \frac{k}{\delta}\int_{-\tau}^{-\tau+\delta} g_b(U_k)e^{\lambda s}\,ds - \lambda\right]\psi = 0.$$

We claim that the characteristic equation has no eigenvalue λ with $\operatorname{Re}\lambda \geq 0$. To see this, we set $\lambda = u + iv$, multiply $\varDelta(k,\lambda,\tau)\psi$ by $\bar{\psi}$ and integrate from 0 to π to obtain:

$$-\int_0^{\pi} \varDelta(k,\lambda,\tau)\psi\bar{\psi}\,dx = \int_0^{\pi}\left[u + iv - \frac{k}{\delta}\int_{-\tau}^{-\tau+\delta} g_b(U_k)e^{\lambda s}\,ds\right]|\psi|^2\,dx - \int_0^{\pi}(A(k)\psi)\bar{\psi}\,dx.$$

For $u = v = 0$ we have (recall that $g_b(\xi) < 0$)

$$-\int_0^{\pi} \varDelta(k,0,\tau)\psi\bar{\psi}\,dx = -\frac{k}{\delta}\int_{-\tau}^{-\tau+\delta} ds\int_0^{\pi} g_b(U_k)|\psi|^2 dx - \int_0^{\pi}(A(k)\psi)\bar{\psi}\,dx$$

$$> -\frac{k}{\delta}\int_{-\tau}^{-\tau+\delta} ds\int_0^{\pi} g_b(U_k)|\psi|^2 dx > 0.$$

Hence, $\varDelta(k,0,\tau)\psi \neq 0$, for all $\psi \in H^2 \cap H_0^1$, $\psi \neq 0$.

If $u \geq 0$, $v \geq 0$, $u + v > 0$, then

$$-\int_0^{\pi} \varDelta(k,\lambda,\tau)\psi\bar{\psi}\,dx = u\int_0^{\pi}|\psi|^2 dx - \int_0^{\pi}(A(k)\psi)\bar{\psi}\,dx$$

$$+ iv\int_0^{\pi}\left[1 - \frac{2k}{\delta\pi}g_b(U_k)\int_{-\tau}^{-\tau+\delta} e^{us}s\,ds\right]|\psi|^2 dx$$

$$-\frac{2k}{\delta\pi}\int_{-\tau}^{-\tau+\delta} e^{us}\left(-ivs + \frac{\pi}{2}e^{ivs}\right)ds\int_0^{\pi} g_b(U_k)|\psi|^2 dx.$$

If we define

$$z_2 = -\frac{2k}{\delta\pi}\int_{-\tau}^{-\tau+\delta} e^{us}\left(-ivs + \frac{\pi}{2}e^{ivs}\right)ds\int_0^{\pi} g_b(U_k)|\psi|^2 dx$$

$$= -\frac{2k}{\delta\pi}\int_{-\tau}^{-\tau+\delta} e^{us}W_{\pi/2}(-vs)ds\int_0^{\pi} g_b(U_k)|\psi|^2 dx,$$

since $W_{\pi/2}(-vs) \in D_{\pi/2}$ and $D_{\pi/2}$ is closed, so $z_2 \in D_{\pi/2}$. We have

$$u\int_0^{\pi}|\psi|^2 dx - \int_0^{\pi}(A(k)\psi)\bar{\psi}\,dx > u\int_0^{\pi}|\psi|^2 dx.$$

By assumption, $(kM(k)/\delta)\int_{-\tau}^{-\tau+\delta} s\,ds > -\pi/2$, so

$$1 - \frac{2k}{\delta\pi}g_b(U_k)\int_{-\tau}^{-\tau+\delta} e^{us}s\,ds \geq 1 + \frac{2kM(k)}{\delta\pi}\int_{-\tau}^{-\tau+\delta} s\,ds = \sigma > 0.$$

Therefore,

$$v \int_0^\pi \left[1 - \frac{2k}{\delta\pi} g_b(U_k) \int_{-\tau}^{-\tau+\delta} e^{us} s \, ds \right] |\psi|^2 \, dx \geq v\sigma \int_0^\pi |\psi|^2 dx > 0, \qquad \text{if } v > 0.$$

Thus

$$z_1 = u \int_0^\pi |\psi|^2 dx - \int_0^\pi (A(k)\psi)\bar{\psi} \, dx + iv \int_0^\pi \left[1 - \frac{2k}{\delta\pi} g_b(U_k) \int_{-\tau}^{-\tau+\delta} e^{us} s \, ds \right] |\psi|^2 dx,$$

$z_1 \in D_\theta$, for all $\theta \in (0, \pi)$, in particular for $\theta = \pi/2$. However, $z_1 \notin \partial D_{\pi/2}$, since $\mathrm{Re}\, z_1 \geq 0$ and $\mathrm{Im}\, z_1 \geq 0$, but $\mathrm{Re}\, z_1 + \mathrm{Im}\, z_1 > 0$. Thus, $z_1 + z_2 \neq 0$. Hence, $\int_0^\pi \Delta(k, \lambda, \tau)\psi\bar{\psi} \neq 0$, $\Delta(k, \lambda, \tau)\psi \neq 0$, for all $u \geq 0$, $v \geq 0$ and $\psi \in H^2 \cap H_0^1$, $\psi \neq 0$.

Finally notice that for $u \geq 0$, $v \leq 0$ and $\psi \in H^2 \cap H_0^1$, $\psi \neq 0$,

$$\Delta(k, u + iv, \tau)\psi = \bar{\Delta}(k, u - iv, \tau)\bar{\psi} \neq 0. \qquad \blacksquare$$

We now analyze the stability of U_k for $k \in (1, k^*)$ fixed and the delay τ being treated as a parameter. To describe the stability of U_k, it is enough investigate how the eigenvalue $\lambda = i\omega$ varies as the delay τ passes through each τ_{k_n}, $n = 0, 1, 2, \ldots.$

Lemma 10. (i) *If* $0 < k^* - 1 \ll 1$, *then for* $k \in (1, k^*]$

$$\text{(11)} \qquad S_{k_n} \stackrel{\text{def}}{=} \int_0^\pi \left[1 + \tau_{k_n} k e^{-i\theta_k} g_b(U_k) \left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right] - k g_b(U_k) e^{-i\theta_k} \frac{e^{i\omega_k\delta}}{i\omega_k} - ik g_b(U_k) \frac{e^{-i\theta_k}}{\omega_k} \left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right] \right] y_k^2(x) dx \neq 0$$

(ii) *For each* $k \in (1, k^*)$, $0 < k^* - 1 \ll 1$, *and* $n = 0, 1, \ldots,$ $\lambda = i\omega_k$ *is a simple eigenvalue of* $A_{\tau_{k_n}}(k)$.

(iii) $\lim_{k \to 1} \bar{S}_{k_n} = [1 - i(2n\pi + \pi/2)] \int_0^\pi \sin^2(x) dx$.

Proof. (i) From last section, we have:

$$y_k = \beta_k \sin(\cdot) + (k - 1) z_k,$$

$$U_k = \alpha_k (k - 1) \sin(\cdot) + \alpha_k (k - 1)^2 \xi_k, \qquad \tau_{k_n} = \frac{\theta_k + 2n\pi}{(k - 1)\rho_k}$$

where $\theta_k \to \pi/2$, $\rho_k \to -\lim_{k \to 1} g_b(U_k) U_k$, $\beta_k \to 1$, $\alpha_k \to \int_0^\pi \sin^2(x) dx / \int_0^\pi \sin^3(x) dx$, when $k \to 1$.

