Funkcialaj Ekvacioj, **47** (2004) 361–394

On a Resolvent Estimate of a System of Laplace Operators with Perfect Wall Condition

By

T. Akiyama[1], H. Kasai[2], Y. Shibata[3*] and M. Tsutsumi[4]

(Waseda University[1,3,4] and Fukushima University[2], Japan)

Abstract. This paper is concerned with the study of the system of Laplace operators with perfect wall condition in the L_p framework. Our study includes a bounded domain, an exterior domain and a domain having noncompact boundary such as a perturbed half space. A direct application of our study is to prove the analyticity of the semigroup corresponding to the Maxwell equation of parabolic type, which appears as a linearized equation in the study of the nonstationary problem concerning the Ginzburg-Landau-Maxwell equation describing the Ginzburg-Landau model for superconductivity, the magnetohydrodynamic equation and the Navier-Stokes equation with Neumann boundary condition. And also, our theory is applicable to some solvability of the stationary problem of these nonlinear equations in the L_p framework.

Key Words and Phrases. A system of Laplace operators, Perfect wall condition, Resolvent estimate.

2000 *Mathematics Subject Classification Numbers.* 35J55, 35J25.

1. Introduction

1.1. Main results

In this paper we investigate the resolvent problem for the system of Laplace operators with perfect wall condition on some domain $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$:

$$(1.1) \qquad \lambda u - \Delta u = f + \nabla \cdot F \qquad \text{in } \Omega,$$

$$-(\operatorname{curl} u)\nu = g + F\nu, \qquad \nu \cdot u = 0 \qquad \text{on } \partial\Omega,$$

where $\partial\Omega$ is the boundary of Ω; ν is the unit outer normal vector to $\partial\Omega$; the resolvent parameter λ is contained in the sector

$$\Sigma_\varepsilon = \{0 \neq z \in \boldsymbol{C} \mid |\arg z| < \pi - \varepsilon\}, \qquad 0 < \varepsilon < \pi/2;$$

$f = {}^t(f_1, \dots, f_n) \in L_p(\Omega)^n$, $F = (F_{ij}) \in W_p^1(\Omega)^{n \times n}$ and $g = {}^t(g_1, \dots, g_n) \in W_p^1(\Omega)^n$ are the prescribed forces which satisfy the conditions:

* This work is partially supported by Grant-in-Aid for Scientific Research (B) 12440045, Ministry of Education, Sciences, Sports and Culture, Japan.

$$(1.2) \qquad \nu \cdot (F\nu)|_{\partial\Omega} = 0 \quad \text{and} \quad \nu \cdot g|_{\partial\Omega} = 0.$$

To state our result precisely, at this point we outline our notation used throughout the paper. Given vector or matrix M, ${}^tM$ means the transposed M. Given Banach space X with norm $\|\cdot\|_X$, we set

$$X^n = \{v = {}^t(v_1, \ldots, v_n) \mid v_j \in X\}, \qquad \|v\|_X = \sum_{j=1}^n \|v_j\|_X;$$

$$X^{n\times n} = \{V = (V_{ij}) \mid V_{ij} \in X\}, \qquad \|V\|_X = \sum_{i,j=1}^n \|V_{ij}\|_X;$$

and $V = (V_{ij})$ means the $n \times n$ matrix whose i-th column and j-th row component is V_{ij}. For the differentiation of the $n \times n$ matrix of functions $F = (F_{ij})$, the n-vector of functions $g = {}^t(g_1, \ldots, g_n)$ and the scalar function f, we use the following notation: $\partial_j f = \partial f/\partial x_j$,

$$\nabla f = {}^t(\partial_1 f, \ldots, \partial_n f), \quad \nabla \cdot g = \sum_{j=1}^n \partial_j g_j, \quad \nabla \cdot F = {}^t\left(\sum_{j=1}^n \partial_j F_{1j}, \ldots, \sum_{j=1}^n \partial_j F_{nj}\right),$$

$$\nabla g = (g_{ij}) \text{ with } g_{ij} = \partial_j g_i, \quad \operatorname{curl} g = \nabla g - {}^t(\nabla g) = (g_{ij}) \text{ with } g_{ij} = \partial_j g_i - \partial_i g_j,$$

where the dot $\cdot$ denotes the inner-product of $\boldsymbol{R}^n$. For the functional space, $L_p(\Omega)$ denotes the usual L_p space on Ω with norm $\|\cdot\|_{L_p(\Omega)}$. Moreover, we set

$$W_p^m(\Omega) = \left\{u \in L_p(\Omega) \mid \|u\|_{W_p^m(\Omega)} = \sum_{|\alpha| \leqq m} \|\partial_x^\alpha u\|_{L_p(\Omega)} < \infty\right\},$$

$$\dot{W}_p^1(\Omega)^n = \{g \in W_p^1(\Omega)^n \mid \nu \cdot g|_{\partial\Omega} = 0\},$$

$$\dot{W}_p^1(\Omega)^{n\times n} = \{F \in W_p^1(\Omega)^{n\times n} \mid \nu \cdot (F\nu)|_{\partial\Omega} = 0\}.$$

Note that what $F \in \dot{W}_p^1(\Omega)^{n\times n}$ and what $g \in \dot{W}_p^1(\Omega)^n$ imply that F and g satisfy the condition (1.2), respectively. $C = C(a, b, \ldots)$ means that the constant C depends on the quantities $a, b, \ldots$ in the parenthesis. Moreover, to denote generic constants we use the same letter C, and therefore the constants C may change from line to line. Given $R > 0$, we set

$$B_R = \{x \in \boldsymbol{R}^n \mid |x| < R\}, \qquad B^R = \boldsymbol{R}^n \backslash B_R.$$

We are interested in $L_p(\Omega)$ estimates of the unknown vector $u = {}^t(u_1, \ldots, u_n)$. To describe our main results we formulate some assumptions on the domain.

Assumption 1.1. Let $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, be a domain with boundary $\partial\Omega \in C^{2,1}$ and suppose one of the following cases:

(i) Ω is bounded.

(ii) Ω is an exterior domain, i.e., there exists a compact set $\mathcal{O}$ such that $\Omega = \boldsymbol{R}^n \backslash \mathcal{O}$.

(iii) Ω is a perturbed half space, i.e., there exists an $R > 0$ such that $\Omega \cap B^R = \boldsymbol{R}^n_+ \cap B^R$, where

$$\boldsymbol{R}^n_+ = \{x = (x_1, \dots, x_n) \in \boldsymbol{R}^n \mid x_n > 0\}.$$

For the notational simplicity, we set

$$\mathscr{I}^1_p(u, \lambda, \Omega) = |\lambda|^{1/2}\|u\|_{L_p(\Omega)} + \|\nabla u\|_{L_p(\Omega)},$$

$$\mathscr{I}^2_p(u, \lambda, \Omega) = |\lambda|\,\|u\|_{L_p(\Omega)} + |\lambda|^{1/2}\|\nabla u\|_{L_p(\Omega)} + \|\nabla^2 u\|_{L_p(\Omega)},$$

where

$$\nabla^k u = (\partial^\alpha_x u \mid |\alpha| = k), \qquad \nabla^1 u = \nabla u, \qquad \nabla^0 u = u.$$

The following theorems are our main results.

Theorem 1.2. *Let* $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $\delta > 0$ *and* $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, *be a domain satisfying the Assumption* 1.1. *Then, we have the following assertions:*

(1) *For every* $\lambda \in \Sigma_\varepsilon$, $f \in L_p(\Omega)^n$, $F \in \dot{W}^1_p(\Omega)^{n\times n}$ *and* $g \in \dot{W}^1_p(\Omega)^n$, *the problem* (1.1) *admits a unique solution* $u \in W^2_p(\Omega)^n$ *which satisfies the estimates*:

$$\mathscr{I}^1_p(u, \lambda, \Omega) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(\Omega)} + \|F\|_{L_p(\Omega)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial\Omega)}\}, \tag{1.3}$$

$$\mathscr{I}^2_p(u, \lambda, \Omega) \leqq C\{\|f\|_{L_p(\Omega)} + \mathscr{I}^1_p((F, g), \lambda, \Omega)\}, \tag{1.4}$$

where $C = C(\Omega, p, \varepsilon, \delta) > 0$ *and* $|\lambda| \geqq \delta$.

(2) *If* f *and* F *satisfy the conditions:* $\nabla\cdot f = 0$ *in* Ω *and* $\nu \cdot f = 0$ *on* $\partial\Omega$, *and* $F + {}^tF = 0$, *respectively, and* $g = 0$, *then* $\nabla \cdot u = 0$ *in* Ω.

(3) *If* $f \in W^m_p(\Omega)^n$, $F \in W^{m+1}_p(\Omega)^{n\times n}$, $g \in W^{m+1}_p(\Omega)$ *and* $\partial\Omega \in C^{m+2,1}$ *for some integer* $m \geqq 1$ *additionally, then* $u \in W^{m+2}_p(\Omega)^n$.

Remark 1.3. Noting the formula:

$$\Delta u = \nabla\cdot(\operatorname{curl} u) + \nabla(\nabla\cdot u), \qquad u = {}^t(u_1, \dots, u_n), \tag{1.5}$$

by Theorem 1.2 (2) we know the unique existence of solution $u \in W^2_p(\Omega)^n$ of the resolvent problem corresponding to the parabolic Maxwell equation:

$$\begin{aligned} \lambda u - \nabla\cdot(\operatorname{curl} u) + \nabla\Phi = f, &\qquad \nabla\cdot u = 0 \qquad \text{in } \Omega,\\ -(\operatorname{curl} u)\nu = 0, &\qquad \nu\cdot u = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

where Φ is a function corresponding to the Helmholtz decomposition of the n-vector f, i.e.

$$f = h + \nabla\Phi \text{ with } \nabla \cdot h = 0 \text{ in } \Omega \text{ and } \nu \cdot h|_{\partial\Omega} = 0.$$

Theorem 1.4. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$ and suppose that Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$. Then, there exists a positive constant δ_1 such that if $\lambda \in \boldsymbol{C}$ satisfies the condition: $|\lambda| \leqq \delta_1$, then for every $f \in L_p(\Omega)^3$, $F \in \dot{W}_p^1(\Omega)^{3\times 3}$ and $g \in \dot{W}_p^1(\Omega)^3$, the problem* (1.1) *admits a unique solution $u \in W_p^2(\Omega)^3$ which satisfies the estimates*:

$$\|u\|_{W_p^1(\Omega)} \leqq C\{\|(f, F)\|_{L_p(\Omega)} + \|g\|_{L_p(\partial\Omega)}\}, \tag{1.7}$$

$$\|u\|_{W_p^2(\Omega)} \leqq C\{\|f\|_{L_p(\Omega)} + \|(F, g)\|_{W_p^1(\Omega)}\}. \tag{1.8}$$

Moreover, if f and F satisfy the conditions: $\nabla \cdot f = 0$ in Ω and $\nu \cdot f = 0$ on $\partial\Omega$, and $F + {}^tF = 0$, respectively, and $g = 0$, then $\nabla \cdot u = 0$ in Ω.

Finally, we consider the weak solution corresponding to (1.1). In view of (1.5), by the divergence theorem of Gauss we have

$$\begin{aligned}(-\Delta u, v)_G = &-((\mathrm{curl}\, u)\nu_{\partial G}, v)_{\partial G} - (\nabla \cdot u, \nu_{\partial G} \cdot v)_{\partial G} \\ &+ \frac{1}{2}(\mathrm{curl}\, u, \mathrm{curl}\, v)_G + (\nabla \cdot u, \nabla \cdot v)_G,\end{aligned} \tag{1.9}$$

where $\nu_{\partial G}$ denotes the unit outer normal to the boundary ∂G of the domain G. Here and hereafter we set

$$(u, v)_G = \int_G u(x) \cdot v(x)dx, \qquad (u, v)_{\partial G} = \int_{\partial G} u(x) \cdot v(x)d\sigma,$$

where $d\sigma$ is the surface element of ∂G. By (1.9) we see that the variational formula corresponding to (1.1) is the following:

$$\lambda(u, v)_\Omega + \frac{1}{2}(\mathrm{curl}\, u, \mathrm{curl}\, v)_\Omega + (\nabla \cdot u, \nabla \cdot v)_\Omega = (f, v)_\Omega - (F, \nabla v)_\Omega - (g, v)_{\partial\Omega} \tag{1.10}$$

for any $v \in \dot{W}_{p'}^1(\Omega)^n$. Here and hereafter p' denotes the dual exponent of p such that $1/p + 1/p' = 1$.

Theorem 1.5. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$ and $\delta > 0$.* (1) *Let $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, be a domain satisfying the Assumption* 1.1. *Then, for every $\lambda \in \Sigma_\varepsilon$, $f \in L_p(\Omega)^n$, $F \in L_p(\Omega)^{n\times n}$ and $g \in L_p(\partial\Omega)^n$, the variational equation* (1.10) *admits a unique solution $u \in \dot{W}_p^1(\Omega)^n$ which satisfies the estimate* (1.3).

(2) *Suppose that Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$. Then, there exists a positive constant δ_1 such that if $\lambda \in \boldsymbol{C}$ satisfies the con-*

dition: $|\lambda| \leqq \delta_1$, *then for every* $f \in L_p(\Omega)^3$, $F \in L_p(\Omega)^{3\times 3}$ *and* $g \in L_p(\partial\Omega)^3$, *the variational equation* (1.10) *admits a unique solution* $u \in \dot{W}^1_p(\Omega)^3$ *which satisfies the estimate* (1.7).

(3) *In addition, we assume that* f *and* F *satisfy the conditions:* $\nabla \cdot f = 0$ *in* Ω *and* $\nu \cdot f = 0$ *on* $\partial\Omega$, *and* $F + {}^tF = 0$, *respectively, and that* $g = 0$ *in the both case of* (1) *and* (2). *Then,* $\nabla \cdot u = 0$ *in* Ω.

1.2. Motivation

In order to deal with some fundamental problems in mathematical physics, we meet the parabolic Maxwell equation:

$$(1.11) \qquad u_t + \nabla \times (\nabla \times u) + \mu u + \nabla \Phi = F, \qquad \nabla \cdot u = 0 \qquad \text{in } \Omega_T,$$
$$(\nabla \times u) \times \nu|_{\partial\Omega} = 0, \qquad \nu \cdot u|_{\partial\Omega} = 0, \qquad u|_{t=0} = a,$$

where Ω is some domain in $\boldsymbol{R}^3$, μ is a non-negative constant, $\Omega_T = \Omega \times [0, T)$ and t denotes the time variable.

In fact, A. Schmid [18] and L. P. Gor'kov and G. M. Eliashberg [10] asserted that the non-stationary Ginzburg-Landau model for superconductivity is described by the TDGLM (time dependent Ginzburg-Landau-Maxwell) equation:

$$(1.12) \qquad \psi_t - i\Phi\psi = (\nabla - iA)^2\psi + \kappa(1 - |\psi|^2\psi) \qquad \text{in } \Omega_T,$$
$$\sigma(A_t - \nabla\Phi) + \nabla \times (\nabla \times A) + \nabla \times H = J_{GL} \qquad \text{in } \Omega_T,$$
$$\nabla \cdot A = 0 \quad (\text{Coulomb gauge}) \qquad \text{in } \Omega_T,$$
$$\partial_\nu \psi|_{\partial\Omega} = 0, \qquad (\nabla \times A + H) \times \nu|_{\partial\Omega} = 0, \qquad \nu \cdot A|_{\partial\Omega} = 0,$$
$$(\psi, A)|_{t=0} = (\psi_0, A_0),$$

where ψ is the complex-valued order parameter; Φ is the scalar electric potential; A is the magnetic vector potential, σ and κ are positive physical constants; H is the external magnetic field; $\partial_\nu = \nu \cdot \nabla$; and

$$J_{GL} = -\frac{i}{2}[\overline{\psi}(\nabla - iA)\psi - \psi\overline{(\nabla - iA)\psi}],$$

which is called the Ginzburg-Landau current. If we linearize (1.12) at the constant state $(\psi, A, H) = (\psi_0, 0, 0)$ with $|\psi_0| = 1$, ψ_0 being a complex number, then we have (1.11) with $\mu > 0$ from the second equation of (1.12) as a linearized equation.

