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Algebraic Independence of Painlevé First Transcendents

By

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Abstract. It will be proved that Painlevé first transcendents and their first derivatives are algebraically independent over the rational function field with complex coefficients, by the use of the irreducibility. A particular case indicates that the group of differential automorphisms of the differential field generated by a Painlevé first transcendent over the rational function field is trivial.

Key Words and Phrases. Painlevé first transcendent, Differential automorphism, Irreducibility.

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1. Introduction

Let K be an ordinary differential field of characteristic 0 with the field of constants $\boldsymbol{C}$, the complex number field, containing an element x with $x' = 1$ and U be a universal differential field extension of K. An element $y \in U$ is called a Painlevé first transcendent, abbr. PI, if it satisfies the differential equation over the rational function field $\boldsymbol{C}(x)$

$$y'' = 6y^2 + x.$$

The "irreducibility" property of PI's is well-known (cf. [N], [U]): If y satisfies an algebraic differential equation of the first order over K, then it is algebraic over K. We here use the term "irreducibility" in this meaning. The proof of this is essentially due to investigation of a weight function w of $K[y, y']$ defined by

$$w(y) = 2, \qquad w(y') = 3, \qquad w(a) = 0 \quad (\text{for all } a \in K \backslash \{0\}).$$

together with a K-derivation X of $K[y, y']$ defined by

$$X = y' \frac{\partial}{\partial y} + 6y^2 \frac{\partial}{\partial y'},$$

and a distinguished polynomial $\gamma = y'^2 - 4y^3 \in K[y, y']$. This method applies to the proof of the following.

Lemma. *If y is a PI and transcendental over K then there exists no K-differential automorphism of the differential field extension $K(y, y')$ of K other than the unit element.*

From this results the following.

Theorem 1. *If y_i $(1 \le i \le n)$ are distinct PI's and each of them is transcendental over K, then y_i, y_i' $(1 \le i \le n)$ are algebraically independent over K.*

For example if as K we take a differential field extension of $\boldsymbol{C}(x)$ generated with solutions of linear differential equations over $\boldsymbol{C}(x)$, every PI is transcendental over K (cf. [N], [U]).

As for the relation between Painlevé's first and second equations, the following indicates their independence.

Theorem 2. *Let y be a PI over K and z satisfy the Painlevé's second equation $z'' = 2z^3 + xz + \alpha$ $(\alpha \in \boldsymbol{C})$ over K. Suppose that y and z are transcendental over K and no irreducible polynomial in $K[y, y']$ or $K[z, z']$ divides its derivative. Then y, y', z, z' are algebraically independent over K.*

We here wish to note most equalities were calculated by the mathematical software *Mathematica*.

Section 2 is devoted to describe some properties of the differential algebra associated with a PI over K, which will be used in the proofs of Theorem 1 and 2, and verify the proof of the "irreducibility" of PI's. In sections 3 and 4 the proof of Lemma will be given. Section 5 and 6 will conclude the proofs of Theorems 1 and 2 respectively.

2. Differential algebra $K[y, y']$

Let y be a PI which is transcendental over K. The polynomial algebra with coefficients from $K, K[y, y']$, is also treated as a differential algebra.

The weight function w of $K[y, y']$ over K which we frequently use satisfies the properties.

1) $w(y) = 2,\ w(y') = 3$.
2) $w(u + v) \le \max\{w(u), w(v)\}$, where the equality holds if $w(u) \ne w(v)$.
3) $w(uv) = w(u) + w(v)$.
4) $w(u') \le w(u) + 1$.

For any nonnegative integer n we denote by V_n the K-vector space generated with power products $y^i y'^j$ $(2i + 3j = n)$. Clearly $V_0 = K,\ V_1 = 0$. Each $u \in K[y, y']$ is described as a unique sum of polynomials in V_n $(n \ge 0)$: $u = \sum_{n \ge 0} u_n$, u_n will be called the n-component of u. For nonzero u the $(w(u) + 1)$-component of u' agrees with $Xu_{w(u)}$, where $X = y'(\partial/\partial y) +$

$6y^2(\partial/\partial y') \in \mathrm{Der}(K[y, y']/K)$ (the set of all derivations of $K[y, y']$ to itself over K), which satisfies $X V_n \subset V_{n+1}$.

