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Well-Posedness for the Boussinesq-Type System Related to the Water Wave

By

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Abstract. This paper studies the initial value problem of Boussinesq-type system which describes the motion of water waves. We show the time local well-posedness in the weighted Sobolev space. This is the generalization of Angulo's work [1] from the view of regularity. Our argument is based on the contraction mapping principle for the integral equations after reducing our problem into the derivative nonlinear Schrödinger system. To overcome the regularity loss in the nonlinearity, we shall apply the smoothing effects of linear Schrödinger group due to Kenig-Ponce-Vega [7]. The gauge transform is also used to remove size restriction on the initial data.

Key Words and Phrases. Time local well-posedness, Derivative nonlinear Schrödinger equations, Smoothing effect, Gauge transform.

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1. Introduction

In this paper, we consider the initial value problem for the Boussinesq-type system:

$$(1.1)\qquad \begin{cases} \partial_t u + \partial_x v + u\partial_x u = 0, & x, t \in \boldsymbol{R}, \\ \partial_t v - \partial_x^3 u + \partial_x u + \partial_x(uv) = 0, & x, t \in \boldsymbol{R}, \\ u(0,x) = u_0(x),\ v(0,x) = v_0(x), & x \in \boldsymbol{R}. \end{cases}$$

Kaup [5] proposed the system (1.1) as a model for the dynamics of the water wave with the surface tension. In the above equations, u and v stand for the horizontal velocity of the fluid and the vertical displacement of the surface from the equilibrium state, respectively. For detail on the physical background, see e.g., Angulo [1] and Kaup [5].

As far as we know, there is only one well-posedness result about (1.1). Angulo [1] obtained the local solution in Sobolev space $H_x^{s,0} \times H_x^{s-1,0}$ $(s > 3/2)$, where

$$H_x^{\sigma,\alpha} = \{f \in \mathscr{S}'(\boldsymbol{R}); \|\langle x\rangle^\alpha \langle D_x\rangle^\sigma f\|_{L_x^2} < \infty\}$$

with $\langle x\rangle^\alpha = (1+x^2)^{\alpha/2}$ and $\langle D_x\rangle^\sigma = \mathscr{F}^{-1}\langle\xi\rangle^\sigma\mathscr{F}$. His idea is based on the energy method with the a priori estimate like

$$\frac{d}{dt}(\|u(t)\|_{H_x^{s,0}}^2 + \|v(t)\|_{H_x^{s-1,0}}^2) \le C\|\partial_x u(t)\|_{L_x^\infty}(\|u(t)\|_{H_x^{s,0}}^2 + \|v(t)\|_{H_x^{s-1,0}}^2).$$

Therefore, one requires $s > 3/2$ at least so that $\|\partial_x u(t)\|_{L_x^\infty}$ is estimated by the Sobolev inequality.

Our concern at present paper is to construct a solution to (1.1) in the function space with less regularity than the Angulo's assumption. More precisely, we show the time local well-posedness in $X^s = (H_x^{s,0} \times H_x^{s-1,0}) \cap (H_x^{s_1,\alpha_1} \times H_x^{s_1-1,\alpha_1})$ with $s > s_1 + \alpha_1 > 1$, $s_1 > 1/2$ and $\alpha_1 > 1/2$, where the well-posedness stands for the existence, uniqueness of the solution and continuous dependence on the initial data. From the view of regularity, this is the generalization of Angulo's work and very close to the desired $H_x^{1,0} \times L_x^2$ well-posedness problem. Our idea is based on the contraction mapping principle of the integral equation after deforming (1.1) into the system of nonlinear Schrödinger equations:

$$i\partial_t \vec{u}^{(2)} + \partial_x^2 \vec{u}^{(2)} + iA(u-\varphi)\partial_x \vec{u}^{(2)} + \vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)}) = \vec{0}, \tag{1.2}$$

where i is the imaginary unit, $\varphi \in C_0^\infty(\boldsymbol{R})$ is independent of time variable, $\vec{u}^{(2)}$ denotes 2×1 matrix whose components belong to $\boldsymbol{C}$, $\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})$ stands for the nonlinear term which includes large order derivatives of φ and $v^{(\ell)}$ (mollification of v) but does not cause the loss of derivative for $\vec{u}^{(2)}$, and the matrix $A(u-\varphi)$ is defined by

$$A(u-\varphi) = \begin{pmatrix} u-\varphi & 0 \\ 0 & \overline{u-\varphi} \end{pmatrix}.$$

For the derivation of (1.2), we refer to section 2. Note that the pair of $\vec{u}^{(2)}$ and $v^{(\ell)}$ is obtained by the invertible transformation of $(u, v)^t$.