If we consider the magnetohydrodynamic equation proposed by T. G. Cowling [5] or L. Landau and E. Lifshitz [11]:

$$
\begin{aligned}
(1.13)\qquad & u_t + (\nabla \cdot u)u = -\nabla p + \frac{1}{\mathrm{Re}}\Delta u + \mathrm{S}\left(\frac{1}{2}\nabla |H|^2 - (H\cdot\nabla)H\right) + F \quad \text{in } \Omega_T,\\
& H_t = (H\cdot\nabla)u - (u\cdot\nabla)H - \frac{1}{\mathrm{Rm}}\nabla\times(\nabla\times H) \quad \text{in } \Omega_T,\\
& \nabla\cdot u = 0, \qquad \nabla\cdot H = 0 \quad \text{in } \Omega_T,\\
& u|_{\partial\Omega} = 0, \qquad (\nabla\times H)\times\nu|_{\partial\Omega} = 0, \qquad \nu\cdot H|_{\partial\Omega} = 0,\\
& (u, H)|_{t=0} = (u_0, H_0),
\end{aligned}
$$

where S, Re and Rm are positive physical constants, then we have (1.11) from the second equation of (1.13) with $\mu = 0$ as a linearized equation at the constant state.

If we consider the Navier-Stokes equation with Neumann boundary condition:

$$
\begin{aligned}
(1.14)\qquad & u_t + (u\cdot\nabla)u = \nabla p + \nu\Delta u + f, \qquad \nabla\cdot u = 0 \quad \text{in } \Omega_T,\\
& \nabla\times u\times\nu|_{\partial\Omega} = 0, \qquad \nu\cdot u|_{\partial\Omega} = 0, \qquad u|_{t=0} = a,
\end{aligned}
$$

where ν is a positive physical constant, then we have (1.11) with $\mu = 0$ as a linearized equation because of the formula: $-\Delta u = \nabla\times(\nabla\times u) - \nabla(\nabla\cdot u)$ with $\nabla\cdot u = 0$.

To solve (1.12), (1.13) and (1.14) by using the usual contraction mapping principle, it is important to show the analyticity of the semigroup corresponding to (1.11). Therefore, according to the well-known analytic semigroup theory (cf. Pazy [17]), it is the most fundamental point that we investigate the theory for the resolvent problem (1.6), which follows from our main results in the subsection 1.1.

Moreover, if we consider the stationary problem corresponding to (1.12):

$$
\begin{aligned}
(1.15)\qquad & -i\Phi\psi = (\nabla - iA)^2\psi + \kappa(1 - |\psi|^2)\psi \quad \text{in } \Omega,\\
& -\sigma\nabla\Phi + \nabla\times(\nabla\times A) + \nabla\times H = J_{GL}, \qquad \nabla\cdot A = 0 \quad \text{in } \Omega,\\
& \partial_\nu\psi|_{\partial\Omega} = 0, \qquad (\nabla\times A + H)\times\nu|_{\partial\Omega} = 0, \qquad \nu\cdot H|_{\partial\Omega} = 0,
\end{aligned}
$$

where $H = {}^t(H_1, H_2, H_3)$ is a given external force, then as a linearized problem at the constant state $\psi = \psi_0 \in \boldsymbol{C}$ with $|\psi_0| = 1$ we have

$$
\begin{aligned}
(1.16)\qquad & A + \nabla\times(\nabla\times A) + \nabla\times H = 0, \qquad \nabla\cdot A = 0 \quad \text{in } \Omega,\\
& (\nabla\times A + H)\times\nu|_{\partial\Omega} = 0 \qquad \nu\cdot A|_{\partial\Omega} = 0.
\end{aligned}
$$

The problem (1.16) is reduced to the equation (1.1) with $\lambda = 1$, $f = 0$, $g = 0$ and

$$F = \begin{pmatrix} 0, & -H_3, & H_2 \\ H_3, & 0, & -H_1 \\ -H_2, & H_1, & 0 \end{pmatrix}.$$

In fact, since $H + {}^tH = 0$, Theorem 1.2 (2) guarantees the unique existence of solution A to (1.1) which satisfies the condition: $\nabla \cdot A = 0$. And therefore, by (1.5) we have $-\nabla \times (\nabla \times A) = \nabla \cdot (\mathrm{curl}\, A) = \Delta A$, which implies that A satisfies (1.16). In view of Theorems 1.4 and 1.5, by using the contraction mapping principle we will be able to solve (1.15) at least for small external forces.

In spite of these backgrounds, to the authors it does not seem to be known the systematic study of the system of Laplace operators with perfect wall condition. And the main results for the strong solutions to (1.1) stated in the subsection 1.1 do not seem to be derived directly from the well-known general theory of elliptic operators like the Agmon, Douglis and Nirenberg [2] and Lions and Magenes [12], [13], [14]. We only know the work due to Miyakawa [15], where he proved Theorem 1.2 with $F = G = 0$ for large $|\lambda|$ and proved the local existence theorem of the Navier-Stokes equation with Neumann boundary condition in the L_p framework when the domain is bounded. In fact, he noticed that (1.6) is reduced to (1.1) when f satisfies the condition: $\nabla \cdot f = 0$ in Ω and $\nu \cdot f|_{\partial\Omega} = 0$, i.e. f is in the solenoidal space, and then he reduced the problem to the model problem in the half-space to get some a priori estimate and used Agmon's trick [1] to get the resolvent estimate for large $|\lambda|$, which is enough to show the analyticity of the Stokes semigroup. But, if we would like to study the asymptotic behavior of solutions as time goes to infinity, it is important to investigate the behavior of the resolvent operator near the origin. Moreover, to consider the stationary solution and its stability concerning the initial disturbance, it is important to show not only Theorem 1.2 but also Theorem 1.5. Concerning the weak solution in the L_p theory, to the authors nothing seems to be known about (1.1) although the L_2 theory is rather well known (cf. Sermange and Temam [19], Duvaut and Lions [6], Georgescu [9]). In the case of Neumann or Dirichlet problem for the Laplace operator, Simader and Sohr [20], [21] gave some new idea concerning the weak solution, which gave us an extension of Gårding inequality to the L_p framework and seems to be applicable also to (1.1) with $\lambda = 0$ only. But our approach is to drive the estimates (1.3) and (1.7) to prove the existence of weak solutions, which seems to be new and completely different from the idea due to Simader and Sohr [21]. Moreover, we think that it is important to consider (1.12), (1.13) and (1.14) in the domains mentioned in Assumption 1.1 in view of several different physical situations. Therefore, we think that it is worth while giving a self-contained and very elementary independent proof of our main results stated in the subsection 1.1 in order to give a foundation of the study of nonlinear problems.

After the preparation of the *a priori* estimates of solutions to the problem (1.1) in the whole space and half-space in §2, we solve the problem (1.1) in the bent half-space in §3, which is the main point in our approach. And, after the discussion of the unique existence theorem of the weak solution in the $L_2(\Omega)$ framework in §4 and the discussion of the reduction of the problem to the whole space, half-space and bent half-space problems in §5, following the argument due to Farwig and Sohr [7] we prove Theorem 1.2 (1) by showing the several lemmas in §6. What the solution u satisfies the divergence free condition: $\nabla \cdot u = 0$ follows from the special structure of the boundary condition and (1.5) (cf. Lemma 6.7 below). To prove Theorem 1.4, our proof relies on the result due to von Wahl [24] (cf. Proposition 4.3 below) which guarantees the estimate:

$$\|\nabla u\|_{L_p(\Omega)} \leqq C\{\|\mathrm{curl}\, u\|_{L_p(\Omega)} + \|\nabla \cdot u\|_{L_p(\Omega)}\} \qquad \text{for any } u \in \dot{W}_p^1(\Omega)^n$$

provided that Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$ and $1 < p < \infty$.

2. The whole space and the half-space problems

We start with the following theorem concerning the whole space problem.

Theorem 2.1. *Let $1 < p < \infty$ and $0 < \varepsilon < \pi/2$. Then, for every $\lambda \in \Sigma_\varepsilon$, $f \in L_p(\boldsymbol{R}^n)$ and $F = {}^t(F_1, \dots, F_n) \in W_p^1(\boldsymbol{R}^n)^n$, there exists a unique $u \in W_p^2(\boldsymbol{R}^n)$ which solves the equation*:

$$(2.1) \qquad (\lambda - \Delta)u = f + \nabla \cdot F \qquad \text{in } \boldsymbol{R}^n$$

and satisfies the estimates:

$$(2.2) \qquad \mathscr{I}_p^1(u, \lambda, \boldsymbol{R}^n) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(\boldsymbol{R}^n)} + \|F\|_{L_p(\boldsymbol{R}^n)}\},$$

$$(2.3) \qquad \mathscr{I}_p^2(u, \lambda, \boldsymbol{R}^n) \leqq C\|f + \nabla \cdot F\|_{L_p(\boldsymbol{R}^n)}$$

for some constant $C = C(\varepsilon, p)$.

Moreover, if $u \in L_p(\boldsymbol{R}^n)$ satisfies the condition: $(\lambda - \Delta)u \in L_q(\boldsymbol{R}^n)$ for some $\lambda \in \Sigma_\varepsilon$ and $1 < q < \infty$, then $u \in W_q^2(\boldsymbol{R}^n)$.

Proof. To solve (2.1), we use the Fourier transform $\mathscr{F}$ and its inverse transform $\mathscr{F}^{-1}$. Since

$$(2.4) \qquad ||\xi|^2 + \lambda| \geqq (\sin \varepsilon/2)(|\xi|^2 + |\lambda|)$$

for any $\lambda \in \Sigma_\varepsilon$ and $\xi \in \boldsymbol{R}^n$, if we define $u(x) = \mathscr{F}^{-1}[(\lambda + |\xi|^2)^{-1}\mathscr{F}(f + \nabla \cdot F)(\xi)](x)$, by the Fourier multiplier theorem (cf. Stein [22], Triebel [23]) we see that u satisfies (2.1), (2.2) and (2.3), because

$$|\partial_\xi^\alpha(\lambda+|\xi|^2)^{-1}| \leqq C_{\alpha,\varepsilon}(|\lambda|+|\xi|^2)^{-1}|\xi|^{-|\alpha|} \tag{2.5}$$

where $C_{\alpha,\varepsilon}$ is a constant depending only on α, ε and n. The uniqueness holds in the class of the Schwartz's tempered distributions when $\lambda \in \Sigma_\varepsilon$, which is proved by using the Fourier transform and (2.4). Moreover, the solution is given exactly by using the Fourier transform, so that in view of the uniqueness in the Schwartz's tempered distribution class what $(\lambda - \Delta)u \in L_q(\boldsymbol{R}^n)$ implies that $u = \mathscr{F}^{-1}[(1+|\xi|^2)^{-1}\mathscr{F}[f](\xi)](x) \in W_q^2(\boldsymbol{R}^n)$, where $f = (\lambda-\Delta)u \in L_q(\boldsymbol{R}^n)$. $\square$

Next, we consider the Neumann problem:

$$(\lambda-\Delta)u = f + \nabla\cdot F \quad \text{in } \boldsymbol{R}_+^n, \qquad \partial_n u|_{x_n=0} = (-F_n+g)|_{x_n=0}, \tag{2.6}$$

where $F = {}^t(F_1, \dots, F_n)$.

Theorem 2.2. *Let $1<p<\infty$ and $0<\varepsilon<\pi/2$. Then, for every $\lambda \in \Sigma_\varepsilon$, $f \in L_p(\boldsymbol{R}_+^n)$, $F \in W_p^1(\boldsymbol{R}_+^n)^n$ and $g \in W_p^1(\boldsymbol{R}_+^n)$, the Neumann problem* (2.6) *admits a unique solution $u \in W_p^2(\boldsymbol{R}_+^n)$ which satisfies the a priori estimates*:

$$\mathscr{I}_p^1(u,\lambda,\boldsymbol{R}_+^n) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(\boldsymbol{R}_+^n)} + \|F\|_{L_p(\boldsymbol{R}_+^n)} + |\lambda|^{-1/(2p)}\|g(\cdot,0)\|_{L_p(\boldsymbol{R}^{n-1})}\}, \tag{2.7}$$

$$\mathscr{I}_p^2(u,\lambda,\boldsymbol{R}_+^n) \leqq C\{\|f\|_{L_p(\boldsymbol{R}_+^n)} + \mathscr{I}_p^1((F,g),\lambda,\boldsymbol{R}_+^n)\}, \tag{2.8}$$

for some constant $C = C(\varepsilon,p) > 0$.

Proof. For the notational simplicity, we set $h = f + \nabla\cdot F$. Let us define the even extension h^e of h by

$$h^e(x) = \begin{cases} h(x) & x_n > 0, \\ h(x', -x_n) & x_n < 0, \end{cases}$$

where $x' = (x_1, \dots, x_{n-1})$. Let us define

$$v(x) = \mathscr{F}^{-1}[(\lambda+|\xi|^2)^{-1}\mathscr{F}[h^e](\xi)](x), \tag{2.9}$$

and then we have

$$\partial_n v|_{x_n=0} = 0, \tag{2.10}$$

$$\mathscr{I}_p^2(v,\lambda,\boldsymbol{R}^n) \leqq C\|f+\nabla\cdot F\|_{L_p(\boldsymbol{R}_+^n)}. \tag{2.11}$$

In fact, (2.11) follows from (2.5) and the Fourier multiplier theorem.

Let us define the solution $w(x)$ of the equation:

$$(\lambda-\Delta)w = 0 \quad \text{in } \boldsymbol{R}_+^n, \qquad \partial_n w|_{x_n=0} = (-F_n+g)|_{x_n=0}. \tag{2.12}$$

Applying the partial Fourier transform $\mathscr{F}'$ with respect to x' to the equation (2.12), we have the ordinary differential equation with respect to x_n:

$$[(\lambda+|\xi'|^2)-\partial_n^2]\mathscr{F}'[w](\xi',x_n)=0, \qquad x_n>0,$$
$$\partial_n\mathscr{F}'[w](\xi',0)=-\mathscr{F}'[F_n](\xi',0)+\mathscr{F}'[g](\xi',0),$$

and therefore we have

$$w(x)=(\mathscr{F}')^{-1}\left[\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}}{\sqrt{\lambda+|\xi'|^2}}\mathscr{F}'[F_n-g](\xi',0)\right](x'). \tag{2.13}$$

Here, $(\mathscr{F}')^{-1}$ is the inversion formula of $\mathscr{F}'$. In particular, if we set

$$u(x)=v(x)+w(x), \tag{2.14}$$

then u solves (2.6). In order to estimate w we shall use the following lemma.