Proposition 1. *Suppose $n \geq 2$ and $u \in V_n$, $u(1,2) \neq 0$. Then for any $s > 0$ we have*

$$X^s u \in V_{n+s}, X^s u(1,2) = n(n+1)\dots(n+s-1)u(1,2) \neq 0.$$

Proof. For power product $y^i y'^j$, $2i + 3j = n$, $Xu = iy^{i-1}y'^{j+1} + 6jy^{i+2}y'^{j-1}$ and $Xu(1,2) = i2^{j+1} + 6j2^{j-1} = nu(1,2)$. Therefore for $u \in V_n$, $Xu(1,2) = nu(1,2)$. In particular $Xu \in V_{n+1}$. Assume the assertion holds in case of $s - 1$ $(s > 1)$. Then $X^{s-1}u \in V_{n+s-1}$ and $X^{s-1}u(1,2) = n(n+1)\dots(n+s-2)\cdot u(1,2) \neq 0$. Applying this for Xu,

$$X^s u(1,2) = (n+1)\dots(n+s-1)Xu(1,2) = n(n+1)\dots(n+s-1)u(1,2).$$

A polynomial $\gamma = y'^2 - 4y^3$ occupies a specific position in the theory of PI's. Clearly it satisfies $\gamma(1,2) = 0$ and $X\gamma = 0$. If $u \in V_n$ satisfies $u(1,2) = 0$, then γ divides u. For we may describe $u = f\gamma + gy' + h$, for some $f \in K[y, y']$, $g, h \in K[y]$. Since $u \in V_n$, g or h must be 0. Evaluating at $(1,2)$ we have $2g(1) + h(1) = 0$ therefore $g = h = 0$.

Proposition 2. *For $u \in V_m$ the following hold.*

1) *If $Xu = a\gamma^n$ or $ay\gamma^n$ $(a \in K, n \geq 0)$, then $u = b\gamma^r$ $(b \in K, r \geq 0)$ and $a = 0$. In particular $w(u) \equiv 0 \mod 6$.*

2) *If $X^2u = 12yu + a\gamma^n$ or $12yu + ay\gamma^n$ $(a \in K, n \geq 0)$, then $u = cy'\gamma^r$ or $c\gamma^r$ $(c \in K, r \geq 0)$. In particular $w(u) \equiv 0, 3 \mod 6$.*

3) *If $X^2u = ay'\gamma^n$ $(a \in K, n \geq 0)$, then $u = b\gamma^r$ $(b \in K, r \geq 0)$ and $a = 0$. In particular $w(u) \equiv 0 \mod 6$.*

Proof. 1) Assume $u = b\gamma^r$ $(r \geq 0, b(1,2) \neq 0)$ and $b \notin K$. Then, $Xb = a\gamma^{n-r}$. Since if $n > r$, $Xb(1,2) = 0$, which is impossible by Proposition 1, it is derived that $n = r$, $Xb = a \in K$, and hence that $b \in K$, $a = 0$. The proof in case $Xu = ay\gamma^n$ is similar.

2) Assume u has the same form as in 1). In the first case, $X^2b = 12yb + a\gamma^{n-r}$. If $n > r$ or $a = 0$, $X^2b(1,2) = 12b(1,2)$ and hence $w(b)(w(b)+1) = 12$, $w(b) = 3$. This implies $a = 0$. Since we also have $a = 0$ even if $n = r$, $X^2b = 12yb$, $w(b) = 3$. In the second case, $X^2b = 12yb + ay\gamma^{n-r}$. If $n > r$, we have $a = 0$, and the same result. If $n = r$, $X^2b = 12yb + ay$ implies $12b + a = 0$.