To solve (1.2), we first transform it into the integral equation and apply the contraction mapping priciple in the Banach space Y_T which is defined in section 5. Since the nonlinearity contains $\partial_x \vec{u}^{(2)}$, we encounter the difficulty called "the loss of derivative". To overcome it, we make use of the smoothing property of the linear Schrödinger group $U(t) = \exp(it\partial_x^2)$ due to Kenig-Ponce-Vega [7] (This is introduced in section 3). In their work, however, one requires the smallness on the initial data since, in the nonlinear estimate, the inclusion $L_x^1 L_T^\infty \cdot L_x^\infty L_T^2 \subset L_x^1 L_T^2$ appears and the quantity $\|\cdot\|_{L_x^1 L_T^\infty}$ is not expected to be small even when $T \downarrow 0$, where $\|g\|_{L_x^1 L_T^\infty} = \|(\sup_{t \in [0,T]} |g(t,x)|)\|_{L_x^1}$. To remove this smallness condition, we take advantage of the explicit appearance of φ in (1.2). More concretely speaking, when $\|u-\varphi\|_{L_x^1 L_T^\infty}$ arises from the nonlinear estimate by applying the smoothing property of $U(t)$, we take φ sufficiently close to u_0 in $H_x^{s,0} \cap H_x^{s_1,\alpha_1}$ and $T > 0$ sufficiently small. Then, we can make

$\|u-\varphi\|_{L_x^1 L_T^\infty}$ so small that the contraction mapping principle successfully works. We also note that the estimate of $\|u\|_{L_x^1 L_T^\infty}$ gives the regularity and weight conditions on the initial data. Although the nonlinearity $\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})$ likely diverges when φ is close to u_0, it is suitably estimated by fixing φ and then taking $T > 0$ rather small. This is because we can explicitly derive the power of T in the estimate of $\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})$.

Let us state our main theorem. (The notations are explained at the end of this section.)

Theorem 1.1. (i) *Let* $(u_0, v_0) \in (H_x^{s,0} \times H_x^{s-1,0}) \cap (H_x^{s_1,\alpha_1} \times H_x^{s_1-1,\alpha_1}) \equiv X^s$ *with* $s > s_1 + \alpha_1$, $s_1 > 1/2$ *and* $\alpha_1 > 1/2$. *Then, for some* $T > 0$, *there exists a unique solution to* (1.1) *such that* $(u(t), v(t)) \in C([0,T]; X^s)$ *and* $\langle x \rangle^{\alpha_1} u \in L_x^2 L_T^\infty$. *Furthermore, this solution satisfies the smoothing properties:*

$$\|D_x^{s-1/2}\partial_x u\|_{L_x^\infty L_T^2} + \|D_x^{s-1/2} v\|_{L_x^\infty L_T^2} < \infty.$$

(ii) *Let* $(u'(t), v'(t))$ *be a solution to* (1.1) *for the initial data* (u_0', v_0') *with* $\|(u_0', v_0') - (u_0, v_0)\|_{X^s} < \delta$. *If* $\delta > 0$ *is sufficiently small, then there exists some* $T' \in (0, T)$ *such that*

$$\|(u', v') - (u, v)\|_{L_{T'}^\infty X^s} \le C\|(u_0', v_0') - (u_0, v_0)\|_{X^s},$$

$$\|D_x^{s-1/2}\partial_x(u' - u)\|_{L_x^\infty L_{T'}^2} \le C\|(u_0', v_0') - (u_0, v_0)\|_{X^s},$$

$$\|D_x^{s-1/2}(v' - v)\|_{L_x^\infty L_{T'}^2} \le C\|(u_0', v_0') - (u_0, v_0)\|_{X^s}.$$

This paper is organized as follows. In section 2, we discuss the transformation of the system (1.1) into the coupled nonlinear Schrödinger equations. In section 3 and 4, we state the preliminary estimates of linear Schrödinger group $U(t)$ in weighted norm spaces. In section 5, the nonlinear estimates are presented. In section 6–8, we show the existence, uniqueness and Lipschitz continuity of the solution on the initial data.

We close this section by introducing several notations. The quantity $\|\cdot\|_X$ denotes the norm of a Banach space X. $\mathscr{B}(X)$ denotes the bounded linear operators on X. Let $L_x^p L_T^r$ and $L_T^r L_x^p$ be the function spaces $L_x^p(\boldsymbol{R}; L^r(0,T))$ and $L^r(0,T; L_x^p(\boldsymbol{R}))$, respectively. The fractional order derivative D_x^σ stands for $\mathscr{F}^{-1}|\xi|^\sigma \mathscr{F}$. In addition, the modified fractional order derivative $\tilde{D}_x^\sigma$ is defined by $\tilde{D}_x^\sigma = \mathscr{F}^{-1}|\xi|^\sigma(1-\eta(\xi))\mathscr{F}$ with $\eta(\xi) \in C_0^\infty(\boldsymbol{R})$ such that $\eta(\xi) = 1$ if $|\xi| < 1$ and $\eta(\xi) = 0$ if $|\xi| > 2$.

We often use 2×1 vector valued functions like $\vec{f}(t,x) = (f_1(t,x), f_2(t,x))^t$ and we let $\|\vec{f}\|_X = \|f_1\|_X + \|f_2\|_X$. The projection P_j $(j = 1, 2)$ is defined by $P_j \vec{f} = f_j$.

2. Transformation of the system

In this section, we transform the system (1.1) into the nonlinear Schrödinger system. Let us proceed in three steps.