Lemma 2.3. *Let $1<p<\infty$, $0<\varepsilon<\pi/2$, $\ell\in\boldsymbol{R}$ and $\Phi(\xi',\lambda)$ be a function in $C^\infty(\boldsymbol{R}^{n-1}\backslash\{0\})$ for each $\lambda\in\Sigma_\varepsilon$ which satisfies the condition*:

$$|\partial_{\xi'}^{\alpha'}\Phi(\xi',\lambda)|\leqq C_{\alpha',\varepsilon}|\lambda|^{\ell}|\xi'|^{-|\alpha'|} \tag{2.15}$$

for any multi-index $\alpha'=(\alpha_1,\dots,\alpha_{n-1})$ with some constant $C_{\alpha',\varepsilon}$ depending only on α' and ε. For every $\lambda\in\Sigma_\varepsilon$ and $k\in W_p^1(\boldsymbol{R}_+^n)$, we set

$$y(x)=(\mathscr{F}')^{-1}[e^{-\sqrt{\lambda+|\xi'|^2}x_n}\Phi(\xi',\lambda)\mathscr{F}'[k](\xi',0)](x').$$

Then, there hold estimates:

$$\|y\|_{L_p(\boldsymbol{R}_+^n)}\leqq|\lambda|^{\ell-1/2p}\|k(\cdot,0)\|_{L_p(\boldsymbol{R}^{n-1})}, \tag{2.16}$$

$$\|\nabla y\|_{L_p(\boldsymbol{R}_+^n)}\leqq C|\lambda|^{\ell}\{\|\nabla k\|_{L_p(\boldsymbol{R}_+^n)}+|\lambda|^{1/2}\|k\|_{L_p(\boldsymbol{R}_+^n)}\}. \tag{2.17}$$

Proof. By (2.4) we have

$$\mathrm{Re}\,\sqrt{\lambda+|\xi'|^2}\geqq 2c_\varepsilon(|\xi'|^2+|\lambda|)^{1/2} \tag{2.18}$$

for any $\lambda\in\Sigma_\varepsilon$ and $\xi'\in\boldsymbol{R}^{n-1}$, where $c_\varepsilon=(2\sqrt{2})^{-1}(\sin(\varepsilon/2))^{3/2}$. By (2.15) and (2.18)

$$|\partial_{\xi'}^{\alpha'}[e^{-\sqrt{\lambda+|\xi'|^2}x_n}\Phi(\xi',\lambda)]|\leqq C_{\alpha',\varepsilon}|\xi'|^{-|\alpha'|}|\lambda|^{\ell}e^{-c_\varepsilon(|\xi'|+|\lambda|^{1/2})x_n} \tag{2.19}$$

for any multi-index $\alpha'=(\alpha_1,\dots,\alpha_{n-1})$ and $x_n>0$, and therefore by the Fourier multiplier theorem

$$\begin{aligned}\|y\|_{L_p(\boldsymbol{R}_+^n)}^p&=\int_0^\infty\|y(\cdot,x_n)\|_{L_p(\boldsymbol{R}^{n-1})}^p dx_n\\&\leqq C|\lambda|^{p\ell}\int_0^\infty e^{-c_\varepsilon|\lambda|^{1/2}px_n}\,dx_n\|k(\cdot,0)\|_{L_p(\boldsymbol{R}^{n-1})}^p\\&\leqq C_{\varepsilon,p}|\lambda|^{p\ell-1/2}\|k(\cdot,0)\|_{L_p(\boldsymbol{R}^{n-1})}^p,\end{aligned} \tag{2.20}$$

which implies (2.16).

To prove (2.17), we take $\delta > 0$ in such a way that $\delta = c_\varepsilon/4$ and we set

$$y(x) = (\mathscr{F}')^{-1}[e^{-\sqrt{\lambda+|\xi'|^2}x_n} e^{\delta|\xi'|x_n} \Phi(\xi',\lambda)\mathscr{F}'[y_\delta](\xi',x_n)](x')$$

where

$$\mathscr{F}'[y_\delta](\xi',x_n) = e^{-\delta|\xi'|x_n}\mathscr{F}'[k](\xi',0).$$

By (2.19) we have

$$|\partial_{\xi'}^{\alpha'}[e^{-\sqrt{\lambda+|\xi'|^2}x_n} e^{\delta|\xi'|x_n}\Phi(\xi',\lambda)]| \leqq C_{\alpha',\varepsilon}|\lambda|^{\ell}|\xi'|^{-|\alpha'|}$$

for any multi-index α', and therefore by the Fourier multiplier theorem we have

$$\|\partial_j y\|_{L_p(\boldsymbol{R}_+^n)} \leqq C_\varepsilon |\lambda|^{\ell}\|\partial_j y_\delta\|_{L_p(\boldsymbol{R}_+^n)}, \qquad j = 1,\dots,n-1. \tag{2.21}$$

On the other hand, since

$$-2\partial_n E(x) = -2\partial_n \mathscr{F}^{-1}[|\xi|^{-2}](x) = (\mathscr{F}')^{-1}[e^{-|\xi'|x_n}](x'), \qquad x_n > 0$$

as follows from the integral formula of Cauchy in the theory of one complex variable where E is the fundamental solution for the Laplacian, by the Agmon-Douglis-Nirenberg lemma for the singular integral operator in the half-space (cf. Agmon-Douglis-Nirenberg [2, Theorem 3.3], Galdi [8, II.9, Theorem 9.6]), we have

$$\|\nabla y_\delta\|_{L_p(\boldsymbol{R}_+^n)} \leqq C_\delta \|\nabla k\|_{L_p(\boldsymbol{R}_+^n)},$$

which combined with (2.21) implies that

$$\|\partial_j y\|_{L_p(\boldsymbol{R}_+^n)} \leqq C_{\delta,\varepsilon}|\lambda|^{\ell}\|\nabla k\|_{L_p(\boldsymbol{R}_+^n)}, \qquad j = 1,\dots,n-1.$$

Observing the identity:

$$\begin{aligned}\partial_n y(x) = &-(\mathscr{F}')^{-1}[e^{-\sqrt{\lambda+|\xi'|^2}x_n}(\lambda+|\xi'|^2)^{-1/2}\lambda\Phi(\xi',\lambda)\mathscr{F}'[k](\xi',0)](x') \\ &- \sum_{j=1}^{n-1} i^{-1}\partial_j(\mathscr{F}')^{-1}[e^{-\sqrt{\lambda+|\xi'|^2}x_n}(\lambda+|\xi'|^2)^{-1/2}\xi_j\Phi(\xi',\lambda)\mathscr{F}'[k](\xi',0)](x')\end{aligned}$$

and noting that $(\lambda+|\xi'|^2)^{-1/2}\sqrt{\lambda}\Phi(\xi',\lambda)$ and $(\lambda+|\xi'|^2)^{-1/2}\xi_j\Phi(\xi',\lambda)$ also satisfy the condition (2.15), we have

$$\|\partial_n y\|_{L_p(\boldsymbol{R}_+^n)} \leqq C_{\delta,\varepsilon}|\lambda|^{\ell}\{\|\nabla k\|_{L_p(\boldsymbol{R}_+^n)} + |\lambda|^{(1/2)(1-1/p)}\|k(\cdot,0)\|_{L_p(\boldsymbol{R}^{n-1})}\}. \tag{2.22}$$

Since

$$|\lambda|^{(1/2)(1-1/p)}\|k(\cdot,0)\|_{L_p(\boldsymbol{R}^{n-1})} \leqq \sigma\|\nabla k\|_{L_p(\boldsymbol{R}_+^n)} + C(\sigma,p)|\lambda|^{1/2}\|k\|_{L_p(\boldsymbol{R}_+^n)} \tag{2.23}$$

for any $\sigma > 0$ as follows from the interpolation inequality:

$$\|k(\cdot, 0)\|_{L_p(\boldsymbol{R}^{n-1})} \leqq p^{1/p}\|k\|_{L_p(\boldsymbol{R}^n_+)}^{1/p'}\|\nabla k\|_{L_p(\boldsymbol{R}^n_+)}^{1/p}, \tag{2.24}$$

combining (2.22) and (2.23) implies (2.17), which completes the proof of the lemma. □

We continue the proof of Theorem 2.2. Applying Lemma 2.3 to (2.13) and using (2.23), we have $\mathscr{I}_p^2(w, \lambda, \boldsymbol{R}^n_+) \leqq C\mathscr{I}_p^1((F_n, g), \lambda, \boldsymbol{R}^n_+)$, which combined with (2.14) and (2.11) implies (2.8).

To prove (2.7), we observe that

(2.25)

$$\mathscr{F}[h^e](\xi) = \mathscr{F}[f^e](\xi) + \sum_{j=1}^{n-1} i\xi_j\mathscr{F}[(F_j)^e](\xi) - 2\mathscr{F}'[F_n](\xi', 0) + i\xi_n\mathscr{F}[(F_n)^o](\xi),$$

where $(F_n)^o(x) = F_n(x', x_n)$ for $x_n > 0$ and $= -F_n(x', -x_n)$ for $x_n < 0$, i.e., the odd extension of F_n to the whole space. By using the integral formula of Cauchy in the theory of one complex variable, we have

$$2\mathscr{F}^{-1}[(\lambda + |\xi|^2)^{-1}\mathscr{F}'[F_n](\xi', 0)](x) = \mathscr{F}'\left[\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}}{\sqrt{\lambda + |\xi'|^2}}\mathscr{F}'[F_n](\xi', 0)\right](x'),$$

which combined with (2.25), (2.14), (2.13) and (2.9) implies that

$$u(x) = \mathscr{F}^{-1}[(\lambda + |\xi|^2)^{-1}\mathscr{F}[f^e](\xi)](x) + \sum_{j=1}^{n-1}\mathscr{F}^{-1}[(\lambda + |\xi|^2)^{-1}i\xi_j\mathscr{F}[(F_j)^e](\xi)](x)$$
$$+ \mathscr{F}^{-1}[(\lambda + |\xi|^2)^{-1}i\xi_n\mathscr{F}[(F_j)^o](\xi)](x) - (\mathscr{F}')^{-1}\left[\frac{e^{-\sqrt{\lambda+|\xi'|^2}x_n}}{\sqrt{\lambda + |\xi'|^2}}\mathscr{F}'[g](\xi', 0)\right](x')$$

Applying Lemma 2.3 and the Fourier multiplier theorem with help of (2.4) and (2.5), we have (2.7). This completes the proof of the theorem. □

Now, we consider the Dirichlet problem:

$$(\lambda - \Delta)u = f + \nabla \cdot F \quad \text{in } \boldsymbol{R}^n_+, \qquad u|_{x_n=0} = 0. \tag{2.26}$$

Theorem 2.4. *Let* $1 < p < \infty$ *and* $0 < \varepsilon < \pi/2$. *Then, for every* $\lambda \in \Sigma_\varepsilon$, $f \in L_p(\boldsymbol{R}^n_+)$ *and* $F = {}^t(F_1, \dots, F_n) \in W_p^1(\boldsymbol{R}^n_+)^n$, *the Dirichlet problem* (2.26) *admits a unique solution* $u \in W_p^2(\boldsymbol{R}^n_+)$ *which satisfies the estimates*:

$$\mathscr{I}_p^1(u, \lambda, \boldsymbol{R}^n_+) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(\boldsymbol{R}^n_+)} + \|F\|_{L_p(\boldsymbol{R}^n_+)}\}, \tag{2.27}$$

$$\mathscr{I}_p^2(u, \lambda, \boldsymbol{R}^n_+) \leqq C\|(f, \nabla F)\|_{L_p(\boldsymbol{R}^n_+)}. \tag{2.28}$$

Proof. Set $k = f + \nabla \cdot F$ and $u(x) = \mathscr{F}^{-1}[(|\xi|^2 + \lambda)^{-1}\mathscr{F}[k^o](\xi)](x)$, where k^o is the odd extension of k to the whole space. Then, we have $u|_{x_n=0} = 0$. In view of (2.5), by the Fourier multiplier theorem we have (2.28). Since

$$\mathscr{F}[k^o](\xi) = \mathscr{F}[f^o](\xi) + \sum_{j=1}^{n-1} i\xi_j \mathscr{F}[F_j^o](\xi) + i\xi_n \mathscr{F}[F_n^e](\xi),$$

in view of (2.5) by the Fourier multiplier theorem we have (2.27). This completes the proof of the theorem. □

Finally, we consider (1.1) in the half-space $\boldsymbol{R}_+^n$:

$$(2.29) \qquad (\lambda - \Delta)u = f + \nabla \cdot F \quad \text{in } \boldsymbol{R}_+^n, \qquad -(\operatorname{curl} u)\nu_0 = g + F\nu_0, \quad \nu_0 \cdot u = 0 \quad \text{on } \boldsymbol{R}_0^n,$$

where $\nu_0 = (0, \dots, 0, -1)$ and $\boldsymbol{R}_0^n = \{(x', x_n) \in \boldsymbol{R}^n \mid x_n = 0\}$. Since $\nu_0 \cdot u = u_n = 0$ on $\boldsymbol{R}_0^n$, we have

$$-(\operatorname{curl} u)\nu_0 = {}^t(\partial_n u_1, \dots, \partial_n u_{n-1}, 0) \quad \text{on } \boldsymbol{R}_0^n.$$

Therefore, if $g_n = 0$ and $F_{nn} = 0$ on $\boldsymbol{R}_0^n$, then (2.29) is equivalent to the problem:

$$(2.30) \qquad \begin{aligned} &(\lambda - \Delta)u_j = f_j + \sum_{k=1}^n \partial_k F_{jk}, && j = 1, \dots, n, && \text{in } \boldsymbol{R}_+^n, \\ &\partial_n u_j = g_j - F_{jn}, && j = 1, \dots, n-1, && \text{on } \boldsymbol{R}_0^n, \\ &u_n = 0 && && \text{on } \boldsymbol{R}_0^n. \end{aligned}$$

Then, combining Theorems 2.2 and 2.4, we have the following theorem.

Theorem 2.5. *Let* $1 < p < \infty$ *and* $0 < \varepsilon < \pi/2$. *Then, the following assertions hold.*

(1) *For every* $\lambda \in \Sigma_\varepsilon$, $f = {}^t(f_1, \dots, f_n) \in L_p(\boldsymbol{R}_+^n)^n$, $F = (F_{ij}) \in W_p^1(\boldsymbol{R}_+^n)^{n \times n}$ *with* $F_{nn}|_{x_n=0} = 0$ *and* $g = {}^t(g_1, \dots, g_{n-1}, 0) \in W_p^1(\boldsymbol{R}_+^n)^n$, *the half-space problem* (2.29) *admits a unique solution* $u \in W_p^2(\boldsymbol{R}_+^n)^n$ *which satisfies the estimates*:

$$(2.31) \qquad \mathscr{I}_p^1(u, \lambda, \boldsymbol{R}_+^n) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(\boldsymbol{R}_+^n)} + \|F\|_{L_p(\boldsymbol{R}_+^n)} + |\lambda|^{-1/(2p)}\|g(\cdot, 0)\|_{L_p(\boldsymbol{R}^{n-1})}\},$$

$$(2.32) \qquad \mathscr{I}_p^2(u, \lambda, \boldsymbol{R}_+^n) \leqq C\{\|f\|_{L_p(\boldsymbol{R}_+^n)} + \mathscr{I}_p^1((F, g), \lambda, \boldsymbol{R}_+^n)\}.$$

(2) *If* $u \in W_p^1(\boldsymbol{R}_+^n)$ *satisfies the homogeneous variational equation*:

$$\lambda(u,\Phi)_{\boldsymbol{R}^n_+} + \frac{1}{2}(\operatorname{curl} u, \operatorname{curl} \Phi)_{\boldsymbol{R}^n_+} + (\nabla\cdot u, \nabla\cdot\Phi)_{\boldsymbol{R}^n_+} = 0 \tag{2.33}$$

for any $\Phi \in \dot{W}^1_{p'}(\boldsymbol{R}^n_+)^n$*, then* $u = 0$ *provided that* $\lambda \in \Sigma_\varepsilon$.

(3) *Let* $u \in \dot{W}^1_p(\boldsymbol{R}^n_+)$ *satisfy the variational problem*:

$$\lambda(u,\Phi)_{\boldsymbol{R}^n_+} + \frac{1}{2}(\operatorname{curl} u, \operatorname{curl} \Phi)_{\boldsymbol{R}^n_+} + (\nabla\cdot u, \nabla\cdot\Phi)_{\boldsymbol{R}^n_+} \tag{2.34}$$
$$= (f,\Phi)_{\boldsymbol{R}^n_+} - (F,\Phi)_{\boldsymbol{R}^n_+} - (g,\Phi)_{\boldsymbol{R}^n_0} \qquad \text{for any } \Phi \in \dot{W}^1_{p'}(\boldsymbol{R}^n_+)^n.$$

If $f \in L_p(\boldsymbol{R}^n_+)^n \cap L_q(\boldsymbol{R}^n_+)^n$, $F \in W^1_p(\boldsymbol{R}^n_+)^{n\times n} \cap W^1_q(\boldsymbol{R}^n_+)^{n\times n}$ *with* $F_{nn}|_{x_n=0} = 0$ *and* $g = (g_1,\dots,g_{n-1},0) \in W^1_p(\boldsymbol{R}^n_+)^n \cap W^1_q(\boldsymbol{R}^n_+)^n$ *for some* $1 < q < \infty$*, then* $u \in W^2_p(\boldsymbol{R}^n_+)^n \cap W^2_q(\boldsymbol{R}^n_+)^n$.