3) Let $u = b\gamma^r$ as above. Then $X^2b = ay'\gamma^{n-r}$. If $n > r$ or $a = 0$, $X^2b(1,2) = 0$, therefore $b \in K$. Assume $n = r$ and $a \neq 0$. Then $X^2b = ay'$, $w(b) = 1$, which is absurd.

As an application of Proposition 2 we include the proof of the fact mentioned in the introduction.

Proposition 3. *If a nonzero polynomial $u \in K[y, y']$ divides its derivative u', then $u \in K$.*

Proof. Let $n = w(u)$ and u_n be the n-component of u. Since $w(u'/u) \leq 1$, $c = u'/u \in K$. Looking at the $(n+1)$-components of the both sides of $u' = cu$, we find $Xu_n = 0$. Hence $u_n = a\gamma^s$ $(s \geq 0, a \in K \backslash \{0\})$ and $n = 6s$. Put $v = u - a\gamma^s$. Then

$$v' = cv + (ac - a')\gamma^s - 2saxy'\gamma^{s-1}.$$

Let $m = w(v)$ and v_m the m-component of v. If $m = n - 1 > 4$

$$Xv_m = (ac - a')\gamma^s,$$

hence, m is divided by 6, which is absurd. If $6s - 5 < m < 6s - 1$ then

$$a' = ac, \qquad m = 6s - 4, \qquad Xv_m = -2saxy'\gamma^{s-1}.$$

It follows $v_m = -2saxy\gamma^{s-1}$, from which

$$v' = cv - 2saxy'\gamma^{s-1}, \qquad v'_m = cv_m - 2sa(xy' + y)\gamma^{s-1} - 4s(s-1)ax^2yy'\gamma^{s-2},$$

and hence

$$Xv_{m-1} = 2say\gamma^{s-1}.$$

We have $a = 0$, a contradiction.

3. Differential automorphism

We here suppose that y is a PI and transcendental over K and consider the differential field $K(y, y')$. This and next sections are devoted to the proof of Lemma.

We denote by O_p the local ring at the prime ideal generated by an irreducible polynomial $p \in K[y, y'] \backslash K$ and $M_p = pO_p$ its maximal ideal. O_p is a differential K-algebra. Every nonzero $u \in K(y, y')$ can be written in a unique shape $p^s v$ with s integer and $v \in O_p \backslash pO_p$. Defining $v_p(u) = s$, we have a discrete valuation v_p of $K(y, y')$. Since p' is not divisible by p, $v_p(u') = v_p(u) - 1$, provided $v_p(u) \neq 0$. In fact, if we let $u = p^s v$ $(s \neq 0, v \in O_p \backslash M_p)$, we have $u' = p^{s-1}(sp'v + pv')$ and $sp'v + pv' \in O_p \backslash M_p$.

Now suppose the converse of Lemma, namely, there exists a K-differential automorphism σ distinct from the unit element, and let $z = \sigma y$.

Assume that $z \in K[y, y']$. Examining the weights of the both sides of $z'' = 6z^2 + x$, we have $w(z) \leq 2$, which implies $z = y$, a contradiction. Thus $v_p(z) < 0$ for some irreducible polynomial p. Since $v_p(z'') = v_p(z) - 2$, $v_p(z) = -2$. We may therefore write as $z = fg^{-2}$, where f, g have no common devisor and g has no multiple factor. The substitution implies

$$g(-4f'g' + gf'' - 2fg'' - xg^3) = 6f(f - g'^2).$$

We know that $f - g'^2$ is divisible by g, namely, $f = g'^2 + gh$ for some $h \in K[y, y']$. Calculating $g^3(z'' - 6z^2 - x) = 0$, we obtain

$$g(-xg^2 - 6h^2 - 2g'h' - hg'' + 2g''^2 + gh'' + 2g'g^{(3)}) = 10g'^2(h + g'').$$