If $t_k = \arg(\int_0^\pi y_k^2(x) dx)$, $-\pi < t_k \leq \pi$, then $|\int_0^\pi y_k^2(x) dx| e^{it_k} = \int_0^\pi y_k^2(x) dx \xrightarrow{k \to 1} \int_0^\pi \sin^2(x) dx$. Therefore, $t_k \to 0$ as $k \to 1$. Besides,

$$\int_0^\pi \left[e^{-it_k} + \tau_{k_n} k e^{-i(\theta_k+t_k)} g_b(U_k)\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right] - k g_b(U_k) e^{-i(\theta_k+t_k)}\frac{e^{i\omega_k\delta}}{i\omega_k}\right.$$

$$\left. - ik g_b(U_k)\frac{e^{-i(\theta_k+t_k)}}{\omega_k}\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right]\right] y_k^2(x)dx$$

$$= \int_0^\pi e^{-it_k} y_k^2(x)dx + \frac{\theta_k+2n\pi}{\rho_k} k \int_0^\pi e^{-i(\theta_k+t_k)}\frac{g_b(U_k)}{k-1}\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right] y_k^2(x)dx$$

$$- k\int_0^\pi g_b(U_k) e^{-i(\theta_k+t_k)}\frac{e^{i\omega_k\delta}}{i\omega_k} y_k^2(x)dx - ik\int_0^\pi g_b(U_k)\frac{e^{-i(\theta_k+t_k)}}{\omega_k}\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right] y_k^2(x)dx$$

$$= \left|\int_0^\pi y_k^2(x)dx\right| + \frac{\theta_k+2n\pi}{\rho_k} k\int_0^\pi e^{-i(\theta_k+t_k)}\frac{g_b(U_k)}{U_k}\frac{U_k}{k-1}\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right] y_k^2(x)dx$$

$$- k\int_0^\pi \frac{g_b(U_k)}{U_k}\frac{U_k}{k-1} e^{-i(\theta_k+t_k)} e^{i\omega_k\delta}\frac{(k-1)}{i\omega_k} y_k^2(x)dx$$

$$- ik\int_0^\pi \frac{g_b(U_k)}{U_k}\frac{U_k}{k-1} e^{-i(\theta_k+t_k)}\frac{(k-1)}{\omega_k}\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right] y_k^2(x)dx$$

$$= \left|\int_0^\pi y_k^2(x)dx\right| + \frac{\theta_k+2n\pi}{\rho_k} k\int_0^\pi e^{-i(\theta_k+t_k)}\frac{g_b(U_k)}{U_k}\alpha_k(\sin(x)+(k-1)\xi_k)(\beta_k \sin(x)$$

$$+ (k-1)z_k)^2\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right]dx - k\int_0^\pi \frac{g_b(U_k)}{U_k} e^{-i(\theta_k+t_k)} e^{i\omega_k\delta}\frac{1}{i\rho_k}\alpha_k(\sin(x)+(k-1)\xi_k)$$

$$\cdot(\beta_k \sin(x)+(k-1)z_k)^2 dx - ik\int_0^\pi \frac{g_b(U_k)}{U_k} e^{-i(\theta_k+t_k)}\frac{1}{\rho_k}\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right]\alpha_k(\sin(x)$$

$$+ (k-1)\xi_k)(\beta_k \sin(x)+(k-1)z_k)^2 dx$$

$$\to \int_0^\pi \sin(x)^2 dx + \left(\frac{\pi}{2}+2n\pi\right) i\int_0^\pi \alpha_1 \sin^3(x)dx - \int_0^\pi \alpha_1 \sin^3(x) - i^2\int_0^\pi \alpha_1 \sin^3(x)dx$$

$$= \int_0^\pi \sin(x)^2 dx + i\left(\frac{\pi}{2}+2n\pi\right)\int_0^\pi \sin^2(x)dx, \qquad \text{as } k\to 1,$$

recall that $t_k \to 0$, $\rho_k \to -\lim_{k\to 1} g_b(U_k)/U_k$ and $\lim_{k\to 1}(e^{i\omega\delta}-1)/(i\omega\delta) = 1$. We used the dominated convergence theorem to calculate those limits. Since

$$\lim_{k\to 1}\int_0^\pi \left[e^{-it_k} + \tau_{k_n} k e^{-i(\theta_k+t_k)} g_b(U_k)\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right] - k g_b(U_k) e^{-i(\theta_k+t_k)}\frac{e^{i\omega_k\delta}}{i\omega_k}\right.$$

$$\left. - ik g_b(U_k)\frac{e^{-i(\theta_k+t_k)}}{\omega_k}\left[\frac{e^{i\omega_k\delta}-1}{i\omega_k\delta}\right]\right] y_k^2(x)dx = \lim_{k\to 1} S_{k_n},$$

for $k \in (1, k^*]$, we have $S_{k_n} \neq 0$.

(ii) From Corollary 4, we have $N[A_{\tau_{k_n}}(k) - i\omega_k] = \text{span}[e^{i\omega_k \cdot} y_k]$. Now, suppose $\phi \in \text{Dom}(A_{\tau_{k_n}}) \cap \text{Dom}([A_{\tau_{k_n}}]^2)$ and $[A_{\tau_{k_n}}(k) - i\omega_k]^2 \phi = 0$. It follows that $[A_{\tau_{k_n}}(k) - i\omega_k]\phi \in N(A_{\tau_{k_n}}(k) - i\omega_k) = \text{span}[e^{i\omega_k(\cdot)} y_k]$. Hence, there is a constant c such that

$$[A_{\tau_{k_n}}(k) - i\omega_k]\phi = ce^{i\omega_k(\cdot)} y_k$$

or

$$\begin{cases} \dot{\phi}(\theta) = i\omega_k \phi(\theta) + ce^{i\omega_k \theta} y_k, \quad \theta \in [-\tau_{k_n}, 0], \\ \dot{\phi}(0) = A(k)\phi(0) + \dfrac{k}{\delta}\displaystyle\int_{-\tau}^{-\tau+\delta} g_b(U_k)\phi(s)ds. \end{cases} \tag{12}$$

The first equation gives:

$$\begin{cases} \phi(\theta) = \phi(0)e^{i\omega_k\theta} + ce^{i\omega_k\theta}\theta y_k, \\ \dot{\phi}(0) = i\omega_k\phi(0) + cy_k. \end{cases} \tag{13}$$

Substituting the last equation into the first one we get:

$$\Delta(k, i\omega_k, \tau_{k_n})\phi(0) = \left(A(k) + kg_b(U_k)e^{-i\omega_k\tau_{k_n}}\left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta}\right] - i\omega_k\right)\phi(0)$$

$$= A(k)\phi(0) + kg_b(U_k)e^{-i\theta_k}\left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta}\right]\phi(0) - i\omega_k\phi(0)$$

$$= \dot{\phi}(0) - \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)\phi(s)ds + kg_b(U_k)e^{-i\theta_k}\left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta}\right]\phi(0) - \dot{\phi}(0) + cy_k$$

$$= -\frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)[\phi(0)e^{i\omega_k s} + ce^{i\omega_k s}sy_k]ds + kg_b(U_k)e^{-i\theta_k}\left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta}\right]\phi(0) + cy_k$$

$$= -kg_b(U_k)e^{-i\theta_k}\left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta}\right]\phi(0) - \frac{k}{\delta}c\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)e^{i\omega_k s}sy_k\, ds$$

$$+ kg_b(U_k)e^{-i\theta_k}\left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta}\right]\phi(0) + cy_k = c\left[1 - \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)e^{i\omega_k s}s\, ds\right]y_k.$$

Therefore, considering that $A(k)$ is self-adjoint:

$$0 = \int_0^\pi \phi(0)[\Delta(k, i\omega_k, \tau_{k_n})y_k](x)dx = \int_0^\pi y_k(x)[\Delta(k, i\omega_k, \tau_{k_n})\phi(0)](x)dx$$

$$= c\int_0^\pi \left(1 - \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)e^{i\omega_k s}s\, ds\right)y_k^2(x)dx$$

$$= c \int_0^\pi \left[1 + \tau_{k_n} k e^{-i\theta_k} g_b(U_k) \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] - k g_b(U_k) e^{-i\theta_k} \frac{e^{i\omega_k \delta}}{i\omega_k} \right.$$
$$\left. - ik g_b(U_k) \frac{e^{-i\theta_k}}{\omega_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] \right] y_k^2(x) dx.$$