Proof. In view of (2.30), the first assertion follows from Theorems 2.2 and 2.4 immediately. The uniqueness assertion (2) follows from the solvability of the dual problem with $f \in C^\infty_0(\boldsymbol{R}^n_+)^n$, $F = 0$ and $g = 0$, which is guaranteed by (1). Finally, we shall show (3). Since the solution to (2.29) can be constructed by using the Fourier transform exactly, we see that there exists a $v \in W^2_p(\boldsymbol{R}^n_+)^n \cap W^2_q(\boldsymbol{R}^n_+)^n$ of the equation (2.29). Since v also satisfies the variational equation (2.34), the uniqueness assertion (2) implies that $u = v$, which implies that $u \in W^2_p(\boldsymbol{R}^n_+)^n \cap W^2_q(\boldsymbol{R}^n_+)^n$. □

3. The bent half-space problem

Let $\omega : \boldsymbol{R}^{n-1} \to \boldsymbol{R}$ be a function in $\mathscr{B}^3(\boldsymbol{R}^{n-1})$, where $\mathscr{B}^k(\boldsymbol{R}^{n-1})$ $(k \geqq 0)$ denotes the set of all bounded functions whose derivatives up to k are also bounded almost everywhere in $\boldsymbol{R}^{n-1}$. Let H be the bent half space with boundary ∂H defined by

$$H = \{x = (x', x_n) \in \boldsymbol{R}^n \mid x_n > \omega(x'), x' = (x_1,\dots,x_{n-1}) \in \boldsymbol{R}^{n-1}\}, \tag{3.1}$$

$$\partial H = \{x = (x', x_n) \in \boldsymbol{R}^n \mid x_n = \omega(x'), x' \in \boldsymbol{R}^{n-1}\}. \tag{3.2}$$

The unit outer normal $\nu = {}^t(\nu_1,\dots,\nu_n)$ of ∂H is defined by

$${}^t\nu = (\nabla'\omega, -1)/\sqrt{1+|\nabla'\omega|^2}, \qquad \nabla'\omega = (\partial_1\omega,\dots,\partial_{n-1}\omega), \qquad \partial_j = \partial/\partial x_j. \tag{3.3}$$

In this section, we consider the bent half-space problem:

$$(\lambda - \Delta)u = f + \nabla\cdot F \qquad \text{in } H, \tag{3.4}$$
$$-(\operatorname{curl} u)\nu|_{\partial H} = (g + F\nu)|_{\partial H}, \qquad \nu\cdot u|_{\partial H} = 0,$$

where F and g satisfy the condition:

$$\nu \cdot (F\nu)|_{\partial H} = 0 \qquad \text{and} \qquad \nu \cdot g|_{\partial H} = 0. \tag{3.5}$$

Theorem 3.1. *Let* $\omega(x') \in \mathscr{B}^3(\boldsymbol{R}^{n-1})$ *and set*

$$|||\omega||| = \sum_{|\alpha'| \leqq 3} \|\partial_{x'}^{\alpha'}\omega\|_{L_\infty(\boldsymbol{R}^{n-1})}, \qquad \partial_{x'}^{\alpha'} = \partial_1^{\alpha_1} \dots \partial_{n-1}^{\alpha_{n-1}}, \qquad \alpha' = (\alpha_1, \dots, \alpha_{n-1}).$$

Let H be a bent half space defined by (3.1). *Let* $1 < p < \infty$ *and* $0 < \varepsilon < \pi/2$. *Then, there exist constants* $K = K(p, \varepsilon, n) \in (0, 1)$ *and* $\lambda_0 = \lambda_0(p, \varepsilon, n, |||\omega|||) \geqq 1$ *possessing the following properties*:

(1) *If* $\|\nabla'\omega\|_{L_\infty(\boldsymbol{R}^{n-1})} \leqq K$, *then for every* $\lambda \in \Sigma_\varepsilon$ *with* $|\lambda| \geqq \lambda_0$, $f = {}^t(f_1, \dots, f_n) \in L_p(H)^n$, $F = (F_{ij}) \in W_p^1(H)^{n\times n}$ *and* $g = {}^t(g_1, \dots, g_n) \in W_p^1(H)^n$ *which satisfy* (3.5), *the bent half-space problem* (3.4) *admits a unique solution* $u \in W_p^2(H)^n$. *This solution satisfies the a priori estimates* (1.3) *and* (1.4) *with* $\Omega = H$ *for some constant* $C = C(p, \varepsilon, n, |||\omega|||) \geqq 1$.

(2) *If we assume that* $f \in W_p^m(H)^n$, $F \in W_p^{m+1}(H)^{n\times n}$, $g \in W_p^{m+1}(H)^n$ *and* $\omega \in \mathscr{B}^{m+3}(\boldsymbol{R}^{n-1})$ *for some integer* $m \geqq 1$ *additionally, then* $u \in W_p^{m+2}(H)^n$.

Proof. Let $u = {}^t(u_1, \dots, u_n)$ be a solution to (3.4), which is written conponentwise as follows:

$$(\lambda - \Delta)u_j = f_j + \sum_{k=1}^n \partial_k F_{jk} \qquad \text{in } H, \tag{3.6}$$

$$-\sum_{k=1}^n (\partial_k u_j - \partial_j u_k)\nu_k = g_j + \sum_{k=1}^n F_{jk}\nu_k \qquad \text{on } \partial H, \tag{3.7}$$

$$\sum_{k=1}^n \nu_k u_k = 0 \qquad \text{on } \partial H, \tag{3.8}$$

where $j = 1, \dots, n$ and $\partial_j = \partial/\partial x_j$. Since $\nu_n \neq 0$, by (3.5) we see that the nth component of equations in (3.7) is automatically satisfied if other boundary conditions are satisfied. Therefore, as the boundary condition, we adopt (3.7) with $j = 1, \dots, n-1$ and (3.8), below. Using the change of variable:

$$y_j = x_j, \quad j = 1, \dots, n-1, \qquad y_n = x_n - \omega(x') = x_n - \omega(y'), \tag{3.9}$$

we will reduce the problem in the bent half space H to that in the half-space. In fact, since

$$\frac{\partial}{\partial x_j} = \frac{\partial}{\partial y_j} - \omega_j \frac{\partial}{\partial y_n}, \quad j = 1, \dots, n-1; \qquad \frac{\partial}{\partial x_n} = \frac{\partial}{\partial y_n} \tag{3.10}$$

where $\omega_j = \partial\omega/\partial x_j$, if we set $u_j(x) = \tilde{u}_j(y)$, $f_j(x) = \tilde{f}_j(y)$, $F_{ij}(x) = \tilde{F}_{ij}(y)$, $g_j(x) = \tilde{g}_j(y)$ and $v_j(x) = \tilde{v}_j(y)$, the problem (3.6), (3.7) with $j = 1, \dots, n-1$ and (3.8) are reduced to the equation:

$$(3.11)\quad (\lambda - \Delta)\tilde{u}_\ell = \tilde{f}_\ell + \sum_{k=1}^{n-1} \partial_k(\tilde{F}_{\ell k} - \omega_k \partial_n \tilde{u}_\ell) + \partial_n\left(\tilde{F}_{\ell n} - \sum_{k=1}^{n-1} \omega_k \tilde{F}_{\ell k} - \sum_{k=1}^{n-1} \omega_k \partial_k \tilde{u}_\ell + |\nabla'\omega|^2 \partial_n \tilde{u}_\ell\right) \quad \text{in } \boldsymbol{R}^n_+,$$

$$(3.12)\quad \partial_n \tilde{u}_j = -\tilde{v}_n^{-1}\tilde{g}_j + \sum_{k=1}^{n-1} \omega_{jk}\tilde{u}_k - \left[\tilde{F}_{jn} - \sum_{k=1}^{n-1} \omega_k \tilde{F}_{jk} - \sum_{k=1}^{n-1} \omega_k \partial_k \tilde{u}_j + |\nabla'\omega|^2 \partial_n \tilde{u}_j + \omega_j \partial_n\left(\tilde{u}_n - \sum_{k=1}^{n-1} \omega_k \tilde{u}_k\right)\right] \quad \text{on } \boldsymbol{R}^n_0,$$

$$(3.13)\quad \tilde{u}_n - \sum_{k=1}^{n-1} \omega_k \tilde{u}_k = 0 \quad \text{on } \boldsymbol{R}^n_0,$$

where $\ell = 1, \dots, n$, $j = 1, \dots, n-1$, $\omega_{jk} = \partial^2\omega/\partial y_j \partial y_k$, $\partial_j = \partial/\partial y_j$, and we have used the formula:

$$\partial_j \tilde{u}_n = \sum_{k=1}^{n-1} \partial_j(\omega_k \tilde{u}_k) \quad \text{on } \boldsymbol{R}^n_0 \quad \text{for } j = 1, \dots, n-1,$$

which follows from (3.13). If we set

$$(3.14)\quad v_j = \tilde{u}_j, \quad j = 1, \dots, n-1; \qquad v_n = \tilde{u}_n - \sum_{k=1}^{n-1} \omega_k \tilde{u}_k,$$

then (3.11) to (3.13) is reduced to the equations:

$$(3.15)\quad (\lambda - \Delta)v_j = \tilde{f}_j + \sum_{k=1}^{n-1} \partial_k(\tilde{F}_{jk} - \omega_k \partial_n v_j) + \partial_n\left(\tilde{F}_{jn} - \sum_{k=1}^{n-1} \omega_k \tilde{F}_{jk} - \sum_{k=1}^{n-1} \omega_k \partial_k v_j + |\nabla'\omega|^2 \partial_n v_j + \omega_j \partial_n v_n\right) - \omega_j \partial_n^2 v_n \quad \text{in } \boldsymbol{R}^n_+,$$

$$(3.16)\quad \partial_n v_j = \sqrt{1 + |\nabla'\omega|^2}\,\tilde{g}_j + \sum_{k=1}^{n-1} \omega_{jk} v_k - \left[\tilde{F}_{jn} - \sum_{k=1}^{n-1} \omega_k \tilde{F}_{jk} - \sum_{k=1}^{n-1} \omega_k \partial_k v_j + |\nabla'\omega|^2 \partial_n v_j + \omega_j \partial_n v_n\right] \quad \text{on } \boldsymbol{R}^n_0,$$

$$(3.17)\qquad (\lambda-\Delta)v_n=\tilde f_n-\sum_{k=1}^{n-1}\omega_k\tilde f_k+\sum_{k=1}^{n-1}\partial_k\left\{\tilde F_{nk}-\sum_{\ell=1}^{n-1}\omega_\ell F_{\ell k}-2\omega_k\partial_n v_n\right\}$$

$$+\partial_n\left[\tilde F_{nn}-\sum_{\ell=1}^{n-1}\omega_\ell(\tilde F_{\ell n}+\tilde F_{n\ell})+\sum_{k,\ell=1}^{n-1}\omega_\ell\omega_k\tilde F_{\ell k}\right]$$

$$+|\nabla'\omega|^2\partial_n^2 v_n+\sum_{k,\ell=1}^{n-1}\omega_{\ell k}\tilde F_{\ell k}-\sum_{k,\ell=1}^{n-1}2\omega_{k\ell}\omega_k\partial_n v_\ell$$

$$+(\Delta'\omega)\partial_n v_n+\sum_{\ell=1}^{n-1}[2(\nabla'\omega_\ell)\cdot(\nabla' v_\ell)+(\Delta'\omega_\ell)v_\ell]\qquad\text{in } \boldsymbol{R}^n_+,$$

$$(3.18)\quad v_n=0 \text{ on } \boldsymbol{R}^n_0,\qquad\qquad \text{on } \boldsymbol{R}^n_0,$$

where

$$\partial_n^2 v_n=\partial^2 v_n/\partial x_n^2,\qquad (\nabla'\omega_\ell)\cdot(\nabla' v_\ell)=\sum_{k=1}^{n-1}(\partial_k\omega_\ell)\partial_k v_\ell,\qquad \Delta'\omega_\ell=\sum_{k=1}^{n-1}\partial_k^2 v_\ell.$$

By (3.17) we write $(1+|\nabla'\omega|^2)\partial_n^2 v_n$ in terms of $\tilde f_j, \tilde F_j, \tilde G_j, \nabla v$ and $\partial_j\nabla v$ $(j=1,\dots,n-1)$ and we insert this formula into the right hand side of (3.15) and (3.17). Using the fact:

$$(3.19)\qquad \tilde F_{nn}-\sum_{k=1}^{n-1}\omega_k(\tilde F_{nk}+\tilde F_{kn})+\sum_{k,\ell=1}^{n-1}\omega_k\omega_\ell\tilde F_{k\ell}|_{y_n=0}=\nu\cdot(F\nu)|_{\partial H}=0$$

which follows from (3.5), finally we arrive at the half space problem:

$$(3.20)\quad (\lambda-\Delta)v_j=A_j(f,F,v)+\sum_{k=1}^{n}\partial_k[B_{jk}(F,v)]\qquad \text{in } \boldsymbol{R}^n_+,\ j=1,\dots,n;$$

$$\partial_n v_j=E_j(g,v)-B_{jn}(F,v),\qquad \text{on } \boldsymbol{R}^n_0,\ j=1,\dots,n-1;$$

$$v_n=0\qquad \text{on } \boldsymbol{R}^n_0,$$

where $\kappa_j=\omega_j(1+|\nabla'\omega|^2)^{-1}$;

$$A_j(f,F,v)=\tilde f_j+\kappa_j\left(\tilde f_n-\sum_{\ell=1}^{n-1}\omega_\ell\tilde f_\ell\right)-\sum_{\ell=1}^{n-1}(\partial_\ell\kappa_j)\left\{\tilde F_{n\ell}-\sum_{m=1}^{n-1}\omega_m\tilde F_{m\ell}\right\}$$

$$+\sum_{\ell,m=1}^{n-1}\kappa_j\omega_{\ell m}\tilde F_{m\ell}-\kappa_j(\lambda v_n)-\sum_{\ell=1}^{n-1}\partial_\ell\kappa_j\partial_\ell v_n+2\sum_{\ell=1}^{n-1}(\partial_\ell\kappa_j)\omega_\ell\partial_n v_n$$

$$-2\sum_{\ell,m=1}^{n-1}\kappa_j\omega_{\ell m}\omega_\ell\partial_n v_m+\kappa_j(\Delta'\omega)\partial_n v_n+\kappa_j\sum_{\ell=1}^{n-1}\{2(\nabla'\omega_\ell)\cdot(\nabla' v_\ell)+(\Delta'\omega_\ell)v_\ell\};$$

$$A_n(f,F,v) = (1+|\nabla'\omega|^2)^{-1}\left[\tilde{f}_n - \sum_{\ell=1}^{n-1}\omega_\ell \tilde{f}_\ell + \sum_{\ell,m=1}^{n-1}\omega_{\ell m}\tilde{F}_{m\ell} - \sum_{\ell,m=1}^{n-1} 2\omega_{\ell m}\omega_\ell \partial_n v_m \right.$$

$$\left. + (\Delta'\omega)\partial_n v_n + \sum_{\ell=1}^{n-1}\{2(\nabla'\omega_\ell)\cdot(\nabla' v_\ell) + (\Delta'\omega_\ell)v_\ell\}\right]$$

$$- \sum_{k=1}^{n-1}(\partial_k(1+|\nabla'\omega|^2)^{-1})\left(\tilde{F}_{nk} - \sum_{\ell=1}^{n-1}\omega_\ell F_{\ell k}\right) + 2\sum_{k=1}^{n-1}(\partial_k(1+|\nabla'\omega|^2)^{-1})\omega_k\partial_n v_n$$

$$+ |\nabla'\omega|^2(1+|\nabla'\omega|^2)^{-1}(\lambda v_n) + \sum_{k=1}^{n-1}(\partial_k|\nabla'\omega|^2(1+|\nabla'\omega|^2)^{-1})\partial_k v_n;$$

$$B_{jk}(F,v) = \tilde{F}_{jk} + \kappa_j\left(\tilde{F}_{nk} - \sum_{\ell=1}^{n-1}\omega_\ell\tilde{F}_{\ell k}\right) + \kappa_j(\partial_k v_n - 2\omega_k\partial_n v_n) - \omega_k\partial_n v_j;$$

$$B_{jn}(F,v) = \tilde{F}_{jn} - \sum_{\ell=1}^{n-1}\omega_\ell\tilde{F}_{j\ell} - \sum_{\ell=1}^{n-1}\omega_\ell\partial_\ell v_j + |\nabla'\omega|^2\partial_n v_j + \omega_j\partial_n v_n$$

$$+ \kappa_j\left[\tilde{F}_{nn} - \sum_{\ell=1}^{n-1}\omega_\ell(\tilde{F}_{\ell n} + \tilde{F}_{n\ell}) + \sum_{\ell,m=1}^{n-1}\omega_\ell\omega_m\tilde{F}_{m\ell}\right];$$

$$B_{nk}(F,v) = (1+|\nabla'\omega|^2)^{-1}\left[\tilde{F}_{nk} - \sum_{\ell=1}^{n-1}\omega_\ell\tilde{F}_{\ell k} - 2\omega_k\partial_n v_n - |\nabla'\omega|^2\partial_k v_n\right];$$

$$B_{nn}(F,v) = (1+|\nabla'\omega|^2)^{-1}\left[\tilde{F}_{nn} - \sum_{\ell=1}^{n-1}\omega_\ell(\tilde{F}_{\ell n} + \tilde{F}_{n\ell}) + \sum_{\ell,m=1}^{n-1}\omega_\ell\omega_m\tilde{F}_{m\ell}\right];$$

$$E_j(g,v) = \sqrt{1+|\nabla'\omega|^2}\,\tilde{g}_j + \sum_{\ell=1}^{n-1}\omega_{j\ell}v_\ell;$$

and $j,k = 1,\dots,n-1$.