It is found that g, g' have no common divisor. In fact, otherwise let p be a common divisor of g, g' and $g = pu$ for $u \in K[y, y']$ with u indivisible by p. Then $g' = p'u + pu'$ and hence $p'u$ is divisible by p, which contradicts that p' is indivisible by p. It follows $h = -g'' + gk$ for some $k \in K[y, y']$, and so

$$z = -t' + k, \qquad t = g'g^{-1}.$$

To investigate k we extend the weight function w to $K(y, y')$ in a usual manner, namely,

$$w(uv^{-1}) = w(u) - w(v), \qquad (u, v \in K[y, y'], v \neq 0).$$

The same properties 1)–4) in section 2 are valid. For z it follows $w(z) \leq 2$ since if $w(z) > 0$ then $w(z) + 2 \geq w(z'') = w(6z^2 + x) = 2w(z)$. Hence $w(k) \leq \max\{w(z), w(t')\} \leq 2$. $k \in K[y, y']$ derives a description $k = ay + b$ $(a, b \in K)$.

Letting newly $t = (\alpha g)'(\alpha g)^{-1}$ with $(\alpha'\alpha^{-1})' = -b$ and adopting $K(\alpha, \alpha')$ as new K, we may assume $b = 0$. This is garanteed because y, y' still remain algebraically independent over $K(\alpha, \alpha')$ (cf. [N], [U]). Now we know

$$z = -t' + ay \qquad (t = g'g^{-1}, a \in K, g \in K[y, y'] \backslash K).$$

Proposition 4. *Suppose* $u, v \in K(y, y')$, $a, b \in K$, $v \neq 0$ *satisfy* $u' + av'v^{-1} + by = 0$. *Then* $a = b = 0$, $u \in \boldsymbol{C}$.

Proof. Let p be an irreducible polynomial in $K[y, y']$ with $v_p(v) \neq 0$. Since $v_p(u') \leq -2$ or $v_p(u') \geq 0$, and $v_p(v'v^{-1}) = -1$, it follows $a = 0$ and hence $u' + by = 0$, which implies $u \in K[y, y']$. If $b \neq 0$, then $w(u) \leq 1$, therefore $u \in K$, a contradiction. Thus $b = 0$ and $u' = 0$. It is known that the latter shows $u \in \boldsymbol{C}$.

By changing the roles of y and z, we have $y = -t_1' + a_1 z$, where $t_1 = g_1' g_1^{-1}$, $g_1 \in K[z, z']$ and $a_1 \in K$. This derives

$$y = -t_1' + a_1(-t' + ay) = -(t_1 + a_1 t)' + a_1' g' g^{-1} + a a_1 y,$$

hence $a_1' = 0$, $aa_1 = 1$, $t_1 + a_1 t \in \boldsymbol{C}$ by Proposition 3. Considering an irreducible factor p of g and coefficients of $p'p^{-1}$ in t and t_1, we see a_1, similarly a, is an integer, hence $a = \pm 1$. We conclude here for the present

$$z = -t' \pm y.$$

4. Two cases

Case $z = -t' + y$.

Let $S(g) = g^2((-t' + y)'' - 6(-t' + y)^2 - x)$. Then

$$S(g) = -3g''^2 + 4g'g^{(3)} - gg^{(4)} - 12g'^2 y + 12gg''y,$$

which must vanish. Note $w(S(\xi)) \le 2w(\xi) + 4$ for $\xi \in K[y, y'], \neq 0$. Define $B(\xi, \eta) = S(\xi + \eta) - S(\xi) - S(\eta)$ $(\xi, \eta \in K[y, y'])$, namely

$$B(\xi, \eta) = -\xi(\eta^{(4)} - 12y\eta'') + 4\xi'\eta^{(3)} - 6\xi''\eta''$$
$$+ 4(\xi^{(3)} - 6y\xi')\eta' + (-\xi^{(4)} + 12y\xi'')\eta.$$

Since $w(X^4 u - 12yX^2 u) = w(u) + 4$ for $u \in V_m$, $m > 0$ by Proposition 2,

$$B(\xi, \eta) = -\xi(\eta^{(4)} - 12y\eta'') + (\text{terms of weight lower than } w(\xi) + w(\eta) + 4)$$

if $w(\xi) \ge w(\xi')$, $w(\eta) > 0$.