From (i), we get $c = 0$. Thus $\phi \in N(A_{\tau_{k_n}} - i\omega_k)$. By induction we have

$$N([A_{\tau_{k_n}} - i\omega_k]^j) = N(A_{\tau_{k_n}} - i\omega_k), \qquad j = 1, 2, \dots \quad \text{and} \quad n = 0, 1, 2, \dots.$$

Therefore $\lambda = i\omega_k$ is a simple eigenvalue of $A_{\tau_{k_n}}$ for $n = 0, 1, 2, \dots$

(iii) It follows from (i) that

$$\lim_{k \to 1} S_{k_n} = \int_0^\pi \sin^2(x) dx + i\left(2n\pi + \frac{\pi}{2}\right) \int_0^\pi \sin^2(x) dx$$

and thus,

$$\lim_{k \to 1} \overline{S}_{k_n} = \left[1 - i\left(2n\pi + \frac{\pi}{2}\right) \right] \int_0^\pi \sin^2(x) dx. \qquad \blacksquare$$

Since $\lambda = i\omega_k$ is a simple eigenvalue of $A_{\tau_{k_n}}$, by using the implicit function theorem we show that there is a neighborhood $O_{k_n} \times C_{k_n} \times H_{k_n} \subset \boldsymbol{R} \times \boldsymbol{C} \times H_0^1 \cap H^2$ of $(\tau_{k_n}, i\omega_k, y_k)$ and a continuously differentiable function $(\lambda, y) : O_{k_n} \to C_{k_n} \times H_{k_n}$ such that for each $\tau \in O_{k_n}$, the only eigenvalue of $A_{\tau_{k_n}}$ in C_{k_n} is $\lambda(\tau)$ and $\lambda(\tau_{k_n}) = i\omega_k$, $y(\tau_{k_n}) = y_k$ and for each $\tau \in O_{k_n}$,

$$\Delta(k, \lambda(\tau), \tau) y(\tau) = \left[A(k) + k e^{-\lambda(\tau)\tau} g_b(U_k) \left[\frac{e^{\lambda(\tau)\delta} - 1}{\lambda(\tau)\delta} \right] - \lambda(\tau) \right] y(\tau) = 0.$$

Differentiating the above equation with respect to τ at $\tau = \tau_{k_n}$, we have

$$\left\{ -\frac{d\lambda(\tau_{k_n})}{d\tau} \tau_{k_n} k g_b(U_k) e^{-\lambda(\tau_{k_n})\tau_{k_n}} \left[\frac{e^{\lambda(\tau_{k_n})\delta} - 1}{\lambda(\tau_{k_n})\delta} \right] - \lambda(\tau_{k_n}) k g_b(U_k) e^{-\lambda(\tau_{k_n})\tau_{k_n}} \left[\frac{e^{\lambda(\tau_{k_n})\delta} - 1}{\lambda(\tau_{k_n})\delta} \right] \right.$$
$$+ k g_b(U_k) e^{-\lambda(\tau_{k_n})\tau_{k_n}} \left[\frac{\frac{d\lambda(\tau_{k_n})}{d\tau} \delta e^{\lambda(\tau_{k_n})\delta} \lambda(\tau_{k_n})\delta - \frac{d\lambda(\tau_{k_n})}{d\tau} \delta (e^{\lambda(\tau_{k_n})\delta} - 1)}{(\lambda(\tau_{k_n})\delta)^2} \right]$$
$$\left. - \frac{d\lambda(\tau_{k_n})}{d\tau} \right\} y(\tau_{k_n}) + \Delta(k, \lambda(\tau_{k_n}), \tau_{k_n}) \frac{dy(\tau_{k_n})}{d\tau}$$

$$= \left\{ -\frac{d\lambda(\tau_{k_n})}{d\tau} \tau_{k_n} k g_b(U_k) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] - i\omega_k k g_b(U_k) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] \right.$$

$$\left. + k g_b(U_k) e^{-i\theta_k} \frac{d\lambda(\tau_{k_n})}{d\tau} \left[\frac{\delta^2 e^{i\omega_k \delta} i\omega_k - \delta(e^{i\omega_k \delta} - 1)}{(i\omega_k)^2 \delta^2} \right] - \frac{d\lambda(\tau_{k_n})}{d\tau} \right\} y_k$$

$$+ \Delta(k, \lambda(\tau_{k_n}), \tau_{k_n}) \frac{dy(\tau_{k_n})}{d\tau}$$

$$= \frac{d\lambda(\tau_{k_n})}{d\tau} \left\{ -1 - \tau_{k_n} k g_b(U_k) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] + k g_b(U_k) e^{-i\theta_k} \frac{e^{i\omega_k \delta}}{i\omega_k} \right.$$

$$\left. - k g_b(U_k) \frac{e^{-i\theta_k}}{i\omega_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] \right\} y_k - i\omega_k k g_b(U_k) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k$$

$$+ \Delta(k, \lambda(\tau_{k_n}), \tau_{k_n}) \frac{dy(\tau_{k_n})}{d\tau} = 0.$$

Taking an inner product with y_k, recalling the self-adjointness of $A(k)$, from the definition of S_{k_n} and the fact that $\Delta(k, i\omega_k, \tau_{k_n}) y_k = 0$, we have:

$$\frac{d\lambda(\tau_{k_n})}{d\tau} = \frac{-\int_0^\pi i\omega_k k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k^2(x) dx}{S_{k_n}}$$

$$= \frac{-1}{|S_{k_n}|^2} \left\{ \int_0^\pi i\omega_k k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k^2(x) dx \overline{S_{k_n}} \right\}$$

$$= \frac{-1}{|S_{k_n}|^2} \left\{ \int_0^\pi i\omega_k k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k^2(x) dx \overline{\int_0^\pi y_k^2(x) dx} \right.$$

$$+ \int_0^\pi i\omega_k k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k^2(x) dx$$

$$\times \overline{\int_0^\pi \tau_{k_n} k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k^2(x) dx} - \int_0^\pi i\omega_k k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right]$$

$$\times y_k^2(x) dx \overline{\int_0^\pi k g_b(U_k(x)) e^{-i\theta_k} \frac{e^{i\omega_k \delta}}{i\omega_k} y_k^2(x) dx} - \int_0^\pi i\omega_k k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right]$$

$$\times y_k^2(x) dx \overline{\int_0^\pi i k g_b(U_k(x)) \frac{e^{-i\theta_k}}{\omega_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k^2(x) dx}$$

$$= \frac{-1}{|S_{k_n}|^2} \left\{ \int_0^\pi i\omega_k k g_b(U_k(x)) e^{-i\theta_k} \left[\frac{e^{i\omega_k \delta} - 1}{i\omega_k \delta} \right] y_k^2(x) dx \left| \int_0^\pi y_k^2(x) dx \right| e^{-it_k} \right.$$

$$+ i\omega_k k^2 \tau_{k_n} \left| \int_0^\pi g_b(U_k(x)) \left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right] y_k^2(x0)dx \right|^2 - i\omega_k k^2 \left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right] \frac{e^{-i\omega_k\delta}}{(-i\omega_k)}$$

$$\times \left| \int_0^\pi g_b(U_k(x)) y_k^2(x)dx \right|^2 - k^2 \left| \int_0^\pi g_b(U_k(x)) \left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right] y_k^2(x)dx \right|^2$$

$$= \frac{-1}{|S_{k_n}|^2} \left\{ i\omega_k k \int_0^\pi g_b(U_k(x)) e^{-i(\theta_k + t_k)} \left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right] y_k^2(x)dx \left| \int_0^\pi y_k^2(x)dx \right| \right.$$

$$+ i\omega_k k^2 \tau_{k_n} \left| \int_0^\pi g_b(U_k(x)) \left[\frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right] y_k^2(x)dx \right|^2$$

$$+ k^2 \left[\frac{1 - e^{-i\omega_k\delta}}{i\omega_k\delta} \right] \left| \int_0^\pi g_b(U_k(x)) y_k^2(x)dx \right|^2 - k^2 \left| \frac{e^{i\omega_k\delta} - 1}{i\omega_k\delta} \right|^2 \left| \int_0^\pi g_b(U_k(x)) y_k^2(x)dx \right|^2.$$

Hence, for each $k \in (1, k^*]$:

$$\operatorname{Re} \frac{d\lambda(\tau_{k_n})}{d\tau} \equiv \frac{1}{|S_{k_n}|^2} k(k-1)\omega_k \rho_1 \left(\int_0^\pi \sin^2(x)dx \right)^2 > 0.$$

Therefore $\operatorname{Re} d\lambda(\tau_{k_n})/d\tau > 0$.