Step 1 *Decomposition in the Fourier space.* Let $\eta(\xi) \in C_0^\infty(\boldsymbol{R})$ with $\eta(\xi) = 1$ if $|\xi| < 1$ and $\eta(\xi) = 0$ if $|\xi| > 2$. In addition, we let $v^{(\ell)} = \mathscr{F}^{-1}\eta(\xi)\mathscr{F}v$ and $v^{(h)} = v - v^{(\ell)}$. Then, from (1.1), it follows that

$$\begin{cases} \partial_t u + \partial_x v^{(h)} + u\partial_x u + \partial_x v^{(\ell)} = 0, \\ \partial_t v^{(h)} + (1 - \mathscr{F}^{-1}\eta\mathscr{F})(-\partial_x^3 u + \partial_x u + \partial_x(uv)) = 0, \\ \partial_t v^{(\ell)} + \mathscr{F}^{-1}\eta\mathscr{F}(-\partial_x^3 u + \partial_x u + \partial_x(uv)) = 0. \end{cases}$$

Let $w = \partial_x^{-1} v^{(h)} (\equiv \int_{-\infty}^x v^{(h)}(y)dy)$. Then, the first two equations in the above system yield

$$\begin{cases} \partial_t u + \partial_x^2 w + u\partial_x u + f = 0, \\ \partial_t w - \partial_x^2 u + u\partial_x w + g = 0, \end{cases} \tag{2.1}$$

where $f = \partial_x v^{(\ell)}$ and $g = u + uv^{(\ell)} + \mathscr{F}^{-1}\eta\mathscr{F}(\partial_x^2 u - u - u(\partial_x w + v^{(\ell)}))$. We observe that f and g do not cause the loss of derivative. Also, since the symbol of $\partial_x^{-1}\mathscr{F}^{-1}(1-\eta)\mathscr{F}$ does not have a singularity, $w \in H_x^{s_1,\alpha_1}$ if $v \in H_x^{s_1,\alpha_1}$. This is why we require the decomposition in Fourier space. Since our aim is the reduction of regularity of the solution, let us mainly consider the transformation of u and w.

Step 2 *Diagonalization.* We next diagonalize the system (2.1). Set

$$\begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} u \\ w \end{pmatrix} \equiv R \begin{pmatrix} u \\ w \end{pmatrix}.$$

Then, (2.1) is transformed into the nonlinear Schrödinger system:

$$\partial_t \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} + \begin{pmatrix} -i\partial_x^2 & 0 \\ 0 & i\partial_x^2 \end{pmatrix} \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} + u\partial_x \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix} + R \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.2}$$

For the simple expression of (2.2), let $\vec{u}^{(1)} = \begin{pmatrix} u^{(1)} \\ \overline{w^{(1)}} \end{pmatrix} \equiv Q \begin{pmatrix} u^{(1)} \\ w^{(1)} \end{pmatrix}$. Then, $\vec{u}^{(1)}$ satisfies

$$\partial_t \vec{u}^{(1)} - i\partial_x^2 \vec{u}^{(1)} + A(u)\partial_x \vec{u}^{(1)} + \vec{f}^{(1)} = \vec{0}, \tag{2.3}$$

where $A(u) = \begin{pmatrix} u & 0 \\ 0 & \bar{u} \end{pmatrix}$, $\vec{f}^{(1)} = QR\begin{pmatrix} f \\ g \end{pmatrix}$ and $u = \frac{1}{\sqrt{2}}(u^{(1)} - iw^{(1)})$.

Step 3 *Gauge Transform.* In this step, we further deform (2.3) by the gauge transformation to make the heavy term $A(u)\partial_x \vec{u}^{(1)}$ small. This kind of transformation sometimes appears in the study of derivative nonlinear Schrödinger equations (see, e.g., Hayashi [3] and Hayashi-Ozawa [4]). In our case, however, the direct application of the known gauge transformation does not work so well, since u in $A(u)$ is not a solution to the nonlinear Schrödinger equation and we can not expect the suitable ellimination of the heavy term.

Let $\varphi(x) \in C_0^\infty(\boldsymbol{R})$ and write (2.3) as

$$\partial_t \vec{u}^{(1)} - i\partial_x^2 \vec{u}^{(1)} + A(\varphi)\partial_x \vec{u}^{(1)} + A(u-\varphi)\partial_x \vec{u}^{(1)} + \vec{f}^{(1)} = \vec{0}. \tag{2.4}$$

To eliminate $A(\varphi)\partial_x \vec{u}^{(1)}$ in (2.4), we make use of the gauge transform defined by

$$\vec{u}^{(2)} = \begin{pmatrix} e^{i\partial_x^{-1}\varphi/2} & 0 \\ 0 & e^{i\partial_x^{-1}\bar{\varphi}/2} \end{pmatrix} \vec{u}^{(1)} \equiv K(\varphi)\vec{u}^{(1)}.$$

Consequently $\vec{u}^{(2)}$ and $v^{(\ell)}$ satisfy

$$\begin{cases} i\partial_t \vec{u}^{(2)} + \partial_x^2 \vec{u}^{(2)} + iA(u-\varphi)\partial_x \vec{u}^{(2)} + \vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)}) = \vec{0}, \\ \partial_t v^{(\ell)} - \partial_x \mathscr{F}^{-1}\eta\mathscr{F}(\partial_x^2 u - u - u(\partial_x w + v^{(\ell)})) = 0. \end{cases} \tag{2.5}$$

where $\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)}) = B(\varphi, u)\vec{u}^{(2)} + iK(\varphi)\vec{f}^{(1)}$ with

$$B(\varphi, u) = \frac{1}{4}\begin{pmatrix} -2i\partial_x\varphi - \varphi^2 + 2\varphi u & 0 \\ 0 & -2i\partial_x\bar{\varphi} - \bar{\varphi}^2 + 2\overline{\varphi u} \end{pmatrix}.$$