Let $n = w(g)$ and g_n be the n-component of g. g_n satisfies the following.

$$-3(X^2 g_n)^2 + 4X(g_n)X^3(g_n) - g_n X^4(g_n) - 12(Xg_n)^2 y + 12g_n X^2(g_n)y = 0.$$

Set $g_n = a\gamma^r$ with $a \in V_m$, $a(1,2) \neq 0$ as usual. Then a satisfies the same equation as g_n. If $m > 0$, evaluating at $(1,2)$, we have $-6m(m-1)a(1,2)^2 = 0$, which is impossible. Thus $a \in K$ and $n = 6r$.

Substitution

$$g = a\gamma^r + h, \qquad h \in K[y, y'], \qquad w(h) < 6r,$$

implies

$$S(a\gamma^r) = (-12a'^2 y + 12aa''y - 3a''^2 + 4a'a^{(3)} - aa^{(4)})\gamma^{2r} + \cdots,$$
$$B(a\gamma^r, h) = a(-h^{(4)} + 12h''y)\gamma^r + \cdots.$$

Let $m = w(h)$ and h_m the m-component of h. We have readily $m \le n - 2$.

Assume $m = n - 2$. Looking at the $(2n + 2)$-component of $S(g)$,

$$X^4 h_m = 12yX^2 h_m - 12(a'^2 - aa'')a^{-1}y\gamma^r.$$

It follows m is 0, 3 modulo 6, which is a contradiction.

By $w(B(a\gamma^r)) = m + n + 4$ we obtain $m < n - 3$.

Let us examine the case where $m = n - 4$. In this case we see $-a'^2 + aa'' = 0$, $c = a'a^{-1} \in \boldsymbol{C}$, and

$$a(-X^4 h + 12X^2 hy)\gamma^r - 24a^2 rxy'^2\gamma^{2r-1} = 0.$$

Suppose $r = 1$ and $m = 2$. We find $h = \alpha y + b$ $(\alpha, b \in K)$. For h_2

$$X^4 h_2 - 12yX^2 h_2 + 24axy'^2 = 0.$$

From $h_2 = \alpha y$ it follows $\alpha = -2ax$. By $g = a\gamma - 2axy + b$,

$$\begin{aligned} S(g) &= 12a(b'' - 2b'c + bc^2)yy'^2 \\ &\quad + a(-10a - b^{(4)} + 4b^{(3)}c + 6b''c^2 + 4b'c^3 - bc^4)y'^2 \\ &\quad + \text{polynomial linear in } y'. \end{aligned}$$

Therefore

$$b'' - 2b'c + bc^2 = 0, \qquad -10a - b^{(4)} + 4b^{(3)}c - 6b''c^2 + 4b'c^3 - bc^4 = 0.$$

The first equality reduces the second one into $-10a = 0$, which is absurd.

Thus $r > 1$, then

$$X^4 h_m = 12yX^2 h_m - 24arxy'^2\gamma^{r-1}.$$

If we put $h_m = e\gamma^s$ $(s \geq 0, e(1,2) \neq 0)$, $X^4 e = 12yX^2 e - 24arxy'^2\gamma^{r-s-1}$. Assume $r - s > 1$. Then $e \in K$ as before, which is absurd. Hence $r - s = 1$ and $X^4 e = 12yX^2 e - 24arxy'^2$, thereby $e = -2arxy$.