We can now describe the positive equilibrium of the system (1).

Theorem 11. *For each $0 < k - 1 \ll 1$, the infinitesimal generator $A_\tau(k)$ has exactly $2(n+1)$ eigenvalues with positive real parts if $\tau \in (\tau_{k_n}, \tau_{k_{n+1}}]$, $n = 0, 1, 2, \ldots,$ and all eigenvalues of $A_\tau(k)$ have negative real parts if $0 \le \tau < \tau_{k_0}$. Hence, the positive equilibrium U_k is locally asymptotically stable if $0 \le \tau < \tau_{k_0}$ and unstable if $\tau > \tau_{k_0}$ and $(kM(k)/\delta) \int_{-\tau_{k_0}}^{-\tau_{k_0}+\delta} s\, ds \le -\pi/2$.*

4. Hopf bifurcation

For fixed $k \in (1, k^*)$ and τ_{k_n}, let $u(t) = U(t, \cdot) - U_k(\cdot)$ and $\alpha = \tau - \tau_{k_n}$. Thus, we obtain the following equation which is equivalent to (1):

$$\dot{u}(t) = A(k)u(t) + \frac{k}{\delta} \int_{-\alpha-\tau_{k_n}}^{-\alpha-\tau_{k_n}+\delta} g_b(U_k)u(t+s)ds \tag{14}$$

$$- kg(U_k) - kg_a(U_k)u(t) - \frac{k}{\delta} \int_{-\alpha-\tau_{k_n}}^{-\alpha-\tau_{k_n}+\delta} g_b(U_k)u(t+s)ds$$

$$+ \frac{k}{\delta} \int_{-\alpha-\tau_{k_n}}^{-\alpha-\tau_{k_n}+\delta} g(u(t) + U_k, u(t+s) + U_k)ds.$$

Let $\beta \in [-1,1]$, $v_k = 2\pi/\omega_k$ and $w(t) = u(t(1+\beta))$. We have that $u(t)$ is a $v_k(1+\beta)$-periodic solution of (14) if and only if $w(t)$ is a v_k-periodic solution for the equation

$$\dot{w}(t) = A(k)w(t) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)w(t+s)ds + G(\alpha,\beta,w_t) \tag{15}$$

where

$$\begin{aligned} G(\alpha,\beta,w_t) = \beta A(k)w(t) &- \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)w(t+s)ds \\ &+ (1+\beta)\frac{k}{\delta}\int_{-\alpha-\tau_{k_n}}^{-\alpha-\tau_{k_n}+\delta} g_b(U_k)w\left(t+\frac{s}{1+\beta}\right)ds \\ &- (1+\beta)kg(U_k) - (1+\beta)kg_a(U_k)w(t) \\ &- (1+\beta)\frac{k}{\delta}\int_{-\alpha-\tau_{k_n}}^{-\alpha-\tau_{k_n}+\delta} g_b(U_k)w\left(t+\frac{s}{1+\beta}\right)ds \\ &+ (1+\beta)\frac{k}{\delta}\int_{-\alpha-\tau_{k_n}}^{-\alpha-\tau_{k_n}+\delta} g\left(w(t)+U_k, w\left(t+\frac{s}{1+\beta}\right)+U_k\right)ds. \end{aligned}$$

We will consider the equation (15) as a perturbation of the linear equation

$$\dot{w}(t) = A(k)w(t) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)w(t+s)ds$$

Remark 12. If we consider the equation (15) as the three-parameter family of abstract semilinear functional differential equation

$$\dot{w} = \tilde{A}(k)w + \mathscr{L}(k, U_k, w_t) + G(\alpha,\beta,w_t),$$

where $\tilde{A}(k) = (D^2 + k)$, $\mathscr{L}$ is the linear operator defined by linear differential equation

$$\dot{w} = kg_a(U_k)w + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)w(t+s)ds,$$

we could try to use the results in [12] about Hopf bifurcation, but the projection of X onto the eigenspace associated to each pair of eigenvalues of $\tilde{A}(k)$ does not commute with the solution operator of the linear functional differential equation given by the operator $\mathscr{L}$, hypotheses necessary there.

We now describe a notation which we will use during the remainder of the paper

(i) $\langle y, z\rangle^* = \int_0^\pi y(x)z(x)dx$, for $y, z \in X = L^2[0,\pi]$

(ii) For $\phi \in C = C([-\tau_{k_n}, 0]; X)$; $\psi \in C^*([0, \tau_{k_n}]; X)$, we define the following bilinear form:

$$(\psi,\phi) = \langle\psi(0),\phi(0)\rangle^* - \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}\left\langle \int_{-\tau_{k_n}}^{\theta} \frac{k}{\delta} g_b(U_k)\psi(\theta-\tau)d\tau, \phi(\theta)\right\rangle^* d\theta$$
$$+ \int_{-\tau_{k_n}+\delta}^{0}\left\langle \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{k}{\delta} g_b(U_k)\psi(-\tau_{k_n}+\delta-\tau)d\tau, \phi(-\tau_{k_n}+\delta)\right\rangle^* d\theta.$$

The operator A^*, relative to that bilinear form, satisfies $(\psi, A\phi) = (A^*\psi, \phi)$.

(iii) Let y_k and S_{k_n} be defined as in (4) and Lemma 10, respectively and let

$$\tilde{\Phi}(\theta) = \left[\, y_k e^{i\omega_k\theta} \quad \bar{y}_k e^{-i\omega_k\theta}\,\right], \quad \theta \in [-\tau_{k_n}, 0] \qquad \text{and}$$
$$\tilde{\Psi}(s) = \begin{bmatrix} y_k e^{-i\omega_k s}/S_{k_n} \\ \bar{y}_k e^{i\omega_k s}/\bar{S}_{k_n}\end{bmatrix}, \quad s \in [0, \tau_{k_n}],$$
$$\Phi(\theta) = [\,\Phi_1(\theta) \quad \Phi_2(\theta)\,] = \tilde{\Phi}(\theta)H \quad \text{and}$$
$$\Psi(s) = \begin{bmatrix}\Psi_1(s)\\ \Psi_2(s)\end{bmatrix} = H^{-1}\tilde{\Psi}(s); \qquad \text{where } H = \frac{1}{2}\begin{bmatrix}1 & -i\\ 1 & i\end{bmatrix}.$$