If v is a solution to (3.20), then defining $\tilde{u}$ by (3.14) in terms of v and setting $u(x) = \tilde{u}(y)$ by (3.9), we see that u is a solution to (3.4). Therefore, we shall solve (3.20) by the contraction mapping principle, below. To do this, given $v \in W_p^2(\boldsymbol{R}_+^n)$, we consider the half-space problem:

$$\text{(3.21)} \qquad (\lambda - \Delta)w = A(f,F,v) + \nabla\cdot B(F,v) \qquad \text{in } \boldsymbol{R}_+^n,$$

$$-(\mathrm{curl}\ w)\nu_0 = E(g,v) + B(F,v)\nu_0 \qquad \text{on } \boldsymbol{R}_0^n,$$

where $\nu_0 = (0,\dots,0,-1)$,

$$A(f, F, v) = {}^t(A_1(f, F, v), \dots, A_n(f, F, v)), \qquad B(F, v) = (B_{jk}(F, v)),$$
$$E(g, v) = {}^t(E_1(g, v), \dots, E_{n-1}(g, v), 0).$$

For the notational simplicity, we set $K_j = \|\nabla^j \omega\|_{L_\infty(\boldsymbol{R}^{n-1})}$, $j = 1, 2, 3$. Since we will choose K_1 small enough, we may assume that $K_1 \leqq 1$. Since $B_{nn}(F, v) = 0$ on $\boldsymbol{R}^n_0$ as follows from (3.19), we can apply Theorem 2.5 to solve (3.21), and therefore there exists a unique solution $w \in W^2_p(\boldsymbol{R}^n_+)^n$ of (3.21) which satisfies the *a priori* estimate:

$$\begin{aligned}(3.22)\quad \mathscr{I}^1_p(w, \lambda, \boldsymbol{R}^n_+) &\leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(H)} + (1 + K_2)\|F\|_{L_p(H)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial H)} \\ &\quad + (K_1 + |\lambda|^{-1/2}K_2 + |\lambda|^{-1}K_3)\mathscr{I}^1_p(v, \lambda, \boldsymbol{R}^n_+)\}; \\ \mathscr{I}^2_p(w, \lambda, \boldsymbol{R}^n_+) &\leqq C\{\|f\|_{L_p(H)} + \|\nabla(F, g)\|_{L_p(H)} + (K_2 + |\lambda|^{1/2})\|(F, g)\|_{L_p(H)} \\ &\quad + (K_1 + |\lambda|^{-1/2}K_2 + |\lambda|^{-1}K_3)\mathscr{I}^2_p(v, \lambda, \boldsymbol{R}^n_+)\};\end{aligned}$$

provided that $|\lambda| \geqq 1$, where we have used (2.23). Let us define the map G by $Gv = w$, and set

$$M_\lambda = C\{\|f\|_{L_p(H)} + \|\nabla(F, g)\|_{L_p(H)} + (K_2 + |\lambda|^{1/2})\|(F, g)\|_{L_p(H)}\}.$$

If we choose $K_1 > 0$ and $\lambda_0 \geqq 1$ in such a way that

$$(3.23)\qquad CK_1 \leqq 1/4, \qquad C(\lambda_0^{-1/2}K_2 + \lambda_0^{-1}K_3) \leqq 1/4,$$

then noting that the equation is linear, by (3.22) we have

$$(3.24)\qquad \mathscr{I}^2_p(Gv, \lambda, \boldsymbol{R}^n_+) \leqq 2M_\lambda \text{ provided that } \mathscr{I}^2_p(v, \lambda, \boldsymbol{R}^n_+) \leqq 2M_\lambda,$$

$$(3.25)\qquad \mathscr{I}^2_p(Gv^1 - Gv^2, \lambda, \boldsymbol{R}^n_+) \leqq \frac{1}{2}\mathscr{I}^2_p(v^1 - v^2, \lambda, \boldsymbol{R}^n_+),$$

for any $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \lambda_0$, which implies that G is the contraction map. Therefore, there exists a fixed point $v \in W^1_p(\boldsymbol{R}^n_+)^n$ with $Gv = v$, which solves (3.20). Inserting $v = w$ into the left hand side of (3.22) and using (3.23) we see that

$$\mathscr{I}^1_p(v, \lambda, \boldsymbol{R}^n_+) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(H)} + \|F\|_{L_p(H)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial H)}\},$$
$$\mathscr{I}^2_p(v, \lambda, \boldsymbol{R}^n_+) \leqq C\{\|f\|_{L_p(H)} + \mathscr{I}^1_p((F, g), \lambda, H)\},$$

for any $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \lambda_0 (\geqq 1)$, where $C = C(p, n, \varepsilon, K_2, K_3)$.

To get the higher regularity of v, we use the following lemma.

Lemma 3.2. *Let* $1 \leqq p < \infty$ *and* $i = 1, \dots, n-1$. (1) *If* $u \in L_p(\boldsymbol{R}^n_+)$ *satisfies the condition*: $\|[u]_{i,h}\|_{L_p(\boldsymbol{R}^n_+)} \leqq C$ *for any* h *with* $0 < |h| \leqq 1$ *with some* *constant C independent of h, then $\partial_i u \in L_p(\boldsymbol{R}^n_+)$ and $\|\partial_i u\|_{L_p(\boldsymbol{R}^n_+)} \leqq C$. Here and hereafter, we set*

$$[u]_{i,h} = \frac{u(x+he_i)-u(x)}{h}, \qquad e_i = (0,\dots,\overset{i\text{-}th}{1},\dots,0).$$

(2) *If $u \in W^1_p(\boldsymbol{R}^n_+)$, then $\|[u]_{i,h}\|_{L_p(\boldsymbol{R}^n_+)} \leqq \|\partial_i u\|_{L_p(\boldsymbol{R}^n_+)}$ for any h with $0 < |h| \leqq 1$.*

In view of Lemma 3.2, if we consider the equation for $[v]_{i,h}$ ($i = 1,\dots,n-1$ and $0 < |h| \leqq 1$), then what $f \in W^1_p(H)$, $F \in W^2_p(H)$, $g \in W^2_p(H)$ and $\omega \in \mathscr{B}^4(\boldsymbol{R}^{n-1})$ implies that $\partial_i v \in W^2_p(\boldsymbol{R}^n_+)$. If we write $\partial^3 v/\partial y_n^3$ by using the equation (3.20), we see also that $\partial^3 v/\partial y_n^3 \in L_p(\boldsymbol{R}^n_+)$. Repeated use of this argument implies the higher regularity of v, which completes the proof of Theorem 3.1. □

For the later use, finally we shall show the following regularity theorem of the weak solution.

Lemma 3.3. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$ and set $K_1 = \min_{q=p,p'} K_0(q,\varepsilon,n) \in (0,1)$, where $K_0(q,\varepsilon,n)$ $(q = p,p')$ are the same constants as in Theorem 3.1. Let $\lambda \in \boldsymbol{C}$ and assume that $v \in \dot{W}^1_p(H)^n$ satisfies the variational equation:*

(3.26)
$$\lambda(v,\Phi)_H + \frac{1}{2}(\mathrm{curl}\, v, \mathrm{curl}\, \Phi)_H + (\nabla\cdot v, \nabla\cdot\Phi)_H = (f,\Phi)_H - (F,\nabla\Phi)_H - (g,\Phi)_{\partial H}$$

for any $\Phi \in \dot{W}^1_{p'}(H)^n$. If $f \in L_p(H)^n$, $F \in W^1_p(H)^{n\times n}$, $g \in W^1_p(H)$, and F and g satisfy (3.5), *then $v \in W^2_p(H)^n$ provided $\|\nabla'\omega\|_{L_\infty(\boldsymbol{R}^{n-1})} \leqq K_1$.*

Proof. Let $\lambda_1 = \max_{q=p,p'} \lambda_0(q,\varepsilon,n,|||\omega|||)$, where $\lambda_0(q,\varepsilon,n,|||\omega|||)$ $(q = p,p')$ are the same numbers as in Theorem 3.1. Then v satisfies the variational equation (3.26), where λ and f are replaced by λ_1 and $f + (\lambda_1 - \lambda)v$, respectively. By the existence theorem of the dual problem with $\lambda = \lambda_1$, we see that the uniqueness of the variational equation (3.26) with $\lambda = \lambda_1$ holds. On the other hand, by Theorem 3.1 with $\lambda = \lambda_1$ we know the existence of $w \in W^2_p(H)^n$ which solves the equation (3.4), where λ and f are replaced by λ_1 and $f + (\lambda_1 - \lambda)v$, respectively. Since w also satisfies the variational equation (3.26) where λ and f are replaced by λ_1 and $f + (\lambda_1 - \lambda)v$, respectively, the uniqueness implies that $v = w$, which means that $v \in W^2_p(H)$. This completes the proof of the lemma. □

4. The unique solvability of the variational equation in the L_2 framework

In this section, we discuss the unique solvability of the variational equation (1.10) in the L_2 framework. We start with the following two lemmas.

Lemma 4.1. *There exists a constant* $C = C(\Omega, n) > 0$ *such that*

(4.1) $$\|u\|^2_{W^1_2(\Omega)} \leqq 2\|\nabla \cdot u\|^2_{L_2(\Omega)} + \|\mathrm{curl}\, u\|^2_{L_2(\Omega)} + C\|u\|^2_{L_2(\Omega)}$$

for any $u \in \dot{W}^1_2(\Omega)^n$.

Proof. See G. Duvaut and J. L. Lions [6].

Lemma 4.2. *If* Ω *is a bounded and simply connected domain in* $\boldsymbol{R}^3$, *then there exists a constant* $C = C(\Omega) > 0$ *such that*

(4.2) $$\|u\|^2_{W^1_2(\Omega)} \leqq C\{\|\nabla \cdot u\|^2_{L_2(\Omega)} + \|\mathrm{curl}\, u\|^2_{L_2(\Omega)}\} \qquad \text{for any } u \in \dot{W}^1_2(\Omega)^n.$$

Proof. To prove Lemma 4.2, we need the following proposition due to von Wahl [24].

Proposition 4.3. *Let* $1 < p < \infty$ *and* G *be a bounded domain in* $\boldsymbol{R}^3$ *with* $\partial G \in C^1$. *Then, there exists a constant* $C > 0$ *such that*

(4.3) $$\|\nabla u\|_{L_p(\Omega)} \leqq C\{\|\nabla \cdot u\|_{L_p(\Omega)} + \|\mathrm{curl}\, u\|_{L_p(\Omega)}\} \qquad \text{for any } u \in \dot{W}^1_p(\Omega)^n$$

if and only if the Betti number of Ω *is equal to zero.*

In view of Lemma 4.1, to prove Lemma 4.2 it suffices to prove that there exists a constant $C = C(\Omega) > 0$ such that

(4.4) $$\|u\|_{L_2(\Omega)} \leqq C\{\|\nabla \cdot u\|_{L_2(\Omega)} + \|\mathrm{curl}\, u\|_{L_2(\Omega)}\} \qquad \text{for any } u \in \dot{W}^1_2(\Omega)^n.$$

We shall show (4.4) by contradiction. Suppose that (4.4) does not hold. Then, there exists a sequence $\{u_j\}$ in $\dot{W}^1_2(\Omega)^n$ such that

(4.5) $$\|u_j\|_{L_2(\Omega)} = 1,$$

(4.6) $$\|\nabla \cdot u_j\|_{L_2(\Omega)} + \|\mathrm{curl}\, u_j\|_{L_2(\Omega)} < 1/j.$$

Combining (4.5), (4.6) and (4.1) implies that $\|u_j\|_{W^1_2(\Omega)} \leqq M$ for any j with some constant M independent of j. Since Ω is bounded, passing to the subsequence if necessary, we may assume that there exists a $u \in \dot{W}^1_2(\Omega)^n$ such that

(4.7) $$u_j \to u \text{ weakly in } W^1_2(\Omega), \qquad u_j \to u \text{ strongly in } L_2(\Omega).$$

By (4.3), (4.5), (4.6) and (4.7), we see that $\nabla u = 0$ and $\|u\|_{L_2(\Omega)} = 1$. What $\nabla u = 0$ implies that u is a constant vector, which combined with the fact that $\nu \cdot u|_{\partial\Omega} = 0$ implies that $u = 0$. This contradicts what $\|u\|_{L_2(\Omega)} = 1$, which completes the proof of Lemma 4.2. □

Set

$$B_\lambda[u, \Phi] = \lambda(u, \Phi)_\Omega + \frac{1}{2}(\mathrm{curl}\, u, \mathrm{curl}\, \Phi)_\Omega + (\nabla \cdot u, \nabla \cdot \Phi)_\Omega,$$

and by Lemma 4.1 and (2.4) we see that for any $\delta > 0$ and $0 < \varepsilon < \pi/2$ there exists a constant $c = C(\delta, \varepsilon, \Omega) > 0$ such that there holds the estimate:

$$(4.8) \quad |B_\lambda[u,u]| \geqq c\|u\|^2_{W_2^1(\Omega)} \qquad \text{for any } \lambda \in \Sigma_\varepsilon \text{ with } |\lambda| \geqq \delta \text{ and } u \in \dot{W}_2^1(\Omega)^n.$$

Moreover, if Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$, then by Lemma 4.2 we see that there exists constant $\delta_1 > 0$ and $c = c(\Omega) > 0$ such that there holds the estimate:

$$(4.9) \quad |B_\lambda[u,u]| \geqq c\|u\|^2_{W_2^1(\Omega)} \qquad \text{for any } \lambda \in \boldsymbol{C} \text{ with } |\lambda| \leqq \delta_1 \text{ and } u \in \dot{W}_2^1(\Omega)^n.$$

In view of (4.8) and (4.9), by the Lax and Milgram theorem we have the following theorem.