We note here that the relation between (u, v) and $(\vec{u}^{(2)}, v^{(\ell)})$ is invertible. In fact, $(u, w) = R^{-1}Q^{-1}K(\varphi)^{-1}\vec{u}^{(2)}$ and hence $(u, v) = (u, \partial_x w + v^{(\ell)}) \in C([0, T]; X^s)$ if and only if both $\vec{u}^{(2)}$ and $v^{(\ell)}$ belong to $C([0, T]; H_x^{s,0} \cap H_x^{s_1,\alpha_1})$. This implies that the solution to (2.5) with $\vec{u}^{(2)}(0, x) = K(\varphi)QR(u_0, \partial_x^{-1}\mathscr{F}^{-1}(1-\eta)\mathscr{F}v_0)^t$ and $v^{(\ell)}(0, x) = \mathscr{F}^{-1}\eta\mathscr{F}v_0$ is immediately transformed into the solution to (1.1). Hereafter, let us mainly seek for the solution to (2.5).

3. Preliminary estimates

In this section, we introduce several key estimates to be frequently used in this paper. In what follows, we use the brief notation GF for $\int_0^t U(t-t') \cdot F(t')dt'$. The smoothing property of $U(t)$ and G plays an important role in recovering the regularity loss of the nonlinearity.

Lemma 3.1. *Let $p \in [2, \infty]$ and $q \in [2, \infty)$. Then, we have*

$$\|D_x^{1/2-1/p} U(t)\phi\|_{L_x^p L_T^2} \le CT^{1/p}\|\phi\|_{L_x^2}, \tag{3.1}$$

$$\|D_x^{1-1/q} GF\|_{L_x^q L_T^2} \le CT^{1/q}\|F\|_{L_x^1 L_T^2}, \tag{3.2}$$

$$\|\partial_x GF\|_{L_x^\infty L_T^2} \le C\|F\|_{L_x^1 L_T^2}, \tag{3.3}$$

$$\|D_x^{1/2} GF\|_{L_T^\infty L_x^2} \le C\|F\|_{L_x^1 L_T^2}. \tag{3.4}$$

Proof of Lemma 3.1. The estimates for the case $p = \infty$ in (3.1), (3.3) and (3.4) are proved in [7; Theorem 2.1, Corollary 2.2 and Theorem 2.3]. The interpolation yields (3.1) and (3.2) (See [2; Proposition 2.3]). □

The following lemma is the well-known Strichartz inequalities.

Lemma 3.2. *Let* $0 \le 2/r_j = 1/2 - 1/q_j \le 1/2$, $(j = 1, 2)$. *Then, we have*

$$\|U(t)\phi\|_{L_T^{r_1} L_x^{q_1}} \le C\|\phi\|_{L_x^2}, \tag{3.5}$$

$$\|GF\|_{L_T^{r_1} L_x^{q_1}} \le C\|F\|_{L_T^{r_2'} L_x^{q_2'}}, \tag{3.6}$$

where $1/r_2 + 1/r_2' = 1/q_2 + 1/q_2' = 1$.

Proof of Lemma 3.2. See, e.g., [12, 14]. □

By interpolating Lemma 3.1 and 3.2, we obtain another type of smoothing property of $U(t)$ and G.

Lemma 3.3. *Let* $p \in [6, \infty)$. *Then, we have*

$$\|D_x^{1/2-3/p} U(t)\phi\|_{L_x^p L_T^{2p/(p-4)}} \le C\|\phi\|_{L_x^2}, \tag{3.7}$$

$$\|D_x^{1-6/p} GF\|_{L_x^p L_T^{2p/(p-4)}} \le CT^{3/2p}\|F\|_{L_x^1 L_T^2}. \tag{3.8}$$

Proof of Lemma 3.3. Let $(i^{-1}\partial_x)^z \equiv \mathscr{F}^{-1}\xi^z\mathscr{F} = \mathscr{F}^{-1}\exp(z \operatorname{Log} \xi)\mathscr{F}$ with $z \in \boldsymbol{C}$ and $\operatorname{Log} \xi = \log|\xi| + i \arg \xi$. Then, by the analogous argument as in [7], we have

$$\|(i^{-1}\partial_x)^{1+i\theta} U(t)\phi\|_{L_x^\infty L_T^2} \le Ce^{\pi|\theta|}\|\phi\|_{L_x^2}, \tag{3.9}$$

$$\|(i^{-1}\partial_x)^{1+i\theta} GF\|_{L_x^\infty L_T^2} \le Ce^{\pi|\theta|/2}\|F\|_{L_x^1 L_T^2} \qquad \text{for } \theta \in \boldsymbol{R}. \tag{3.10}$$

On the other hand, by Lemma 3.2 with $p_1 = r_1 = 6$, $p_2' = 1$ and $r_2' = 4/3$, we see that

$$\|(i^{-1}\partial_x)^{i\theta} U(t)\phi\|_{L_x^6 L_T^6} \le Ce^{\pi|\theta|}\|\phi\|_{L_x^2}, \tag{3.11}$$

$$\begin{aligned} \|(i^{-1}\partial_x)^{i\theta} GF\|_{L_x^6 L_T^6} &\le Ce^{\pi|\theta|}\|GF\|_{L_T^6 L_x^6} \\ &\le Ce^{\pi|\theta|}\|F\|_{L_T^{4/3} L_x^1} \\ &\le Ce^{\pi|\theta|}T^{1/4}\|F\|_{L_x^1 L_T^2}. \end{aligned} \tag{3.12}$$

Note that, to show the first inequality of (3.12), we used the Fourier multiplier theorem for $(i^{-1}\partial_x)^{i\theta}$. Applying Stein's interpolation [11] to the combination of (3.9) with (3.11), and (3.10) with (3.12), we obtain Lemma 3.3. □

Let us call $\|f(\cdot,x)\|_{L_T^\infty}$ "the maximal function of $f(t,x)$". We next give the estimates for the maximal function.