(iv) Let Λ be the eigenfunction space of $A_{\tau_{k_n}}(k)$ corresponding to $\lambda = \pm i\omega_k$. With the above notations, Φ is a real basis of Λ, and Ψ is a real basis of the eigenfunction space of the formal adjoint $A^*_{\tau_{k_n}}(k)$ of $A_{\tau_{k_n}}(k)$, corresponding to $\lambda = \pm i\omega_k$. Furthermore, using the fact that $(\psi, A\phi) = (A^*\psi, \phi)$, we have

$$i\omega_k(y_k e^{-i\omega_k\cdot}, \bar{y}_k e^{-i\omega_k\cdot}) = (A^*_{\tau_{k_n}} y_k e^{-i\omega_k\cdot}, \bar{y}_k e^{-i\omega_k\cdot}) = (y_k e^{-i\omega_k\cdot}, A_{\tau_{k_n}}\bar{y}_k e^{-i\omega_k\cdot})$$
$$= (y_k e^{-i\omega_k\cdot}, -i\omega_k \bar{y}_k e^{-i\omega_k\cdot}) = -i\omega_k(y_k e^{-i\omega_k\cdot}, \bar{y}_k e^{-i\omega_k\cdot})$$

and, consequently, $(y_k e^{-i\omega_k\cdot}, \bar{y}_k e^{-i\omega_k\cdot}) = 0$. This remark tells us that

$$\langle y_k, \bar{y}_k\rangle^* + \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}\left\langle \int_{-\tau_{k_n}}^{\theta} \frac{k}{\delta} g_b(U_k) y_k e^{-i\omega_k(\theta-\tau)}d\tau, \bar{y}_k e^{-i\omega_k\theta}\right\rangle^* d\theta$$
$$+ \int_{-\tau_{k_n}+\delta}^{0}\left\langle \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{k}{\delta} g_b(U_k) y_k e^{-i\omega_k(-\tau_{k_n}+\delta-\tau)}d\tau, \bar{y}_k e^{-i\omega_k(-\tau_{k_n}+\delta)}\right\rangle^* d\theta = 0,$$

that is,

$$\int_0^\pi y_k \bar{y}_k \, dx + \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \int_0^\pi \int_{-\tau_{k_n}}^{\theta} \frac{k}{\delta} g_b(U_k) y_k e^{(-2i\omega_k\theta + i\omega_k\tau)} \, d\tau \bar{y}_k dx d\theta$$

$$+ (\tau_{k_n} - \delta) \int_0^\pi \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{k}{\delta} g_b(U_k) y_k e^{-2i\omega_k(-\tau_{k_n}+\delta)} e^{i\omega_k\tau} \, d\tau \bar{y}_k dx = 0.$$

Hence,

$$\int_0^\pi y_k \bar{y}_k \, dx + \int_0^\pi \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{k}{\delta} g_b(U_k) \frac{e^{i\omega_k\theta}}{i\omega_k} e^{-2i\omega_k\theta} \, d\theta y_k \bar{y}_k dx$$

$$+ \int_0^\pi \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} k g_b(U_k) \frac{e^{-i\omega_k}}{i\omega_k} e^{-2i\omega_k\theta} \, d\theta y_k \bar{y}_k dx$$

$$+ \tau_{k_n} \int_0^\pi \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{k}{\delta} g_b(U_k) e^{i\omega_k\tau} e^{2i\theta_k} e^{-2i\omega_k\delta} \, d\tau y_k \bar{y}_k dx$$

$$- \int_0^\pi \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} k g_b(U_k) e^{i\omega_k\tau} e^{2i\theta_k} e^{-2i\omega_k\delta} \, d\tau y_k \bar{y}_k dx = 0.$$

With that, we can verify that $(\Psi, \Phi) = I$, where $I \in \boldsymbol{R}^{2\times 2}$ is the identity. In fact,

$$(\Psi, \Phi) = \langle \Psi(0), \Phi(0) \rangle^* + \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \left\langle \int_{-\tau_{k_n}}^{\theta} \frac{k}{\delta} g_b(U_k) \Psi(\theta - \tau) d\tau, \Phi(\theta) \right\rangle^* d\theta$$

$$+ \int_{-\tau_{k_n}+\delta}^{0} \left\langle \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{k}{\delta} g_b(U_k) \Psi(-\tau_{k_n} + \delta - \tau) d\tau, \Phi(-\tau_{k_n} + \delta) \right\rangle^* d\theta$$

$$= H^{-1} \left\{ \int_0^\pi \begin{bmatrix} y_k^2 / S_{k_n} & y_k \bar{y}_k / S_{k_n} \\ \bar{y}_k y_k / \bar{S}_{k_n} & \bar{y}_k^2 / \bar{S}_{k_n} \end{bmatrix} dx + \int_0^\pi \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \int_{-\tau_{k_n}}^{\theta} \frac{k}{\delta} g_b(U_k) \right.$$

$$\times \begin{bmatrix} y_k^2 e^{i\omega_k\tau} / S_{k_n} & y_k \bar{y}_k e^{-2i\omega_k\theta} e^{i\omega_k\tau} / S_{k_n} \\ \bar{y}_k y_k e^{2i\omega_k\theta} e^{-i\omega_k\tau} / \bar{S}_{k_n} & \bar{y}_k^2 e^{-i\omega_k\tau} / \bar{S}_{k_n} \end{bmatrix} d\tau d\theta dx$$

$$+ \int_0^\pi \int_{-\tau k_n+\delta}^{0} \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{k}{\delta} g_b(U_k)$$

$$\left. \times \begin{bmatrix} y_k^2 e^{i\omega_k\tau} / S_{k_n} & y_k \bar{y}_k e^{-2i\omega_k(-\tau_{k_n}+\delta)} e^{i\omega_k\tau} / S_{k_n} \\ \bar{y}_k y_k e^{2i\omega_k(-\tau_{k_n}+\delta)} e^{-i\omega_k\tau} / \bar{S}_{k_n} & \bar{y}_k^2 e^{-i\omega_k\tau} / \bar{S}_{k_n} \end{bmatrix} d\tau d\theta dx \right\} H.$$

Solving all that integrals, we have that the second entry is exactly the term obtained at the above remark, which is null, and the third entry is the conjugate of this term which is null too. Now, the numerator of the first entry is exactly

S_{k_n} and the numerator of the fourth entry is $\bar{S}_{k_n}$. We obtain the following conclusion:

$$(\Psi, \Phi) = H^{-1}\left\{\begin{bmatrix} S_{k_n}/S_{k_n} & 0 \\ 0 & \bar{S}_{k_n}/\bar{S}_{k_n} \end{bmatrix}\right\}H = I_{2\times 2}.$$

For $\phi \in \mathscr{C}$, we have $\pi_\Lambda(\phi) = \Phi(\Psi, \phi) = (\Psi_1, \phi)\Phi_1 + (\Psi_2, \phi)\Phi_2$.

(v) Let $\mathscr{P}_{v_k} = \{f \in \mathscr{C}(\boldsymbol{R}, X); f(t + v_k) = f(t), t \in \boldsymbol{R}\}$ with the norm defined by

$$\|f\|_{\mathscr{P}_{v_k}} = \sup_{t\in[0, v_k]} \|f(t)\|_X, \qquad \text{for } f \in \mathscr{P}_{v_k}.$$

(vi) Let $\langle \Psi, f\rangle^* = \begin{bmatrix} \langle \Psi_1, f\rangle^* \\ \langle \Psi_2, f\rangle^* \end{bmatrix}$ and define $\mathbf{I} : \mathscr{P}_{v_k} \to \boldsymbol{R}^2$ by

$$\mathbf{I}f = \int_0^{v_k} \langle \Psi(s), f(s)\rangle^* ds. \tag{16}$$

We know by [9] that for each $f \in \mathscr{P}_{v_k}$, the equation

$$\dot{w}(t) = A(k)w(t) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)w(t+s)ds + f(t) \tag{17}$$

has a v_k-periodic solution if and only if $f \in N(\mathbf{I})$. Besides, there exists a continuous projection $\mathscr{J} : \mathscr{P}_{v_k} \to \mathscr{P}_{v_k}$ such that the collection of f in $\mathscr{P}_{v_k}$, satisfying (16) is $(I - \mathscr{J})\mathscr{P}_{v_k}$ and there is a continuous linear operator $\mathscr{K} : (I - \mathscr{J})\mathscr{P}_{v_k} \to (I - \pi_\Lambda)\mathscr{P}_{v_k}$ such that $\mathscr{K}f$ is a solution of (17) for each $f \in (I - \mathscr{J})\mathscr{P}_{v_k}$. This is the same as $\mathbf{I}f = 0$ if and only if $\mathscr{J}f = 0$; $\mathscr{K} : N(\mathbf{I}) \to \mathscr{P}_{v_k}$ and $\mathscr{K}f$ is solution of (17) if $\pi_\Lambda(\mathscr{K}f) = 0$ or $(\Psi, \mathscr{K}(f)) = 0$.