Theorem 4.4. *Let $0 < \varepsilon < \pi/2$, $\delta > 0$ and $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, be a domain satisfying the Assumption* 1.1. *Then, for every $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \delta$, $f \in W_2^{-1}(\Omega)^n = (\dot{W}_2^1(\Omega)^n)^*$, $F \in L_2(\Omega)^{n\times n}$ and $g \in L_2(\partial\Omega)^n$, the variational problem* (1.10) *admits a unique solution $u \in \dot{W}_2^1(\Omega)^n$.*

Moreover, if Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$, then there exists a constant $\delta_1 > 0$ such that for every $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \delta_1$, $f \in W_2^{-1}(\Omega)^n$, $F \in L_2(\Omega)^{n\times n}$ and $g \in L_2(\partial\Omega)^n$, the variational problem (1.10) *admits a unique solution $u \in \dot{W}_2^1(\Omega)^n$.*

5. Localization of the problem (1.1)

Let u be a solution to (1.1) and $\varphi \in C^\infty(\boldsymbol{R}^n)$ a cut-off function. Then, we have

$$(5.1) \qquad (\lambda - \Delta)(\varphi u) = A_\varphi + \nabla \cdot F_\varphi \qquad \text{in } \Omega,$$

$$(5.2) \qquad -\mathrm{curl}(\varphi u)\nu|_{\partial\Omega} = (B_\varphi + F_\varphi \nu)|_{\partial\Omega}, \qquad \nu \cdot (\varphi u)|_{\partial\Omega} = 0,$$

where

$$(5.3) \quad A_\varphi = \varphi f - F(\nabla\varphi) + (\Delta\varphi)u, \quad F_\varphi = \varphi F - 2u^t(\nabla\varphi), \quad B_\varphi = \varphi g - [(\nabla\varphi)\cdot\nu]u,$$

and ν is suitably extended to the whole $\overline{\Omega}$. In particular,

$$(5.4) \qquad \nu \cdot (F_\varphi \nu)|_{\partial\Omega} = B_\varphi \cdot \nu|_{\partial\Omega} = 0$$

which follows from (1.2) and the fact that $\nu \cdot u|_{\partial\Omega} = 0$. If $\mathrm{supp}\, \varphi \cap \partial\Omega = \emptyset$, then the problem (5.1) to (5.2) is the equation in $\boldsymbol{R}^n$:

$$(5.5) \qquad (\lambda - \Delta)(\varphi u) = A_\varphi + \nabla \cdot F_\varphi \qquad \text{in } \boldsymbol{R}^n.$$

If $\mathrm{supp}\, \varphi \cap \partial\Omega = \mathrm{supp}\, \varphi \cap \boldsymbol{R}^n_0$, then the problem (5.1) to (5.2) is the boundary value problem in $\boldsymbol{R}^n_+$:

$$(5.6) \qquad (\lambda - \Delta)(\varphi u) = A_\varphi + \nabla \cdot F_\varphi \qquad \text{in } \boldsymbol{R}^n_+,$$
$$-\mathrm{curl}(\varphi u)\nu_0|_{\partial\Omega} = (B_\varphi + F_\varphi \nu_0)|_{\boldsymbol{R}^n_0}, \qquad \nu_0 \cdot (\varphi u)|_{\boldsymbol{R}^n_0} = 0.$$

Below, we discuss the case where $\mathrm{supp}\,\varphi \cap \partial\Omega$ is a really curved boundary. In this case, we reduce the problem to that in the bent half space. Let $x_0 \in \partial\Omega$ and η be a positive number determined later. Suppose that φ satisfies the condition:

$$(5.7) \qquad \varphi(x) = \begin{cases} 1 & |x - x_0| < \eta, \\ 0 & |x - x_0| > 2\eta. \end{cases}$$

Let $T = (a_{ij})$ be an orthogonal matrix such that $(a_{n1}, \dots, a_{nn}) = -{}^t\nu(x_0)$. If we set $y = T(x - x_0)$, then $\nabla_x = {}^tT\nabla_y$ with $\nabla_x = {}^t(\partial/\partial x_1, \dots, \partial/\partial x_n)$ and $\nabla_y = {}^t(\partial/\partial y_1, \dots, \partial/\partial y_n)$. Moreover, we see easily that the problem (5.1) to (5.2) is reduced to the following equation:

$$(5.8) \qquad (\lambda - \Delta)v = T\tilde{A}_\varphi + \nabla \cdot [T\tilde{F}_\varphi {}^tT] \qquad \text{in } \tilde{\Omega},$$
$$-(\mathrm{curl}\, v)\mu|_{\partial\tilde{\Omega}} = (T\tilde{B}_\varphi + [T\tilde{F}_\varphi {}^tT]\mu)|_{\partial\tilde{\Omega}}, \qquad \mu \cdot v|_{\partial\tilde{\Omega}} = 0,$$

where $\tilde{\Omega} = T(\Omega - \{x_0\})$, $\tilde{A}_\varphi(y) = A_\varphi(x)$, $\tilde{F}_\varphi(y) = F_\varphi(x)$, $\tilde{B}_\varphi(y) = B_\varphi(x)$, $\mu(y) = T\tilde{\nu}(y) = T\nu(x)$ and $v(y) = T\tilde{u}(y) = Tu(x)$. By the implicit function theorem, we see that there exist small numbers $\varepsilon_0, \varepsilon_1$ with $0 < \varepsilon_1 < \varepsilon_0 \leqq 1$ and a function $\rho(y') \in C^{2,1}(B'_{\varepsilon_0}(0))$ such that

$$(5.9) \qquad \partial\tilde{\Omega} \cap B_{\varepsilon_1}(0) \subset \{y = (y', y_n) \in \boldsymbol{R}^n \mid y_n = \rho(y'), y' \in B'_{\varepsilon_0}(0)\},$$
$$\tilde{\Omega} \cap B_{\varepsilon_1}(0) \subset \{y = (y', y_n) \in \boldsymbol{R}^n \mid y_n > \rho(y'), y' \in B'_{\varepsilon_0}(0)\},$$
$${}^t\mu = (\nabla'\rho(y'), -1)/\sqrt{1 + |\nabla'\rho(y')|^2}, \qquad \nabla'\rho = (\partial_1\rho, \dots, \partial_{n-1}\rho),$$

$\rho(0) = 0$ and $(\nabla'\rho)(0) = (0, \dots, 0)$, where

$$B_{\varepsilon_1}(0) = \{y \in \boldsymbol{R}^n \mid |y| < \varepsilon_1\}, \qquad B'_{\varepsilon_0}(0) = \{y' \in \boldsymbol{R}^{n-1} \mid |y'| < \varepsilon_0\}.$$

Let $\psi(y')$ be a function in $C_0^\infty(\boldsymbol{R}^{n-1})$ such that $\psi(y') = 1$ for $|y'| \leqq 1/2$ and $\psi(y') = 0$ for $|y'| \geqq 1$, and set $\omega(y') = \psi(y'/\varepsilon_2)\rho(y')$ for $0 < \varepsilon_2 < \varepsilon_0 < 1$. Then, we see that

$$(5.10) \qquad \|\nabla'\omega\|_{L_\infty(\boldsymbol{R}^{n-1})} \leqq c_\rho^1 \varepsilon_2, \qquad \|\!|\omega|\!\| \leqq c_\rho^2 \varepsilon_2^{-1},$$

where

$$c_\rho^1 = \left(\|\psi\|_{L_\infty(\boldsymbol{R}^{n-1})} + \frac{1}{2}\|\nabla\psi\|_{L_\infty(\boldsymbol{R}^{n-1})} \right) \sup_{|y'| \leqq \varepsilon_1} |(\nabla')^2\rho(y')|$$

and c_ρ^2 is a positive number depending on $\|\!|\psi|\!\|$ and $\|\!|\rho|\!\|$ but independent of ε_2. Under these preparations, we set

(5.11)
$$H = \{(y', y_n) \in \boldsymbol{R}^n \mid y_n > \omega(y'), y' \in \boldsymbol{R}^{n-1}\},$$
$$\partial H = \{(y', y_n) \in \boldsymbol{R}^n \mid y_n = \omega(y'), y' \in \boldsymbol{R}^{n-1}\},$$
$$^t\mu = (\nabla'\omega, -1)/\sqrt{1 + |\nabla'\omega|^2}.$$

If we choose η such as $0 < 4\eta < \varepsilon_2$, then by (5.7) to (5.9) and (5.11) we see that $v(y) = T(\varphi u)(x)$ satisfies the bent half-space problem:

(5.12)
$$(\lambda - \Delta)v = T\tilde{A}_\varphi + \nabla \cdot [T\tilde{F}_\varphi{}^tT] \qquad \text{in } H,$$
$$-(\text{curl } v)\mu|_{\partial H} = (T\tilde{B}_\varphi + [T\tilde{F}_\varphi{}^tT]\mu)|_{\partial H}, \qquad \mu \cdot v|_{\partial H} = 0.$$

Moreover, it follows from (5.4) and the fact that ${}^tT\mu = \tilde{\nu}$ that
$$\mu \cdot ([T\tilde{F}_\varphi{}^tT]\mu)|_{\partial H} = \mu \cdot (T\tilde{B}_\varphi)|_{\partial H} = 0.$$

Finally, we consider the localization of the solutions to the variational equation (1.10). Let $u \in \dot{W}_p^1(\Omega)$ satisfy the variational equation (1.10). When $\operatorname{supp} \varphi \cap \Omega = \emptyset$, φu satisfies the equation:

(5.13)
$$(\lambda - \Delta)(\varphi u) = A_\varphi + \nabla \cdot F_\varphi$$

in the sense of tempered distribution in $\boldsymbol{R}^n$. When $\operatorname{supp} \varphi \cap \Omega = \operatorname{supp} \varphi \cap \boldsymbol{R}_0^n$, φu satisfies the variational equation:

(5.14)
$$\lambda(\varphi u, \Psi)_{\boldsymbol{R}_+^n} + \frac{1}{2}(\text{curl}(\varphi u), \text{curl } \Psi)_{\boldsymbol{R}_+^n} + (\nabla \cdot (\varphi u), \nabla \cdot \Psi)_{\boldsymbol{R}_+^n}$$
$$= (A_\varphi, \Psi)_{\boldsymbol{R}_+^n} - (F_\varphi, \nabla\Psi)_{\boldsymbol{R}_+^n} - (B_\varphi, \Psi)_{\boldsymbol{R}_0^n} \qquad \text{for any } \Psi \in W_{p'}^1(\boldsymbol{R}_+^n).$$

When $\operatorname{supp} \varphi \cap \partial\Omega$ is a really curved boundary, $v(y) = Tu(x)$ satisfies the variational equation:

(5.15)
$$\lambda(v, \Psi)_H + \frac{1}{2}(\text{curl } v, \text{curl } \Psi)_H + (\nabla \cdot v, \nabla \cdot \Psi)_H$$
$$= (T\tilde{A}_\varphi, \Psi)_H - (T\tilde{F}_\varphi{}^tT, \nabla\Psi)_H - (T\tilde{B}_\varphi, \Psi)_{\partial H}$$
$$\text{for any } \Psi \in \dot{W}_{p'}^1(H).$$

6. The proof of theorems stated in subsection 1.1

First, we shall show some *a priori* estimates of the solution $u \in W_p^2(\Omega)^n$ to (1.1). Let R be a positive large number such as $\Omega \backslash B_{R-1} = \boldsymbol{R}_+^n \backslash B_{R-1}$ when Ω is a perturbed half-space and $\partial\Omega \subset B_{R-1}$ when Ω is a bounded domain or an exterior domain in $\boldsymbol{R}^n$.

Lemma 6.1. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$ and $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, be a domain satisfying the Assumption* 1.1. *Let $f \in L_p(\Omega)^n$, $F \in \dot{W}_p^1(\Omega)^{n\times n}$ and $g \in \dot{W}_p^1(\Omega)^n$. Let $u \in W_p^2(\Omega)^n$ satisfy the equation* (1.1) *for some $\lambda \in \boldsymbol{C}$. Then, the following estimates hold*:

(1) *There exists a constant $\lambda_0 \geqq 1$ such that*

$$(6.1) \qquad \mathscr{I}_p^1(u,\lambda,\Omega) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(\Omega)} + \|F\|_{L_p(\Omega)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial\Omega)}\},$$

$$(6.2) \qquad \mathscr{I}_p^2(u,\lambda,\Omega) \leqq C\{\|f\|_{L_p(\Omega)} + \mathscr{I}_p^1((F,g),\lambda,\Omega)\},$$

for $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \lambda_0$, where $C = C(p,\varepsilon,\Omega,n) > 0$.

(2) *When Ω is an exterior domain or a perturbed half-space, for any $\delta \in (0,\lambda_0]$ there exists a constant $C = C(\delta,p,\varepsilon,\Omega,n) > 0$ such that there hold*

$$(6.3) \qquad \|u\|_{W_p^1(\Omega)} \leqq C\{\|(f,F)\|_{L_p(\Omega)} + \|g\|_{L_p(\partial\Omega)} + \|u\|_{L_p(\Omega_{R+2})}\},$$

$$(6.4) \qquad \|u\|_{W_p^2(\Omega)} \leqq C\{\|f\|_{L_p(\Omega)} + \|(F,g)\|_{W_p^1(\Omega)} + \|u\|_{L_p(\Omega_{R+2})}\},$$

for any $\lambda \in \Sigma_\varepsilon$ with $\delta \leqq |\lambda| \leqq \lambda_0$, where $\Omega_{R+1} = \Omega \cap B_{R+1}$.

(3) *When Ω is a bounded domain, the estimates* (6.3) *and* (6.4) *also hold for any $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_0$, where $C = C(p,\varepsilon,\Omega,n) > 0$.*

Proof. First, we shall estimate u on $\Omega_R = \Omega \cap B_R$. Given $x_0 \in \overline{\Omega_R}$, let φ be a cut-off function whose support is contained in some neighborhood of x_0. We apply Theorem 2.1 to (5.5) when $\operatorname{supp}\varphi \cap \partial\Omega = \emptyset$; Theorem 2.5 to (5.6) when $\operatorname{supp}\varphi \cap \partial\Omega = \operatorname{supp}\varphi \cap \boldsymbol{R}_+^n$; and Theorem 3.1 to (5.12) when $\operatorname{supp}\varphi \cap \Omega$ is a really curved boundary. In the last case, we choose ε_2 so small that $c_p^1\varepsilon_2 \leqq K(p,\varepsilon,n)$ in (5.10) where $K(p,\varepsilon,n)$ is the constant given in Theorem 3.1. Since

$$(6.5) \qquad \|A_\varphi\|_{L_p(\Omega)} \leqq C_\varphi\|(f,F,u)\|_{L_p(\Omega_\varphi)}, \qquad \|F_\varphi\|_{L_p(\Omega)} \leqq C_\varphi\|(F,u)\|_{L_p(\Omega_\varphi)},$$

$$\|\nabla F_\varphi\|_{L_p(\Omega)} \leqq C_\varphi\|(F,u)\|_{W_p^1(\Omega_\varphi)}, \qquad \|B_\varphi\|_{L_p(\partial\Omega)} \leqq C_\varphi(\|g\|_{L_p(\partial\Omega)} + \|u\|_{L_p(\partial\Omega\cap\operatorname{supp}\varphi)}),$$

$$\|B_\varphi\|_{L_p(\Omega)} \leqq C_\varphi\|(g,u)\|_{L_p(\Omega_\varphi)}, \qquad \|\nabla B_\varphi\|_{L_p(\Omega)} \leqq C_\varphi\|(g,u)\|_{W_p^1(\Omega_\varphi)},$$

where $\Omega_\varphi = \Omega \cap \operatorname{supp}\varphi$, by Theorems 2.1, 2.5 and 3.1 we see that there exist positive constants $\lambda(x_0) \geqq 1$ and $C(x_0) > 0$ depending on x_0 such that

$$(6.6) \qquad \mathscr{I}_p^1(\varphi u,\lambda,\Omega) \leqq C(x_0)\{|\lambda|^{-1/2}\|f\|_{L_p(\Omega_\varphi)} + \|F\|_{L_p(\Omega_\varphi)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial\Omega)} + \|u\|_{L_p(\Omega_\varphi)} + |\lambda|^{-1/(2p)}\|u\|_{L_p(\partial\Omega\cap\operatorname{supp}\varphi)}\},$$

$$\mathscr{I}_p^2(\varphi u,\lambda,\Omega) \leqq C(x_0)\{\|f\|_{L_p(\Omega_\varphi)} + \mathscr{I}_p^1((F,g),\lambda,\Omega_\varphi) + \mathscr{I}_p^1(u,\lambda,\Omega_\varphi)\},$$

where $\lambda \in \Sigma_\varepsilon$ and $|\lambda| \geqq \lambda_0(x_0)$. Since $\overline{\Omega_R}$ is compact, there exists a finite number of points $x_j \in \overline{\Omega_R}$, $j = 1, \dots, N$, such that $\overline{\Omega_R} \subset \bigcup_{j=1}^N \Omega_{x_j}$. Set $\lambda_1 = \max_{1 \leqq j \leqq N} \lambda_0(x_j)$ and $C = \max_{1 \leqq j \leqq N} C(x_j)$. Since we may assume that $\varphi(x) = 1$ on Ω_{x_0} for some neighborhood Ω_{x_0} of x_0, we have $\mathscr{I}_p^j(u, \lambda, \Omega_{x_j}) \leqq \mathscr{I}_p^j(\varphi u, \lambda, \Omega)$. And therefore, if we set

$$D = \bigcup_{j=1}^N \overline{(\Omega \cap \operatorname{supp} \varphi_j)} \subset \overline{\Omega_{R+1}}. \tag{6.7}$$

then by (6.6) we have

$$\begin{aligned}\mathscr{I}_p^1(u, \lambda, \Omega_R) &\leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(D)} + \|F\|_{L_p(D)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial\Omega)} \\ &\qquad + \|u\|_{L_p(D)} + |\lambda|^{-1/(2p)}\|u\|_{L_p(\partial\Omega \cap D)}\}, \\ \mathscr{I}_p^2(u, \lambda, \Omega_R) &\leqq C\{\|f\|_{L_p(D)} + \mathscr{I}_p^1((F, g), \lambda, D) + \mathscr{I}_p^1(u, \lambda, D)\},\end{aligned} \tag{6.8}$$

for $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \lambda_1$, where $C = C(p, \varepsilon, n, R, \Omega) > 0$. Next, given any positive number λ_2 we consider the case when $|\lambda| \leqq \lambda_2$. Replacing λ and f by λ_1 and $f + (\lambda_1 - \lambda)u$ in the above argument, respectively, by (6.8) we have

$$\begin{aligned}\|u\|_{W_p^1(\Omega_R)} &\leqq C\{\|(f, F)\|_{L_p(\Omega)} + \|g\|_{L_p(\partial\Omega)} + \|u\|_{L_p(D)} + \|u\|_{L_p(\partial\Omega \cap D)}\}, \\ \|u\|_{W_p^2(\Omega_R)} &\leqq C\{\|f\|_{L_p(\Omega)} + \|(F, g)\|_{W_p^1(\Omega)} + \|u\|_{W_p^1(D)}\},\end{aligned} \tag{6.9}$$

for any $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_2$, where $C = C(p, n, R, \Omega, \lambda_2) > 0$.