Lemma 3.4. *Let $\sigma > 1/2$ and $T \in (0,1]$. Then, we have*

$$\|U(t)\phi\|_{L_x^2 L_T^\infty} \le C\|\phi\|_{H_x^{\sigma,0}}, \tag{3.13}$$

$$\|GF\|_{L_x^2 L_T^\infty} \le CT^{1/2}\|F\|_{L_T^2 H_x^{\sigma,0}}. \tag{3.14}$$

Proof of Lemma 3.4. For the estimate (3.13), see [7; Theorem 3.1]. By (3.13), we can show (3.14) since

$$\begin{aligned}\|GF\|_{L_x^2 L_T^\infty} &\le \int_0^T \|U(t)U(-t')F(t')\|_{L_x^2 L_T^\infty} dt' \\ &\le C\|F\|_{L_T^1 H_x^{\sigma,0}} \\ &\le CT^{1/2}\|F\|_{L_T^2 H_x^{\sigma,0}}. \quad \square\end{aligned}$$

For the estimate of $U(t)$ in weighted norm spaces, we often require the commutator estimate of $\tilde{D}_x^\sigma \equiv \mathscr{F}^{-1}|\xi|^\sigma(1-\eta)\mathscr{F}$ and $\langle x\rangle^\alpha$. This is described as

Lemma 3.5. *Let $\sigma, \alpha \in [0,1)$. Then, $[\tilde{D}_x^\sigma, \langle x\rangle^\alpha]$ belongs to $\mathscr{B}(L_x^p)$ and $\mathscr{B}(L_x^p L_T^r)$ for $p, r \in [1,\infty]$.*

Proof of Lemma 3.5. The commutator $[\tilde{D}_x^\sigma, \langle x\rangle^\alpha]$ has the integral kernel like

$$K(x,y) = (2\pi)^{-1} Os - \int e^{i(x-y)\xi}|\xi|^\sigma(1-\eta(\xi))d\xi(\langle x\rangle^\alpha - \langle y\rangle^\alpha),$$

where $Os - \int$ stands for the oscillatory integral. Since $|\langle x\rangle^\alpha - \langle y\rangle^\alpha| \le C|x-y|$, we see that

$$|K(x,y)| \le \begin{cases} C|x-y|^{-\sigma} & \text{if } |x-y| \le 1, \\ C_N|x-y|^{-N} & \text{if } |x-y| > 1. \end{cases}$$

Hence, Young's inequality yields Lemma 3.5. $\square$

We next show the commutator estimate of D_x^σ and the gauge transform.

Lemma 3.6. *Let $p \in [1,\infty]$, $\sigma \in (0,1)$ and $\varphi \in C_0^\infty(\boldsymbol{R})$. Then, we have*

$$\|D_x^\sigma e^{\partial_x^{-1}\varphi} f\|_{L_x^p} \le Ce^{\|\varphi\|_{L_x^1}}(1+\|\varphi\|_{L_x^\infty})(\|f\|_{L_x^p} + \|D_x^\sigma f\|_{L_x^p}), \tag{3.15}$$

where $\partial_x^{-1}\varphi = \int_{-\infty}^x \varphi(y)dy$.

Proof of Lemma 3.6. We write $D_x^{\sigma} e^{\partial_x^{-1}\varphi} f$ as

$$D_x^{\sigma} e^{\partial_x^{-1}\varphi} f = e^{\partial_x^{-1}\varphi} D_x^{\sigma} f + [D_x^{\sigma}, e^{\partial_x^{-1}\varphi}] f \equiv f_1 + f_2.$$

It is easy to see that $\|f_1\|_{L_x^p} \le e^{\|\varphi\|_{L_x^1}} \|D_x^{\sigma} f\|_{L_x^p}$.

To estimate f_2, we note that $[D_x^{\sigma}, e^{\partial_x^{-1}\varphi}]$ possesses the integral kernel like

$$K(x,y) = (2\pi)^{-1} Os - \int e^{i\xi} |\xi|^{\sigma} d\xi (x-y)^{-1} |x-y|^{-\sigma}$$
$$\times \left\{ \exp\left(\int_{-\infty}^{y} \varphi(z) dz \right) - \exp\left(\int_{-\infty}^{x} \varphi(z) dz \right) \right\}.$$

This yields

$$|K(x,y)| \le C e^{\|\varphi\|_{L_x^1}} \begin{cases} |x-y|^{-\sigma} \|\varphi\|_{L_x^{\infty}} & \text{if } |x-y| \le 1, \\ |x-y|^{-1-\sigma} & \text{if } |x-y| \le 1. \end{cases}$$

Applying Young's inequality, we have $\|f_2\|_{L_x^p} \le C e^{\|\varphi\|_{L_x^1}} (1 + \|\varphi\|_{L_x^{\infty}}) \|f\|_{L_x^p}$. Hence, we obtain the desired result. □

When we apply the fractional order derivative to the nonlinear term, we require Leibniz' type rule described in the following.