Then, up to a time translation, equation (15) has a v_k-periodic solution $w(t)$ if and only if there is a constant c such that

$$\begin{cases} (a)\ \ \mathbf{I}G(\alpha, \beta, w) = 0 \\ (b)\ \ w(t) = c\Phi_1(t) + \mathscr{K}G(\alpha, \beta, w)(t); \quad t \in \boldsymbol{R} \end{cases} \tag{18}$$

where $\Phi_1(t) = [y_k e^{i\omega_k t} + \bar{y}_k e^{-i\omega_k t}]/2 = \operatorname{Re}(y_k e^{i\omega_k t})$, $t \in \boldsymbol{R}$.

Using the implicit function theorem we solve

$$w(t) = c\Phi_1(t) + \mathscr{K}(I - \mathscr{J})G(\alpha, \beta, w)(t); \qquad t \in \boldsymbol{R}$$

for $w^* = w^*(c, \alpha, \beta)$ and c, α, β in a sufficiently small neighborhood of zero, $w^*(c, 0, 0) - c\Phi_1 = o(|c|)$ when $|c| \to 0$. We can also verify that the function $w^*(c, \alpha, \beta)(t)$ is continuously differentiable in c, α, β and t (see [17]). Since $G(0, 0, 0) = 0$ and $\partial G(0, 0, 0)/\partial\omega = 0$, we have that $G(0, 0, w) = O(\|\omega\|^2)$, and hence,

$$w^*(c,0,0) - c\Phi_1 = \mathscr{K}(I-\mathscr{J})G(0,0,w^*(c,0,0)) = O(\|\omega\|^2), \qquad \text{as } \|\omega\| \to 0.$$

With this information, we have $w^*(c,0,0) - c\Phi_1 = o(|c|^2)$, as $|c| \to 0$.

For each c let be μ and γ such that $\alpha = c\mu$ and $\beta = c\gamma$. If we make $w^*(c,c\mu,c\gamma)(t) - c\Phi_1(t) = c^2 W(c,t)$, we have $W(c,\cdot) \in \mathscr{P}_{v_k}$. So, the two equations in (18) are equivalent to

$$\text{(19)} \qquad \begin{cases} \mathscr{F}(c,\mu,\gamma,W) = \int_0^{v_k} \langle \Psi, N(c,\mu,\gamma,W_s)\rangle^* ds = 0 \\ W = \mathscr{K}N(c,\mu,\gamma,W) \end{cases}$$

where

$$\begin{aligned} G(c\mu, c\gamma, w_t) = c^2 \Bigg\{ &\gamma A(k)[\Phi_1(t) + cW(c,t)] - \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k) W(c, t+s) ds \\ &+ (1+c\gamma)\frac{k}{\delta}\int_{-c\mu-\tau_{k_n}}^{-c\mu-\tau_{k_n}+\delta} g_b(U_k) W\left(c, t + \frac{s}{1+c\gamma}\right) ds \\ &+ \gamma \frac{k}{\delta}\int_{-c\mu-\tau_{k_n}}^{-c\mu-\tau_{k_n}+\delta} g_b(U_k)\Phi_1\left(t + \frac{s}{1+c\gamma}\right) ds \\ &- \frac{k}{\delta}\int_0^1 \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)\dot{\Phi}_1\left(t + s - \theta\frac{c\mu + sc\gamma}{1+c\gamma}\right)\left(\frac{\mu + s\gamma}{1+c\gamma}\right) ds d\theta \\ &- (1+c\gamma)k\frac{g(U_k)}{c^2} - (1+c\gamma)k\frac{g_a(U_k)}{c^2}[c\Phi_1(t) + c^2 W(c,t)] \\ &- (1+c\gamma)\frac{k}{\delta}\int_{-c\mu-\tau_{k_n}}^{-c\mu-\tau_{k_n}+\delta} \frac{g_b(U_k)}{c^2}\Bigg[c\Phi_1\left(t + \frac{s}{1+c\gamma}\right) \\ &+ c^2 W\left(c, t + \frac{s}{1+c\gamma}\right)\Bigg] ds \\ &+ (1+c\gamma)\frac{k}{\delta}\int_{-c\mu-\tau_{k_n}}^{-c\mu-\tau_{k_n}+\delta} \frac{g}{c^2}\Bigg(c\Phi_1(t) + c^2 W(c,t) \\ &+ U_k, c\Phi_1\left(t + \frac{s}{1+c\gamma}\right) + c^2 W\left(c, t+\frac{s}{1+c\gamma}\right) + U_k\Bigg) ds \Bigg\} \\ = c^2 N(c,\mu,\gamma,W_t).& \end{aligned}$$

Since a periodic solution of equation (15) is C^1 function, in (19), we can restrict our discussion to $W \in \mathscr{P}_{v_k}^1$, where $\mathscr{P}_{v_k}^1 = \{f \in \mathscr{P}_{v_k}; \dot{f} \in \mathscr{P}_{v_k}\}$ endowed with the norm $\|f\|_{\mathscr{P}_{v_k}^1} = \|f\|_{\mathscr{P}_{v_k}} + \|\dot{f}\|_{\mathscr{P}_{v_k}}$. Then it is not difficult to see that

$\mathscr{F} : I_{\bar{\rho}} \times \boldsymbol{R} \times I_{\bar{\rho}} \times \mathscr{P}^1_{v_k} \to \boldsymbol{R}^2$ is a continuously differentiable function, where $I_{\bar{\rho}} = [-\bar{\rho}, \bar{\rho}]$, with $0 < \bar{\rho} < 1$. Since $c^2 W(c, \cdot) = o(|c|)$, by Taylor's formula we have

$$\begin{aligned}
&g\left(c\Phi_1(t) + c^2 W(c,t) + U_k, c\Phi_1\left(t + \frac{s}{1+c\gamma}\right) + c^2 W\left(c, t + \frac{s}{1+c\gamma}\right) + U_k\right) \\
&\quad = g(U_k) + g_a(U_k)[c\Phi_1(t) + c^2 W(c,t)] \\
&\qquad + g_b(U_k)\left[c\Phi_1\left(t + \frac{s}{1+c\gamma}\right) + c^2 W\left(c, t + \frac{s}{1+c\gamma}\right)\right] + \frac{\partial^2 g(U_k)}{\partial a \partial b} \\
&\qquad \times [c\Phi_1(t) + c^2 W(c,t)]\left[c\Phi_1\left(t + \frac{s}{1+c\gamma}\right) + c^2 W\left(c, t + \frac{s}{1+c\gamma}\right)\right] + o(|c^2|),
\end{aligned}$$

we see that

$$N(0,0,0,W) = \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{\partial^2 g(U_k)}{\partial a \partial b} \Phi_1(t)\Phi_1(t+s)ds.$$

Thus,

$$\mathscr{F}(0,0,0,W) = \int_0^{v_k} \left\langle \Psi(s), \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{\partial^2 g(U_k)}{\partial a \partial b} \Phi_1(s)\Phi_1(s+\theta)d\theta \right\rangle^* ds. \tag{20}$$