Now, we shall discuss the estimate of u on $\Omega^R = \Omega \cap B^R$. Let φ be a function in $C^\infty(\boldsymbol{R}^n)$ such that $\varphi(x) = 1$ for $|x| \geqq R$ and $\varphi(x) = 0$ for $|x| \leqq R - 1$. Since $\Omega_R = \Omega$ when Ω is a bounded domain, the argument below is necessary only for Ω being an exterior domain or a perturbed half space. Note that $\operatorname{supp} \varphi \cap \Omega = \operatorname{supp} \varphi \cap \boldsymbol{R}^n$ when Ω is an exterior domain, and $\operatorname{supp} \varphi \cap \Omega = \operatorname{supp} \varphi \cap \boldsymbol{R}^n_+$ when Ω is a perturbed half-space. Applying Theorems 2.1 and 2.5 to (5.5) and (5.6), respectively, using (6.5) and noting that $\varphi u = u$ on Ω^R, we have

$$\begin{aligned}\mathscr{I}_p^1(u, \lambda, \Omega^R) &\leqq C(\delta)\{|\lambda|^{-1/2}\|f\|_{L_p(\Omega)} + \|F\|_{L_p(\Omega)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial\Omega)} \\ &\qquad + \|u\|_{L_p(\Omega_{R+1})} + |\lambda|^{-1/(2p)}\|u\|_{L_p(\partial\Omega \cap B_{R+1})}\}, \\ \mathscr{I}_p^2(u, \lambda, \Omega^R) &\leqq C(\delta)\{\|f\|_{L_p(\Omega)} + \mathscr{I}_p^1((F, g), \lambda, \Omega) + \mathscr{I}_p^1(u, \lambda, \Omega_{R+1})\},\end{aligned} \tag{6.10}$$

for $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \delta$, and therefore combining (6.8) and (6.10) and inserting the estimates: $\|u\|_{L_p(\partial\Omega \cap B_{R+1})} \leqq C\|u\|_{W_p^1(\Omega_{R+2})}$ and $\mathscr{I}_p^1(u, \lambda, \Omega_{R+1}) \leqq |\lambda|^{-1/2}\mathscr{I}_p^2(u, \lambda, \Omega_{R+1})$ into the resultant inequalities, we have

$$(6.11)\quad \mathscr{I}_p^1(\lambda, u, \Omega) \leqq C\{|\lambda|^{-1/2}\|f\|_{L_p(\Omega)} + \|F\|_{L_p(\Omega)} + |\lambda|^{-1/(2p)}\|g\|_{L_p(\partial\Omega)} + |\lambda|^{-1/(2p)}\mathscr{I}_p^1(u, \lambda, \Omega)\},$$

$$\mathscr{I}_p^2(\lambda, u, \Omega) \leqq C\{\|f\|_{L_p(\Omega)} + \mathscr{I}_p^1((F, g), \lambda, \Omega) + |\lambda|^{-1/2}\mathscr{I}_p^1(u, \lambda, \Omega_{R+1})\},$$

for $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \lambda_1 \geqq 1$. Choosing λ_0 ($\geqq \lambda_1$) so large that $C\lambda_0^{-1/(2p)} \leqq 1/2$ in (6.11), we have (6.1) and (6.2).

To complete the proofs of (2) and (3), first we note that for any small $\sigma > 0$ there exists a constant $C = C(\sigma, p, n, \Omega) > 0$ such that

$$(6.12)\quad \|u\|_{L_p(\partial\Omega\cap D)} \leqq \sigma\|\nabla u\|_{L_p(\Omega_{R+2})} + C\|u\|_{L_p(\Omega_{R+2})},$$

$$(6.13)\quad \|u\|_{W_p^1(\Omega_{R+1})} \leqq \sigma\|u\|_{W_p^2(\Omega_{R+2})} + C\|u\|_{L_p(\Omega_{R+2})}.$$

In fact, (6.12) follows from the estimate of the trace operator:

$$(6.14)\quad \|u\|_{L_p(\partial\Omega\cap D)} \leqq C\|\nabla u\|_{L_p(\Omega_{R+2})}^{1/p}\|u\|_{L_p(\Omega_{R+2})}^{1-1/p} + C\|u\|_{L_p(\Omega_{R+2})}$$

which follows from (2.24). And, (6.13) follows from the classical interpolation inequality:

$$(6.15)\quad \|u\|_{W_p^1(\Omega_{R+1})} \leqq C\|u\|_{W_p^2(\Omega_{R+1})}^{1/2}\|u\|_{L_p(\Omega_{R+1})}^{1/2}.$$

When Ω is an exterior domain or a perturbed half-space, combining (6.8), (6.9) and (6.10), inserting (6.12) and (6.13) into the resultant estimates and choosing σ small enough, we have (6.3) and (6.4) for any $\lambda \in \Sigma_\varepsilon$ with $\delta \leqq |\lambda| \leqq \lambda_0$. When Ω is a bounded, combining (6.8) and (6.9), inserting (6.12) and (6.13) into the resultant estimates and choosing σ small enough, we have also (6.3) and (6.4) for $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \lambda_0$. This completes the proof of the lemma. □

In the course of the proof of main results, for the notational simplicity we set

$$C_{(0)}^k(\bar{\Omega}) = \{v \in C^k(\bar{\Omega}) \mid \operatorname{supp} v \text{ is compact}\}, \qquad k \geqq 0,$$

$$C_{(0),\nu}^1(\bar{\Omega})^{n\times n} = \{F = (F_{ij}) \in C_{(0)}^1(\bar{\Omega})^{n\times n} \mid \nu\cdot(F\nu)|_{\partial\Omega} = 0\},$$

$$C_{(0),\nu}^1(\bar{\Omega})^n = \{g = {}^t(g_1, \ldots, g_n) \in C_{(0)}^1(\bar{\Omega})^n \mid \nu\cdot g|_{\partial\Omega} = 0\}.$$

Lemma 6.2. *Let $1 < p < \infty$, $0 < \varepsilon < \pi/2$ and $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, be a domain satisfying the Assumption* 1.1. *Let $f \in C_{(0)}^0(\bar{\Omega})^n$, $F \in C_{(0),\nu}^1(\bar{\Omega})^{n\times n}$ and $g \in C_{(0),\nu}^1(\bar{\Omega})^n$. Then, for any $\lambda \in \Sigma_\varepsilon$ the problem* (1.1) *admits a solution $u \in W_p^2(\Omega)^n$.*

Moreover, if Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$, then the problem (1.1) *also admits a solution $u \in W_p^2(\Omega)^3$ when $\lambda \in \boldsymbol{C}$ and $|\lambda| \leqq \delta_1$, where δ_1 is the same constant as in Theorem* 4.4.

Proof. In view of Theorem 4.4, the variational equation (1.10) admits a unique solution $u \in \dot{W}_2^1(\Omega)$ for any $\lambda \in \Sigma_\varepsilon$. Moreover, if Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$, then the solution $u \in \dot{W}_2^1(\Omega)$ also exists for $|\lambda| \leqq \delta_1$. What we have to prove is that the u belongs to $W_p^2(\Omega)$. We start with the case where $1 < p \leqq 2$. First of all, we shall prove that $u \in W_2^2(\Omega_R) \cap W_p^2(\Omega_R)$. In this case, from (1.10) it follows that u also satisfies the variational equation:

$$(6.16)\quad (u, \Phi)_\Omega + \frac{1}{2}(\operatorname{curl} u, \operatorname{curl} \Phi)_\Omega + (\nabla \cdot u, \nabla \cdot \Phi)_\Omega = (f + (1-\lambda)u, \Phi)_\Omega - (F, \Phi)_\Omega - (g, \Phi)_{\partial\Omega} \qquad \text{for any } \Phi \in \dot{W}_{p'}^1(\Omega).$$

Let $x_0 \in \overline{\Omega_R}$ and φ be a cut-off function whose support is contained in some neighborhood of x_0. In view of (6.16), we replace A_φ and λ by $A'_\varphi = \varphi(f + (1-\lambda)u) - F(\nabla\varphi) + (\Delta\varphi)u$ and 1 in (5.1) and (5.2), respectively. Then, we apply Theorem 2.1 to (5.13) when $\operatorname{supp} \varphi \cap \Omega = \emptyset$; Theorem 2.5 to (5.14) when $\operatorname{supp} \varphi \cap \Omega = \operatorname{supp} \varphi \cap \boldsymbol{R}_+^n$; and Lemma 3.3 to (5.15) when $\operatorname{supp} \varphi \cap \Omega$ is a really curved boundary. In the last case, we choose $\varepsilon_2 > 0$ so small that $c_p^1 \varepsilon_2 \leqq K_1$ in (5.10), where K_1 is the same constant as in Lemma 3.3. Therefore, we see that $\varphi u \in W_2^2(\Omega)$, which implies immediately that $u \in W_2^2(\Omega_R)$. Since Ω_R is bounded and $1 < p \leqq 2$, in particular we have $u \in W_2^2(\Omega_R) \cap W_p^2(\Omega_R)$. Especially, when Ω is bounded we have $u \in W_2^2(\Omega) \cap W_p^2(\Omega)$, because $\Omega_R = \Omega$.

When Ω is an exterior domain or a perturbed half-space, we take $\varphi \in C^\infty(\boldsymbol{R}^n)$ such a way that $\varphi(x) = 1$ for $|x| \geqq R$ and $\varphi(x) = 0$ for $|x| \leqq R - 1$. Since we already know that $u \in W_2^2(\Omega_R)$ and since both $\operatorname{supp} \Delta\varphi$ and $\operatorname{supp} \partial_k \varphi$ are contained in B_R, we have $A_\varphi \in L_p(\Omega)^n \cap L_2(\Omega)^n$, $F_\varphi \in W_p^1(\Omega)^{n \times n} \cap W_2^1(\Omega)^{n \times n}$ and $B_\varphi \in W_p^1(\Omega)^n \cap W_2^1(\Omega)^n$, where A_φ, F_φ and B_φ are the same as in (5.3). Applying Theorems 2.1 and 2.5 to (5.13) and (5.14), respectively, we see that $\varphi u \in W_2^2(\Omega)^n \cap W_p^2(\Omega)^n$ for $\lambda \in \Sigma_\varepsilon$. Therefore, we have $u \in W_2^2(\Omega)^n \cap W_p^2(\Omega)^n$.

Next we consider the case where $2 < p < \infty$. Let us set $q = 2n/(n-2)$ when $n \geqq 3$ and $q = p$ when $n = 2$. Since we know that $u \in W_2^2(\Omega)^n$ from the previous argument, by Sobolev's imbedding theorem we see that $u \in W_q^1(\Omega)^n$. Then, $A'_\varphi, A_\varphi \in L_q(\Omega)^n \cap L_2(\Omega)^n$, $F_\varphi \in W_q^1(\Omega)^{n \times n} \cap W_2^1(\Omega)^{n \times n}$ and $B_\varphi \in W_q^1(\Omega)^n \cap W_2^1(\Omega)^n$. Employing the same argument as above, by Theorems 2.1 and 2.5, and Lemma 3.3 we see that $u \in W_q^2(\Omega)^n \cap W_2^2(\Omega)^n$. In particular, we have the lemma when $n = 2$ or $2 < p \leqq q$ and $n \geqq 3$. When $n \geqq 3$ and $q < p < \infty$, we set $q_1 = 2n/(n-3)$ when $n \geqq 4$ and $q_1 = p$ when $n = 3$. Since we already know that $u \in W_q^2(\Omega)^n \cap W_2^2(\Omega)^n$, by Sobolev's imbedding theorem we see that $u \in W_{q_1}^1(\Omega)^n$. Therefore, repeating the same argument as above, we see that $u \in W_{q_1}^2(\Omega) \cap W_2^2(\Omega)^n$. In particular, we have the lemma when $n = 3$ or $q < p \leqq q_1$ and $n \geqq 4$. Repeated use of this argument finally implies that $u \in W_p^2(\Omega)$. This completes the proof of the lemma. □

Lemma 6.3. *Let* $1 < p < \infty$ *and* $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$, *be a domain satisfying the Assumption* 1.1. *Let* $u \in W_p^1(\Omega)^n$ *satisfy the homogeneous equation*:

$$\lambda(u, \Phi)_\Omega + \frac{1}{2}(\mathrm{curl}\, u, \mathrm{curl}\, \Phi)_\Omega + (\nabla \cdot u, \nabla \cdot \Phi)_\Omega = 0 \tag{6.17}$$

for any $\Phi \in \dot{W}_{p'}^1(\Omega)^n$. *If* $\lambda \in \boldsymbol{C} \backslash (-\infty, 0]$, *then* $u = 0$.

Moreover, when Ω *is a bounded and simply connected domain in* $\boldsymbol{R}^3$, *if* $\lambda \in \boldsymbol{C}$ *and* $|\lambda| \leqq \delta_1$, *then* $u = 0$, *where* δ_1 *is the same constant as in Theorem* 4.4.

Proof. Given $f = {}^t(f_1, \dots, f_n)$, $f_j \in C_0^\infty(\Omega)$, by Lemma 6.2 there exists a $\Phi \in W_{p'}^2(\Omega)^n$ which solves the equation:

$$(\lambda - \Delta)\Phi = f \quad \text{in } \Omega, \qquad -(\mathrm{curl}\, \Phi)\nu|_{\partial\Omega} = 0, \qquad \nu \cdot \Phi|_{\partial\Omega} = 0. \tag{6.18}$$

Using (6.17) and (6.18), we have

$$0 = (\lambda u, \Phi)_\Omega + \frac{1}{2}(\mathrm{curl}\, u, \mathrm{curl}\, \Phi)_\Omega + (\nabla \cdot u, \nabla \cdot \Phi)_\Omega = (u, (\lambda - \Delta)\Phi)_\Omega = (u, f)_\Omega,$$

which combined with arbitrariness of choice of f implies that $u = 0$. □

Lemma 6.4. *Let* $1 < p < \infty$, $0 < \varepsilon < \pi/2$, $\delta > 0$ *and* $\Omega \subset \boldsymbol{R}^n$, $n \geqq 2$ *be a domain satisfying the Assumption* 1.1. *Let* $f \in L_p(\Omega)^n$, $F \in \dot{W}_p^1(\Omega)^{n \times n}$ *and* $g \in \dot{W}_p^1(\Omega)^n$. *Let* $u \in W_p^2(\Omega)^n$ *satisfy the equation* (1.1). *Then, there hold the a priori estimates* (1.3) *and* (1.4) *for any* $\lambda \in \Sigma_\varepsilon$ *with* $|\lambda| \geqq \delta$ *with some constant* $C = C(p, \varepsilon, \delta, \Omega, n) > 0$.