Lemma 3.7. *Let* $\sigma \in (0,1)$, $\sigma_1, \sigma_2 \in [0, \sigma]$ *with* $\sigma = \sigma_1 + \sigma_2$. *Also, let* $p, r \in [1, \infty)$ *and* $p_1, p_2, r_1, r_2 \in (1, \infty)$ *with* $1/p = 1/p_1 + 1/p_2$ *and* $1/r = 1/r_1 + 1/r_2$. *Then, we have*

$$\|D_x^{\sigma}(fg) - (D_x^{\sigma} f) g - f (D_x^{\sigma} g)\|_{L_x^p L_T^r} \le C \|D_x^{\sigma_1} f\|_{L_x^{p_1} L_T^{r_1}} \|D_x^{\sigma_2} g\|_{L_x^{p_2} L_T^{r_2}}. \tag{3.16}$$

Proof of Lemma 3.7. For the proof of this Lemma, see [8; Appendix]. □

4. Estimates in weighted norm spaces

We derive the Strichartz type estimate and the estimate of maximal function in the weighted norm spaces. It suffices to consider the case $T \in [0,1]$.

Lemma 4.1. *Let* $\sigma \in [0,1)$, $\alpha \in [1/2, 1)$, $\sigma' > \sigma + \alpha$ *and* $0 \le 2/r = 1/2 - 1/p \le 1/2$. *Then, we have*

$$\|D_x^{\sigma} \langle x \rangle^{\alpha} U(t) \phi\|_{L_T^r L_x^p} \le C T^{1/2} (\|\phi\|_{H_x^{\sigma',0}} + \|\phi\|_{H_x^{\sigma,\alpha}}), \tag{4.1}$$

$$\|D_x^{\sigma} \langle x \rangle^{\alpha} G F\|_{L_T^r L_x^p} \le C T^{1/2} (\|D_x^{\sigma'-1/2} F\|_{L_x^1 L_T^2} + \|F\|_{L_T^2 H_x^{\sigma,\alpha}}), \tag{4.2}$$

$$\|D_x^{\sigma} \langle x \rangle^{\alpha} G F\|_{L_T^r L_x^p} \le C T^{1/2} (\|F\|_{L_T^{\infty} H_x^{\sigma',0}} + \|F\|_{L_T^{\infty} H_x^{\sigma,\alpha}}). \tag{4.3}$$

For the case $\alpha < 1/2$, we can show the following.

Lemma 4.2. *Let $\sigma \in [0,1)$, $\alpha \in [0,1/2)$ and $0 \le 2/r = 1/2 - 1/p \le 1/2$. Then, we have*

$$\|D_x^\sigma \langle x\rangle^\alpha U(t)\phi\|_{L_T^r L_x^p} \le CT^{1/2}(\|\phi\|_{H_x^{\sigma+1/2,0}} + \|\phi\|_{H_x^{\sigma,\alpha}}), \tag{4.4}$$

$$\|D_x^\sigma \langle x\rangle^\alpha GF\|_{L_T^r L_x^p} \le CT^{1/2}(\|D_x^\sigma F\|_{L_x^1 L_T^2} + \|F\|_{L_T^2 H_x^{\sigma,\alpha}}), \tag{4.5}$$

$$\|D_x^\sigma \langle x\rangle^\alpha GF\|_{L_T^r L_x^p} \le CT^{1/2}(\|F\|_{L_T^\infty H_x^{\sigma+1/2,0}} + \|F\|_{L_T^\infty H_x^{\sigma,\alpha}}). \tag{4.6}$$

Proof of Lemma 4.1. Let $f(t,x) = U(t)\phi(x)$. Then f satisfies

$$\begin{cases} i\partial_t f = -\partial_x^2 f, \\ f(0,x) = \phi(x). \end{cases} \tag{4.7}$$

Multiplying $\langle x\rangle^\alpha$ on both hand sides of (4.7), we have

$$i\partial_t(\langle x\rangle^\alpha f) = -\partial_x^2(\langle x\rangle^\alpha f) + 2(\partial_x\langle x\rangle^\alpha)\partial_x f + (\partial_x^2\langle x\rangle^\alpha f).$$

Applying D_x^σ and rewriting the above relation by Duhamel's principle, we see that

$$\begin{aligned} D_x^\sigma\langle x\rangle^\alpha U(t)\phi &= U(t)D_x^\sigma\langle x\rangle^\alpha\phi - 2iGD_x^\sigma(\partial_x\langle x\rangle^\alpha\partial_x f) - iGD_x^\sigma(\partial_x^2\langle x\rangle^\alpha f) \\ &\equiv f_1(t,x) + f_2(t,x) + f_3(t,x). \end{aligned} \tag{4.8}$$

By making use of Lemma 3.2, f_1 and f_3 are easily estimated as

$$\begin{aligned} \|f_1\|_{L_T^r L_x^p} &\le C\|\phi\|_{H_x^{\sigma,\alpha}}, \\ \|f_3\|_{L_T^r L_x^p} &\le C\|D_x^\sigma(\partial_x^2\langle x\rangle^\alpha)U(t)\phi\|_{L_T^1 L_x^2} \\ &\le CT\|\phi\|_{H_x^{\sigma,0}} \le CT\|\phi\|_{H_x^{\sigma,\alpha}}. \end{aligned}$$