Using the definitions of Ψ, Φ_1, we have:

$$\mathscr{F}(0,0,0,W) = \frac{k}{\delta} H^{-1} \int_0^{v_k} \int_0^{\pi} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} dx ds H,$$

with

$$\begin{aligned}
a_{11} = \frac{\partial^2 g(U_k)}{\partial a \partial b} \cdot \frac{1}{4S_{k_n}} \Bigg[& y_k^3 e^{i\omega_k s} \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{i\omega_k \theta}\, d\theta + y_k^2 \bar{y}_k e^{-i\omega_k s} \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{-i\omega_k \theta}\, d\theta \\
& + \bar{y}_k y_k^2 e^{-i\omega_k s} \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{i\omega_k \theta}\, d\theta + y_k \bar{y}_k^2 e^{-3i\omega_k s} \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{-i\omega_k \theta}\, d\theta \Bigg],
\end{aligned}$$

$a_{21} = \bar{a}_{11}$

Recalling that $\int_0^{v_k} e^{ni\omega_k s}\, ds = 0$ for all $n \neq 0$, we obtain

$$\mathscr{F}(0,0,0,W) \equiv 0. \tag{21}$$

This implies that

$$\tilde{f}(s) = \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{\partial^2 g(U_k)}{\partial a \partial b} \Phi_1(s)\Phi_1(s+\theta)d\theta \in N(\mathbf{I}).$$

Next we solve the bifurcation equation (19) in a suitable neighborhood of the origin.

Lemma 13.

$$\frac{\partial \mathscr{F}(0,0,0,W)}{\partial(\mu,\gamma)} = v_k \begin{bmatrix} \mathrm{Re}(\dot{\lambda}(\tau_{k_n})) & 0 \\ -\mathrm{Im}(\dot{\lambda}(\tau_{k_n})) & -\omega_k \end{bmatrix}$$

with $\dot{\lambda}(\tau_{k_n})$ *previously defined.*

Proof. First of all, we calculate $\partial \mathscr{F}(0,0,0,W)/\partial\mu$. We have

$$N(0,\mu,0,W_t) = \frac{-k}{\delta} g_b(U_k) \int_0^1 \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \dot{\Phi}_1(t+s)\mu \, ds d\theta + \frac{k}{\delta} \frac{\partial^2 g(U_k)}{\partial a \partial b} \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \Phi_1(t)\Phi_1(t+s)ds$$

and thus,

$$\frac{\partial N(0,\mu,0,W_t)}{\partial\mu} = \frac{-k}{\delta} g_b(U_k) \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \dot{\Phi}_1(t+s)ds.$$

We remark that $\dot{\Phi}_1(t+\theta) = (i\omega_k/2)(\tilde{\Phi}_1(t+\theta) - \tilde{\Phi}_2(t+\theta))$. Then,

$$\frac{\partial \mathscr{F}(0,0,0,W_s)}{\partial\mu} = \frac{-ki\omega_k}{2\delta} \int_0^{v_k} \int_0^{\pi} H^{-1} \begin{bmatrix} \tilde{\Psi}_1(s) \\ \tilde{\Psi}_2(s) \end{bmatrix} g_b(U_k) \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} (\tilde{\Phi}_1(s+\theta) - \tilde{\Phi}_2(s+\theta)) d\theta dx ds.$$

If we let

$$M = \begin{bmatrix} \tilde{\Psi}_1(s) \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \tilde{\Phi}_1(s+\theta)d\theta & -\tilde{\Psi}_1(s) \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \tilde{\Phi}_2(s+\theta)d\theta \\ \tilde{\Psi}_2(s) \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \tilde{\Phi}_1(s+\theta)d\theta & -\tilde{\Psi}_2(s) \int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \tilde{\Phi}_2(s+\theta)d\theta \end{bmatrix}_{2\times 2}$$

then

$$\frac{\partial \mathscr{F}(0,0,0,W_s)}{\partial\mu} = \frac{-ki\omega_k}{\delta} \int_0^{v_k} \int_0^{\pi} g_b(U_k) H^{-1} M H \begin{pmatrix} 1 \\ 0 \end{pmatrix} dx ds.$$

It is not difficult to see that

$$\frac{\partial \mathscr{F}(0,0,0,W_s)}{\partial\mu} = H^{-1} \begin{bmatrix} v_k \dot{\lambda}(\tau_{k_n}) & 0 \\ 0 & v_k \dot{\lambda}(\tau_{k_n}) \end{bmatrix} H \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

thus,

$$\frac{\partial \mathscr{F}(0,0,0,W_s)}{\partial \mu} = v_k \begin{bmatrix} \text{Re}\ \dot{\lambda}(\tau_{k_n}) \\ -\text{Im}\ \dot{\lambda}(\tau_{k_n}) \end{bmatrix}.$$

Now we calculate $\partial \mathscr{F}(0,0,0,W_s)/\partial \gamma$.

$$N(0,0,\gamma,W_t) = \gamma A(k)\Phi_1(t) + \frac{\gamma k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)[\Phi_1(t+s) - \dot{\Phi}_1(t+s)s]ds$$
$$+ \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \frac{\partial^2 g(U_k)}{\partial a \partial b}\Phi_1(t)\Phi_1(t+s)ds$$

and so,

$$\frac{\partial N(0,0,\gamma,W_t)}{\partial \gamma} = A(k)\Phi_1(t) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)[\Phi_1(t+s) - \dot{\Phi}_1(t+s)s]ds.$$

Thus,

$$\frac{\partial \mathscr{F}(0,0,0,W_s)}{\partial \gamma} = \int_0^{v_k}\int_0^{\pi} H^{-1}\begin{bmatrix}\tilde{\Psi}_1(s) \\ \tilde{\Psi}_2(s)\end{bmatrix} A(k)[\,\tilde{\Phi}_1(s) \quad \tilde{\Phi}_2(s)\,]H\begin{pmatrix}1\\0\end{pmatrix}dxds$$
$$+ \frac{k}{\delta}\int_0^{v_k}\int_0^{\pi} H^{-1}\begin{bmatrix}\tilde{\Psi}_1(s) \\ \tilde{\Psi}_2(s)\end{bmatrix} g_b(U_k)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}[\,\tilde{\Phi}_1(s+\theta) \quad \tilde{\Phi}_2(s+\theta)\,]H\begin{pmatrix}1\\0\end{pmatrix}d\theta dxds$$
$$- \frac{k}{\delta}\int_0^{v_k}\int_0^{\pi} H^{-1}\begin{bmatrix}\tilde{\Psi}_1(s) \\ \tilde{\Psi}_2(s)\end{bmatrix} g_b(U_k)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}\frac{i\omega_k\theta}{2}(\tilde{\Phi}_1(s+\theta) - \tilde{\Phi}_2(s+\theta))d\theta dxds.$$