By substitution $g = a\gamma^r - 2arxy\gamma^{r-1} + h$, we have

$$S(a\gamma^r - 2arxy\gamma^{r-1}) = 384a^2 r(r-1)(r-2)xy^5(-y'^2 + 2y^3)\gamma^{2r-3} + \cdots.$$

$$B(a\gamma^r - 2arxy\gamma^{r-1}, h) = a(-h^{(4)} + 12yh'')\gamma^r + \cdots.$$

If $r > 3$, $X^4 h_m - 12yX^2 h_m$ is divided by γ, where h_m denotes the $m(= w(h))$-component of h. But this is impossible. If $r = 3$, $X^4 h_m - 12yX^2 h_m = 2304axy^5(-y'^2 + 2y^3)$, which has no solution. Hence $r = 2$.

Then

$$S(a\gamma^2 - 4axy\gamma + h) = a(-h^{(4)} + 12h''y)\gamma^2 + 96a^2x^2y(4y'^2 + 9y^3)\gamma^2 + \cdots,$$

which yields $w(h) = 6$. But there is no solution $u \in V_6$ with $X^4 u - 12yX^2 u = 96ax^2y(4y'^2 + 9y^3)$, which completes the proof in case $z = -t' + y$.

Case $z = -t' - y$.

Let again $S = g^2((-t' - y)'' - 6(-t' - y)^2 - x)$. Then

$$S = -3g''^2 + 4g'g^{(3)} - gg^{(4)} + 12g'^2 y - 12gg''y - 2g^2 x - 12g^2 y^2.$$

Let $n = w(g)$ and g_n be the n-component of g. g_n satisfies the following.

$$-3(X^2 g_n)^2 + 4X(g_n)X^3(g_n) - g_n X^4(g_n) + 12(Xg_n)^2 y$$
$$- 12X(g_n)X^2(g_n)y - 12g_n^2 y^2 = 0.$$

Set $g_n = a\gamma^r$ with $a \in V_m$, $a(1,2) \neq 0$ as usual. We have $-6(m+1)(m+2) \cdot a(1,2)^2 = 0$, this is impossible, completing the proof of Lemma.

5. Proof of Theorem 1

The Theorem for $n = 1$ is clearly valid. Suppose that the Theorem holds for $n - 1$. Let y_i $(1 \leq i \leq n)$ be distinct PI's and transcendental over K. We shall deduce a contradiction under the assumption that y_i, y_i' $(1 \leq i \leq n)$ are algebaically dependent over K. Let L denote the algebraic closure of differential extension field $K(y_1, y_1', \dots, y_{n-2}, y_{n-2}')$ in U and set $y = y_{n-1}$, $z = y_n$ for simplicity. Then, by our assumption, y, y', z, z' depend algebraically over L, which indicates z is algebraic over $L(y, y')$. We shall show $z \in L(y, y')$.

Regarding z as algebraic over $L_1(y)$, where L_1 denotes the algebraic closure of $L(y')$ in U, we have expansions in a local parameter t at $\alpha \in L_1$

$$y = \alpha + t^e, \qquad z = \sum_{i=r}^{\infty} a_i t^i \qquad (a_i \in L_0, a_r \neq 0)$$

with e the ramification exponent. Differentiation of the first yields

$$et^{e-1}t' = y' - \alpha^* - \alpha_{y'}(6y^2 + x) = y' - \alpha^* - \alpha_{y'}(6\alpha^2 + x) - 12\alpha_{y'}t^e - 6\alpha_{y'}t^{2e}.$$

Here "$*$" indicates the extension of the derivation of $L[y']$ defined by $(\sum c_i y'^i)^* = \sum c_i' y'^i$, and $\alpha_{y'} = \partial/\partial y'$ the derivation with respect to y'.