As for the estimate of f_2, we apply Lemma 3.2 and we have

$$\begin{aligned} \|f_2\|_{L_T^r L_x^p} &\le C\|D_x^\sigma(\partial_x\langle x\rangle^\alpha)\partial_x U(t)\phi\|_{L_T^1 L_x^2} \\ &\le CT^{1/2}(\|(\partial_x\langle x\rangle^\alpha)D_x^\sigma\partial_x U(t)\phi\|_{L_x^2 L_T^2} \\ &\qquad + \|[D_x^\sigma, \partial_x\langle x\rangle^\alpha]\partial_x U(t)\phi\|_{L_T^2 L_x^2}) \\ &\le CT^{1/2}(\|D_x^\sigma\partial_x U(t)\phi\|_{L_x^q L_T^2} + T^{1/2}\|\phi\|_{H_x^{\sigma,0}}), \end{aligned} \tag{4.9}$$

where $1/q \in (\alpha - 1/2, 1/2)$. Note that, in the above estimate, the commutator $[D_x^\sigma, \partial_x\langle x\rangle^\alpha]$ is the $\sigma - 1$th order pseudo-differential operator. Using Lemma 3.1, we see that

$$\|f_2\|_{L_T^r L_x^p} \le CT^{1/2}\|\phi\|_{H_x^{\sigma+1/q+1/2,0}}.$$

As a result, we obtain (4.1).

To prove (4.2), we first follow the analogous derivation of (4.8). Then, we have

$$\begin{aligned} D_x^\sigma\langle x\rangle^\alpha GF &= G(D_x^\sigma\langle x\rangle^\alpha F) - 2iG(D_x^\sigma(\partial_x\langle x\rangle^\sigma)\partial_x GF) \\ &\quad - iG(D_x^\sigma(\partial_x^2\langle x\rangle^\alpha)GF) \\ &\equiv g_1(t,x) + g_2(t,x) + g_3(t,x). \end{aligned}$$

According to Lemma 3.2, we have

$$\begin{aligned} \|g_1\|_{L_T^r L_x^p} &\le C\|D_x^\sigma\langle x\rangle^\alpha F\|_{L_T^1 L_x^2} \\ &\le CT^{1/2}\|F\|_{L_T^2 H_x^{\sigma,\alpha}}, \\ \|g_3\|_{L_T^r L_x^p} &\le CT\|D_x^\sigma(\partial_x^2\langle x\rangle^\alpha)GF\|_{L_T^\infty L_x^2} \\ &\le CT^{3/2}\|F\|_{L_T^2 H_x^{\sigma,0}} \\ &\le CT^{3/2}\|F\|_{L_T^2 H_x^{\sigma,\alpha}}. \end{aligned}$$

It remains to estimate g_2. Lemma 3.2 gives

$$\begin{aligned} (4.10)\quad \|g_2\|_{L_T^r L_x^p} &\le CT^{1/2}(\|(\partial_x\langle x\rangle^\alpha)D_x^\sigma\partial_x GF\|_{L_x^2 L_T^2} + \|[D_x^\sigma, \partial_x\langle x\rangle^\alpha]\partial_x GF\|_{L_T^2 L_x^2}) \\ &\le CT^{1/2}\|D_x^\sigma\partial_x GF\|_{L_x^q L_T^2} + CT\|GF\|_{L_T^\infty H_x^{\sigma,0}}, \end{aligned}$$

where $1/q \in (\alpha - 1/2, 1/2)$. Applying Lemma 3.1 to the first term and Lemma 3.2 to the second, we have

$$\|g_2\|_{L_T^r L_x^p} \le CT^{1/2}(\|D_x^{\sigma+1/q}F\|_{L_x^1 L_T^2} + \|F\|_{L_T^2 H_x^{\sigma,\alpha}}).$$

Hence, we obtain (4.2). The estimate (4.3) follows by using

$$\begin{aligned} \|D_x^\sigma\partial_x GF\|_{L_x^q L_T^2} &\le \int_0^T \|D_x^\sigma\partial_x U(t)U(-t')F(t')\|_{L_x^q L_T^2}dt' \\ &\le CT\|F\|_{L_T^\infty H_x^{\sigma+1/q+1/2,0}} \end{aligned}$$

in (4.10). □

Proof of Lemma 4.2. This is almost similar to the proof of Lemma 4.1 except for using the following estimates in (4.9) and (4.10):

$$\|f_2\|_{L_T^r L_x^p} \le CT^{1/2}(\|D_x^\sigma \partial_x U(t)\phi\|_{L_x^\infty L_T^2} + \|\phi\|_{H_x^{\sigma,0}})$$

$$\le CT^{1/2}\|\phi\|_{H^{\sigma+1/2,0}},$$

and

$$\|g_2\|_{L_T^r L_x^p} \le CT^{1/2}(\|D_x^\sigma \partial_x GF\|_{L_x^\infty L_T^2} + \|GF\|_{L_T^\infty H_x^{\sigma,0}})$$

$$\le CT^{1/2}(\|D_x^\sigma F\|_{L_x^1 L_T^2} + \|F\|_{L_T^2 H_x^{\sigma,\alpha}}).$$

Hence, we obtain Lemma 4.2. □

We next prove the estimates of maximal function in the weighted norm space. These estimates give the regularity constraint on the initial data.