Moreover, if Ω *is a bounded and simply connected domain in* $\boldsymbol{R}^3$, *then for* $\lambda \in \boldsymbol{C}$ *with* $|\lambda| \leqq \delta_1$ *there hold the a priori estimates* (1.7) *and* (1.8), *where* δ_1 *is the same constant as in Theorem* 4.4.

Proof. In view of Lemma 6.1, (1.3) and (1.4) hold for $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \lambda_0$, where $\lambda_0 \geqq 1$ is the constant given in Lemma 6.1. Therefore, since $\|g\|_{L_p(\partial\Omega)} \leqq C\|g\|_{W_p^1(\Omega)}$ as follows from (2.23), in view of (6.3) and (6.4) it suffices to prove that there exists a constant C such that

$$\|u\|_{L_p(\Omega_{R+2})} \leqq C\{\|(f, F, G)\|_{L_p(\Omega)} + \|g\|_{L_p(\partial\Omega)}\} \tag{6.19}$$

for any $\lambda \in \Sigma_\varepsilon$ with $\delta < |\lambda| \leqq \lambda_0$. We shall show (6.19) by contradiction. Suppose that (6.19) does not hold. Then, there exist sequences $\{\lambda_j\} \subset \Sigma_\varepsilon$ with $\delta \leqq |\lambda_j| \leqq \lambda_0$, $\{u_j\} \subset W_p^2(\Omega)^n$, $\{f_j\} \subset L_p(\Omega)^n$, $\{F_j\} \subset \dot{W}_p^1(\Omega)^{n \times n}$ and $g_j \subset \dot{W}_p^1(\Omega)^n$ such that u_j solves (1.1) with $f = f_j$, $F = F_j$ and $g = g_j$, and

(6.20) $$\|u_j\|_{L_p(\Omega_{R+2})} = 1,$$

(6.21) $$\|(f_j, F_j, G_j)\|_{L_p(\Omega)} + \|g_j\|_{L_p(\partial\Omega)} < 1/j.$$

By (1.1), u_j satisfies also the variational equation:

(6.22) $$\lambda(u_j, \Phi)_\Omega + \frac{1}{2}(\mathrm{curl}\, u_j, \mathrm{curl}\, \Phi)_\Omega + (\nabla\cdot u_j, \nabla\cdot\Phi)_\Omega$$
$$= (f_j, \Phi)_\Omega - (F_j, \nabla\Phi)_\Omega + (g_j, \Phi)_{\partial\Omega} \qquad \text{for any } \Phi \in \dot{W}^1_{p'}(\Omega).$$

By (6.20), (6.21) and (6.3) we have $\|u_j\|_{W^1_p(\Omega)} \leqq M$ for any j with some constant $M > 0$ independent of j, which implies that passing to subsequences if necessary, we may assume that there exists a $\lambda \in \Sigma_\varepsilon$ with $\delta \leqq |\lambda| \leqq \lambda_0$ and a $u \in W^1_p(\Omega)^n$ such that

(6.23) $u_j \to u$ weakly $*$ in $W^1_p(\Omega)^n$, $u_j \to u$ strongly in $L_p(\Omega)$, $\lambda_j \to \lambda$

as $j \to \infty$. Passing j to ∞ in (6.22) and using (6.21) and (6.23), we see that u satisfies (6.17), which combined with Lemma 6.3 implies that $u = 0$. On the other hand, by (6.20) we see that $\|u\|_{L_p(\Omega_R)} = 1$, which contradicts what $u = 0$. Therefore, (6.19) holds.

When Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$, in view of Lemma 6.3 employing the same argument as above we also see that (6.19) holds for $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \delta_1$. This completes the proof of the lemma. □

Since $C^0_{(0)}(\bar{\Omega})^n, C^1_{(0),\nu}(\bar{\Omega})^n$ and $C^1_{(0),\nu}(\bar{\Omega})^{n\times n}$ are dense in $L_p(\Omega)^n, \dot{W}^1_p(\Omega)^n$ and $\dot{W}^1_p(\Omega)^{n\times n}$, respectively, combining Lemmas 6.2 and 6.4 we see easily the unique existence of solutions to (1.1) which satisfy the estimates (1.3) and (1.4) for any $\lambda \in \Sigma_\varepsilon$ with $|\lambda| \geqq \delta$. Moreover, when Ω is a bounded and simply connected domain in $\boldsymbol{R}^3$, we also see the unique existence of solutions to (1.1) which satisfy the estimates (1.7) and (1.8) when $\lambda \in \boldsymbol{C}$ and $|\lambda| \leqq \delta_1$, where δ_1 is the same constant as in Theorem 4.4. Since $C^\infty_0(\Omega)^n$ and $C^\infty_0(\Omega)^{n\times n}$ are dense in $L_p(\Omega)^n$ and $L_p(\Omega)^{n\times n}$, respectively and since given $g \in L_p(\partial\Omega)$ with $\nu\cdot g|_{\partial\Omega} = 0$ we can construct a sequence $\{g_j\} \subset W^1_p(\Omega)$ such that $\nu\cdot g_j|_{\partial\Omega} = 0$ and $g_j \to g$ in $L_p(\Omega)$ as $j \to \infty$, by (1.3), (1.7) and Lemma 6.3 we see easily the unique existence of solutions to the variational equation (1.10) having the estimates (1.3) and (1.7).

What we have to prove finally is that the conditions: $\nabla\cdot f = 0$ in Ω, $\nu\cdot f|_{\partial\Omega} = 0$ and $F + {}^tF = 0$ imply that $\nabla\cdot u = 0$ in Ω when $g = 0$. Note that what $F + {}^tF = 0$ implies that F satisfies the condition in (1.2), and therefore the existence theorems which we have already proved hold under the assumption that $F + {}^tF = 0$. To prove that $\nabla\cdot u = 0$, we start with the following definition.

Definition 6.5. Let $1 < p < \infty$ and $\lambda \in \boldsymbol{C}$. We say that the domain Ω is (λ, p)-unique if the following uniqueness assertion holds: Let $u \in W_p^1(\Omega)$ satisfy the homogeneous equation:

$$(\lambda - \Delta)u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u|_{\partial\Omega} = 0. \tag{6.24}$$

When Ω is a bounded domain, in addition we assume that

$$\int_\Omega u\, dx = 0. \tag{6.25}$$

Then, $u = 0$.

Remark 6.6. (1) As we know well (cf. Miyakawa [16]), when $v \in L_p(\Omega)^n$ and $\nabla \cdot v \in L_p(\Omega)$, the trace $\nu \cdot u|_{\partial\Omega}$ to the boundary $\partial\Omega$ is well-defined and belongs to $W_p^{-1/p}(\partial\Omega)$. Moreover, we have the generalized Gauss formula:

$$(\nu \cdot v, \psi)_{\partial\Omega} = (\nabla \cdot v, \psi)_\Omega + (v, \nabla\psi)_\Omega \qquad \text{for any } \psi \in C^1_{(0)}(\overline{\Omega}). \tag{6.26}$$

(2) When $1 < p < \infty$ and $\lambda \in \boldsymbol{C} \setminus (-\infty, 0]$, if Ω is one of domains stated in Assumption 1.1, then Ω is (λ, p)-unique. Moreover, if Ω is a bounded domain, then there exists a constant δ_2 such that Ω is (λ, p)-unique for $\lambda \in \boldsymbol{C}$ with $|\lambda| \leqq \delta_2$. These results were proved by many authors (cf. Galdi [8], Farwig and Sohr [7], Simader and Sohr [21]).

Lemma 6.7. *Let $1 < p < \infty$ and $\lambda \in \boldsymbol{C}$. Assume that Ω is (λ, p)-unique. Let $f \in L_p(\Omega)^n$, $F \in W_p^1(\Omega)^{n \times n}$ and $g = 0$ in the equation* (1.1). *Assume that $\nabla \cdot f = 0$ in Ω, $\nu \cdot f|_{\partial\Omega} = 0$ and $F + {}^tF = 0$. Then, the solution u of* (1.1) *satisfies the property: $\nabla \cdot u = 0$ in Ω.*

Proof. By the assumption we have

$$(\lambda - \Delta)(\nabla \cdot u) = \nabla \cdot (f + \nabla \cdot F) = 0 \qquad \text{in } \Omega, \tag{6.27}$$

because

$$\nabla \cdot (\nabla \cdot F) = \sum_{j,k=1}^n \frac{\partial^2 F}{\partial x_j \partial x_k} = 0$$

as follows from the antisymmetricity of F. To prove that $\partial_\nu(\nabla \cdot u)|_{\partial\Omega} = (\nu \cdot \nabla)(\nabla \cdot u)|_{\partial\Omega} = 0$, we take any $\varphi \in C_0^\infty(\partial\Omega)$ and consider the form: $((\nu \cdot \nabla)(\nabla \cdot u), \varphi)_{\partial\Omega}$. Let ψ be a function in $C^2_{(0)}(\overline{\Omega})$ such that $\psi|_{\partial\Omega} = \varphi$. By (6.26) we have

$$\begin{aligned}((\nu \cdot \nabla)(\nabla \cdot u), \varphi)_{\partial\Omega} &= ((\nabla \cdot \nabla)(\nabla \cdot u), \psi)_\Omega + (\nabla(\nabla \cdot u), \nabla\psi)_\Omega \\ &= (\lambda(\nabla \cdot u), \psi)_\Omega + (\Delta u - \nabla \cdot (\operatorname{curl} u), \nabla\psi)_\Omega\end{aligned}$$

where we have used (6.27) and (1.5). Since $\nu \cdot u|_{\partial\Omega} = 0$ and $(\operatorname{curl} u)\nu|_{\partial\Omega} = -F\nu|_{\partial\Omega}$, we can proceed as follows:

$$= -((\lambda - \Delta)u, \nabla\psi)_\Omega + (F\nu, \nabla\psi)_{\partial\Omega} + (\operatorname{curl} u, \nabla(\nabla\psi))_\Omega$$
$$= -(f + \nabla \cdot F, \nabla\psi)_\Omega + (F\nu, \nabla\psi)_{\partial\Omega},$$

where we have used the antisymmetricity of $\operatorname{curl} u$ which implies that $(\operatorname{curl} u, \nabla(\nabla\psi))_\Omega = 0$. Since $\nabla \cdot f = 0$ in Ω and $\nu \cdot f|_{\partial\Omega} = 0$, we have $(f, \nabla\psi)_\Omega = 0$. On the other hand, since $(F, \nabla(\nabla\psi))_\Omega = 0$ as follows from the antisymmetricity of F, we have $(\nabla \cdot F, \nabla\psi)_\Omega = (F\nu, \nabla\psi)_{\partial\Omega}$, and therefore $((\nu \cdot \nabla)(\nabla \cdot u), \varphi)_{\partial\Omega} = 0$ for any $\varphi \in C_0^\infty(\partial\Omega)$. This implies that $(\nu \cdot \nabla)(\nabla \cdot u)|_{\partial\Omega} = 0$. When Ω is a bounded domain,

$$\int_\Omega \nabla \cdot u \, dx = \int_{\partial\Omega} \nu \cdot u \, d\sigma = 0,$$

which follows from the fact that $\nu \cdot u|_{\partial\Omega} = 0$. Therefore, the (λ, p)-uniqueness of Ω implies that $\nabla \cdot u = 0$ in Ω. This completes the proof of the lemma.

Combining Remark 6.6 and Lemma 6.7, we can complete the proof of Theorems 1.2, 1.4 and 1.5. □

References

[1] S. Agmon, *On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems*, Comm. Pure Appl. Math., **15**, pp. 119–147 (1978).

[2] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions I*, ibid, **12**, pp. 623–727 (1959).

[3] T. Akiyama, *On the existence of L^p solutions of the magnetohydrodynamics equations in a bounded domain*, Nonlinear Analysis, **54** (2003), 1165–1174.

[4] T. Akiyama and Y. Shibata, *On an L_p approach to the stationary and non-stationary problems of the Ginzburg-Landau-Maxwell equations*, to appear in J. Diff. Eqns.

[5] T. G. Cowling, *Magnetohydrodynamics*, Interscience Tracts on Physics and Astronomy, New York, 1957.

[6] G. Duvaut and J. L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, Berlin, New York etc., 1976.

[7] R. Farwig and H. Sohr, *Generalized resolvent estimates for the Stokes system in bounded and unbounded domains*, J. Math. Soc. Japan, **46**, pp. 607–643 (1994).

[8] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, I: Linearized Steady Problems*, Springer Tracts in Natural Philosophy Vol. 38, New York, 1994.

[9] V. Georgescu, *Some boundary value problems for differential forms on compact Riemannian manifolds*, Annali di Matematica, Pura ed. Applicata, serie 4, **122**, pp. 158–198 (1979).

[10] L. P. Gor'kov and G. M. Eliashberg, *Generalization of Ginzburg-Landau equations for non-stationary problems in the case of alloys with paramagnetic impurities*, Soviet Phys. J.E.T.P., **27**, pp. 328–334 (1968).

[11] L. Landau and E. Lifshitz, *Electrodynamique des milieux continus*, Physique théorique, tome VIII, MR, Moscow, 1969.

[12] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non lineaires*, Dunod, Paris, 1969.

[13] J. L. Lions and E. Magenes, *Problemi ai limiti non omogenei (III)*, Ann. Scuola Norm. Sup. Pisa, **15**, pp. 41–103 (1961).

[14] ——— and ———, *Problemi ai limiti non omogenei (V)*, ibid. **16**, pp. 1–44 (1962).

[15] T. Miyakawa, *The L^p approach to the Navier-Stokes equations with the Neumann boundary condition*, Hiroshima Math. J., **10**, pp. 517–537 (1980).

[16] ———, *On nonstationary solutions of the Navier-Stokes equations in an exterior domain*, ibid **112**, pp. 115–140 (1982).

[17] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. **44**, Springer-Verlag, New York, 1983.

[18] A. Schmid, *A time dependent Ginzburg-Landau equation and its application to the problem of resistivity in the mixed state*, Phys. Kondens. Materie **5**, pp. 302–317 (1966).

[19] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. **XXXVI**, pp. 635–664 (1983).

[20] C. G. Simader, *On Dirichlet's boundary value problem*, Lecture Notes in Math., **268**, Springer-Verlag, New York, 1972.

[21] C. G. Simader and H. Sohr, *The Helmholtz decomposition in L^q and related topics*, Mathematical Problems Related to the Navier-Stokes Equation, G. P. Galdi, Ed., Advances in Mathematics for Applied Science, **11**, World Scientific, pp. 1–35 (1992). 1972.

[22] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, 1970.

[23] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, 2nd edition, Johann Ambrosius Barth, Heidelberg, 1995.

[24] W. von Wahl, *Estimating ∇u by* $\operatorname{div} u$ *and* $\operatorname{curl} u$, Math. Meth. Appl. Sci., **15**, pp. 123–143 (1992).

nuna adreso:
T. Akiyama
Department of Mathematical Sciences
Waseda University
Tokyo 169-8555
Japan
E-mail: akiyama@akane.waseda.jp

H. Kasai
Faculty of Education
Fukushima University
Fukushima 960-1296
Japan
E-mail: kasai@educ.fukushima-u.ac.jp

Y. Shibata
Department of Mathematical Sciences
Waseda University
Tokyo 169-8555
Japan
E-mail: yshibata@waseda.jp

M. Tsutsumi
Department of Applied Physics
Waseda University
Tokyo 169-8555
Japan
E-mail: tutumi@tutumi.phys.waseda.ac.jp

(Ricevita la 8-an de majo, 2003)
(Reviziita la 1-an de marto, 2004)