Developing this expressions and letting

$$M_\theta = \begin{bmatrix} \tilde{\Psi}_1(s)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}\tilde{\Phi}_1(s+\theta)\theta\, d\theta & -\tilde{\Psi}_1(s)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}\tilde{\Phi}_2(s+\theta)\theta\, d\theta \\ \tilde{\Psi}_2(s)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}\tilde{\Phi}_1(s+\theta)\theta\, d\theta & -\tilde{\Psi}_2(s)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta}\tilde{\Phi}_2(s+\theta)\theta\, d\theta \end{bmatrix}_{2\times 2}$$

we have

$$\frac{\partial \mathscr{F}(0,0,0,W_s)}{\partial \gamma} = H^{-1}\left\{\int_0^{v_k}\int_0^{\pi}\begin{bmatrix}\tilde{\Psi}_1(s)A_{\tau_{k_n}}\tilde{\Phi}_1(s) & -\tilde{\Psi}_1(s)A^*_{\tau_{k_n}}\tilde{\Phi}_2(s) \\ \tilde{\Psi}_2(s)A_{\tau_{k_n}}\tilde{\Phi}_1(s) & -\tilde{\Psi}_2(s)A^*_{\tau_{k_n}}\tilde{\Phi}_2(s)\end{bmatrix}dxds\right.$$
$$\left. - \frac{ki\omega_k}{\delta}\int_0^{v_k}\int_0^{\pi} g_b(U_k)M_\theta\, dxds\right\}H\begin{pmatrix}1\\0\end{pmatrix}$$

which provides us

$$\frac{\partial \mathscr{F}(0,0,0,W_s)}{\partial \gamma} = H^{-1}\begin{bmatrix} i\omega_k v_k S_{k_n}/S_{k_n} & 0 \\ 0 & -i\omega_k v_k \overline{S}_{k_n}/\overline{S}_{k_n} \end{bmatrix} H\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -v_k \begin{bmatrix} 0 \\ \omega_k \end{bmatrix},$$

and the proof of the lemma is complete. ■

Lemma 14. *Let* $f(\cdot) = (k/\delta)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} (\partial^2 g(U_k)/\partial a \partial b)\Phi_1(\cdot)\Phi_1(\cdot+\theta)d\theta$. *Then, if* $W_k = \mathscr{K}(f)$,

$$\text{(22)} \quad W_k(t) = -\frac{\partial^2 g(U_k)}{\partial a \partial b}\left[\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{i\omega_k\theta}\, d\theta Y_k e^{2i\omega_k t} + \frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{-i\omega_k\theta}\, d\theta \overline{Y}_k e^{-2i\omega_k t} + \frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \cos\omega_k\theta\, d\theta Z_k\right] + \Phi(t)b$$

where

$$Y_k = \frac{k}{4}\left[A(k) + \frac{k}{\delta}g_b(U_k)\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{2i\omega_k\theta}\, d\theta - 2i\omega_k\right]^{-1} y_k^2,$$

$$Z_k = \frac{k}{2}[A(k) + kg_b(U_k)]^{-1} y_k \overline{y}_k e,$$

$$b = -\left(\Psi, -\frac{\partial^2 g(U_k)}{\partial a \partial b}\left[\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{i\omega_k\theta}\, d\theta Y_k e^{2i\omega_k \cdot} + \frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{-i\omega_k\theta}\, d\theta \overline{Y}_k e^{-2i\omega_k \cdot} + \frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \cos\omega_k\theta\, d\theta Z_k\right]\right).$$

Proof. Obviously $W_k(t)$ is ω_k-periodic. We now show that $W_k(t)$ is a solution of

$$\dot{w}(t) = A(k)w(t) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)w(t+s)ds + f(t)$$

with $f(t)$ given above. In fact,

$$A(k)w(t) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)w(t+s)ds - \dot{w}(t)$$

$$= -\frac{\partial^2 g(U_k)}{\partial a \partial b}\left[A(k) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)e^{2i\omega_k s}\, ds - 2i\omega_k\right]\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{i\omega_k\theta}\, d\theta Y_k e^{2i\omega_k t}$$

$$-\frac{\partial^2 g(U_k)}{\partial a \partial b}\left[A(k) + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)e^{-2i\omega_k s}\, ds + 2i\omega_k\right]\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{-i\omega_k\theta}\, d\theta \overline{Y}_k e^{-2i\omega_k t}$$

$$- \frac{\partial^2 g(U_k)}{\partial a \partial b}[A(k) + kg_b(U_k)]\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \cos \omega_k \theta \, d\theta Z_k$$

$$+ A(k)\Phi(t)b + \frac{k}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} g_b(U_k)\Phi(t+s)b \, ds - \dot{\Phi}(t)b$$

$$= -\frac{\partial^2 g(U_k)}{\partial a \partial b}\frac{k}{4}y_k^2\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{i\omega_k\theta} \, d\theta e^{2i\omega_k t} - \frac{\partial^2 g(U_k)}{\partial a \partial b}\frac{k}{4}\bar{y}_k^2\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} e^{-i\omega_k\theta} \, d\theta e^{-2i\omega_k t}$$

$$- \frac{\partial^2 g(U_k)}{\partial a \partial b}\frac{k}{2}y_k\bar{y}_k\frac{1}{\delta}\int_{-\tau_{k_n}}^{-\tau_{k_n}+\delta} \cos \omega_k \theta \, d\theta = -f(t),$$

recalling that $\Phi(t)b$ is a solution of the homogeneous equation. Also $(\Psi, \Phi) = I$, and by definition of b, we have that $(\Psi, W_k) = 0$. Therefore, $W_k = \mathscr{K}(f)$. ■

Now we can state our main theorem:

Theorem 15. *For each fixed k, $k \in (1, k^*)$, a Hopf bifurcation will occur as delay τ passes through each of the points τ_{k_n}, $n = 0, 1, 2, \ldots$, arising a periodic solution $U_{k_n,\tau}$ near U_k with period $T(\tau_{k_n}) \approx 2\pi/\omega_k$.*

Proof. We know by (21) that $\mathscr{F}(0,0,0,W) \equiv 0$, and by (13)

$$\frac{\partial \mathscr{F}(0,0,0,W)}{\partial(\mu,\gamma)} = v_k \begin{bmatrix} \mathrm{Re}(\dot{\lambda}(\tau_{k_n})) & 0 \\ -\mathrm{Im}(\dot{\lambda}(\tau_{k_n})) & -\omega_k \end{bmatrix}.$$

Since $\mathrm{Re}\,\dot{\lambda}(\tau_{k_n}) > 0$, there are a neighborhood $\boldsymbol{B} \subset \boldsymbol{R}$ of the origin, a neighborhood $V_0 \subseteq \mathscr{P}^1_{v_k}$ of W_k, $c_0 > 0$, and continuously differentiable functions $\mu : [-c_0, c_0] \times V_0 \to \boldsymbol{B}$, $\gamma : [-c_0, c_0] \times V_0 \to \boldsymbol{B}$ such that $\mu(0, W_k) = \gamma(0, W_k) = 0$ and for each $(c, W) \in [-c_0, c_0] \times V_0$, $(\mu, \gamma) \in \boldsymbol{B} \times \boldsymbol{B}\mathscr{F}(c, \mu, \gamma, W) = 0$ if and only if $\mu = \mu(c, W)\gamma = \gamma(c, W)$. We define $\Omega : [-c_0, c_0] \times V_0 \to \mathscr{P}^1_{v_k}$ by

$$\Omega(c, W) = W - \mathscr{K}(N(c, \mu(c, W), \gamma(c, W), W)). \tag{23}$$

Then $\Omega(0, W_k) = 0$. By (21), $\partial\mathscr{F}(0,0,0,W_k)/\partial W = 0$. By differentiating the equation (23) with respect to W at $c = 0$, we obtain $(\partial\Omega/\partial W)(c, W)|_{c=0} = I$. Thus, $(\partial\Omega/\partial W)(0, W_k) : \mathscr{P}^1_{v_k} \to \mathscr{P}^1_{v_k}$ is bijective and there are $c_1 \in (0, c_0]$, $V_1 \subseteq V_0$ neighborhood of W_k and $W^* : [-c_1, c_1] \to V_1$ such that $W^*(0) = W_k$ and for $(c, W) \in [-c_1, c_1] \times V_1$, $\Omega(c, W) = 0$ if and only if $W = W^*(c)$. Therefore, there is a v_k-periodic solution for the equation (15), with $W(t)$ near zero for α, β sufficiently small if and only if

$$W(t) = c\Phi_1(t) + [\mathscr{K}G(\alpha, \beta, w)](t)$$

with $\alpha = c\mu(c, W^*(c))$, $\beta = c\gamma(c, W^*(c))$, for some value $c \in [-c_1, c_1]$, with $W^*(c, t) = W^*(c)(t)$. ■

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