Assume that

$$\beta = y' - \alpha^* - \alpha_{y'}(6\alpha^2 + x) = 0.$$

Let $F \in L[y, y']$ be an irreducible polynomial with $F(\alpha, y') = 0$. Then $F^*(\alpha, y') + \alpha^* F_y(\alpha, y') = 0$ and $F_{y'}(\alpha, y') + \alpha_{y'} F_y(\alpha, y') = 0$. Accordingly

$$y' F_y(\alpha, y') + F^*(\alpha, y') + (6\alpha^2 + x)F_{y'}(\alpha, y') = 0,$$

which implies $F'(\alpha, y') = 0$, and hence, F divides F', noting y, y' are algebraically independent over L. This is absurd, therefore we have $\beta \neq 0$ and so $t' = e^{-1}\beta t^{1-e} + \cdots$. If $r < 0$, then $r = -2e$, $a_r = \beta^2$. Now, let j be the least index indivisible by e with $a_j \neq 0$. Equating the coefficients of t^{j-2e} in z'' and $6z^2 + x$

$$j(j-e)a_je^{-2}\beta^2 = 12a_ra_j = 12\beta^2a_j, \qquad j = -3e \text{ or } 4e.$$

This is absurd, giving $e = 1$. Since for every α, the ramification exponent is 1, $z \in L_1(y)$ follows.

Let L_0 denote the algebraic closure of $L(y)$ in U. The argument similar to the above applies to prove $z \in L_0(y)$. According to $L_1(y) \cap L_0(y') = L(y, y')$, $z \in L(y, y')$.

Changing the roles of y, z in the bove argument, we also have $y \in L(z, z')$, hence $L(y, y') = L(z, z')$. This is a contradiction on account of Lemma, completing the proof of Theorem 1.

6. Proof of Theorem 2

Suppose conversely that y, y', z, z' are algebraically dependent over K. We assume further K is algebraically closed. Then y, y' is algebraically dependent over $K(z, z')$ and z, z' over $K(y, y')$. By the irreducibility y is algebraic over $K(y, y')$, and z over $K(z, z')$. Using the same argument as in the preceding section, we have $K(y, y') = K(z, z')$. As usual we adopt the weight function w of $K(y, y')$ defined by $w(y) = 2$, $w(y') = 3$, $w(a) = 0$ for $a \in K$. Investigating the weight in the equation of PII, we know $w(z) \leq 1$. In fact if $w(z) \geq 2$,

$$w(z) + 2 \leq w(z'') = w(2z^3 + xz + \alpha) = 3w(z),$$

which is absurd. Let $p \in K[y, y']$ be any irreducible devisor of z. Since p' is not divisable by p, it follows $v_p(p') = 0$ and $v_p(f') = v_p(f) - 1$ holds for $f \in K(y, y')$ with $v_p(f) < 0$. Let $v_p(z) < 0$. Then

$$v_p(z) - 2 = v_p(z'') = v_p(2z^3 + xz + \alpha) = 3v_p(z),$$

which derives $v_p(z) = -1$. We can describe as $z = fp^{-1}$ with $f \in K(y, y')$. Putting it into equation of PII,

$$f''p^{-1} - 2f'p'p^{-2} + f(-p''p^{-2} + 2p'^2p^{-3}) = 2f^3p^{-3} + xfp^{-1} + \alpha.$$

From this

$$f''p^2 - 2f''p'p - fp''p + 2fp'^2 = 2f^3 + xfp^2 + \alpha p^3,$$

and hence p divides $2f(f^2 - p'^2)$, namely, $v_p(f - \varepsilon(p)p') > 0$, where $\varepsilon(p)$ indicates ± 1 depending on p. Thus $v_p(z - \varepsilon(p)p'p^{-1}) \geq 0$. On letting $h = \Pi p^{\varepsilon(p)}$, we find

$$v_p(z - h'h^{-1}) \geq 0,$$

and hence $z - h'h^{-1} \in K[y, y']$. Viewing $w(z - h'h^{-1}) \leq \max\{w(z), w(h'h^{-1})\}$

≤ 1, we see $z - h'h^{-1} = a \in K$. The h as an element of $K(z, z')$ satisfies $h' = (z - a)h$, which contradicts, however, the irreducibility of PII.

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