Lemma 4.3. *Let* $\sigma \in [0,1)$, $\alpha \in [1/2, 1)$, $\sigma' > \sigma + \alpha$ *and* $\sigma'' > \sigma + 1/2$. *Then, we have*

$$\|D_x^\sigma \langle x\rangle^\alpha U(t)\phi\|_{L_x^2 L_T^\infty} \le C(\|\phi\|_{H_x^{\sigma'+1/2,0}} + \|\phi\|_{H_x^{\sigma'',\alpha}}), \tag{4.11}$$

$$\|D_x^\sigma \langle x\rangle^\alpha GF\|_{L_x^2 L_T^\infty} \le CT^{1/2}(\|D_x^{\sigma'} F\|_{L_x^1 L_T^2} + \|F\|_{L_T^2 H_x^{\sigma'',\alpha}}), \tag{4.12}$$

$$\|D_x^\sigma \langle x\rangle^\alpha GF\|_{L_x^2 L_T^\infty} \le CT^{1/2}(\|F\|_{L_T^\infty H_x^{\sigma'+1/2,0}} + \|F\|_{L_T^\infty H_x^{\sigma'',\alpha}}). \tag{4.13}$$

Proof of Lemma 4.3. We only prove the estimate (4.11) since the other estimates in Lemma 4.3 follows similarly to the proof of Lemma 4.1. According to the derivation of (4.8), we see that

$$\|D_x^\sigma \langle x\rangle^\alpha U(t)\phi\|_{L_x^2 L_T^\infty} \le \|U(t) D_x^\sigma \langle x\rangle^\alpha \phi\|_{L_x^2 L_T^\infty} + 2\|GD_x^\sigma(\partial_x\langle x\rangle^\alpha)\partial_x U(t')\phi\|_{L_x^2 L_T^\infty}$$

$$+ \|GD_x^\sigma(\partial_x^2\langle x\rangle^\alpha)U(t')\phi\|_{L_x^2 L_T^\infty}$$

$$\equiv I_1 + I_2 + I_3.$$

By Lemma 3.4, it follows that

$$I_1 \le C\|\phi\|_{H_x^{\sigma'',\alpha}},$$

$$I_3 \le C\|(\partial_x^2\langle x\rangle^\alpha)U(t)\phi\|_{L_T^\infty H_x^{\sigma'',0}}$$

$$\le C\|\phi\|_{H_x^{\sigma'',\alpha}}.$$

On the other hand, applying Lemma 3.1, 3.4 and the fact that $[\langle D_x\rangle^{\sigma''}, \partial_x\langle x\rangle^\alpha]$ is the $\sigma'' - 1$th order pseudo-differential operator, we see that

$$\begin{aligned} I_2 &\le C\|\langle D_x\rangle^{\sigma+1/2+\varepsilon}(\partial_x\langle x\rangle^{\alpha})\partial_x U(t)\phi\|_{L_T^2L_x^2} \\ &\le C(\|(\partial_x\langle x\rangle^{\alpha})D_x^{\sigma+1/2+\varepsilon}\partial_x U(t)\phi\|_{L_x^2L_T^2} + \|\phi\|_{H_x^{\sigma',0}}) \\ &\le C(\|D_x^{\sigma+3/2+\varepsilon}U(t)\phi\|_{L_x^qL_T^2} + \|\phi\|_{H_x^{\sigma',0}}) \\ &\le C\|\phi\|_{H_x^{\sigma+1+\varepsilon+1/q,0}}, \end{aligned}$$

where $1/q > \alpha - 1/2$. Hence, we obtain (4.11). □

5. Estimates of nonlinearity

When we show the contraction mapping principle, we will require the estimates of $\|GA(u-\varphi)\partial_x\vec{u}^{(2)}\|_{L_T^\infty(H_x^{s,0}\cap H_x^{s_1,\alpha_1})}$ and $\|G\vec{f}^{(2)}\|_{L_T^\infty(H_x^{s,0}\cap H_x^{s_1,\alpha_1})}$. In this section, we derive these nonlinear estimates. Since so many kinds of norms appear in our argument, we first define new notations for the simple description.

Definition 5.1. Let $|||g|||_{Y_T} = |||g|||_{\text{initial}} + |||g|||_{\text{smooth}} + |||g|||_{\text{lower}} + |||g|||_{\text{maxim}}$, where

$$|||g|||_{\text{initial}} \equiv \|g\|_{L_T^\infty H_x^{s,0}} + \|g\|_{L_T^\infty H_x^{s_1,\alpha_1}},$$

$$\begin{aligned} |||g|||_{\text{smooth}} \equiv{}& \|D_x^{s-1/2}\partial_x g\|_{L_x^\infty L_T^2} \\ &+ \sup_{\sigma_0\le\sigma\le 1}\|D_x^{s-1/2-\sigma}\partial_x g\|_{L_x^{6/\sigma}L_T^{6/(3-2\sigma)}} \qquad \text{with } \sigma_0>0 \text{ small}, \end{aligned}$$

$$|||g|||_{\text{lower}} \equiv \|D_x^{s-1/2}\langle x\rangle^{1/2-\nu}g\|_{L_T^4L_x^\infty} + \|D_x^{s_1}\langle x\rangle^{\alpha_1}g\|_{L_T^4L_x^\infty} \qquad \text{with } \nu = \frac{s-s_1-\alpha_1}{2(s-s_1)},$$

$$|||g|||_{\text{maxim}} \equiv \sup_{0\le\sigma\le\sigma_0}\|D_x^{\sigma}\langle x\rangle^{\alpha_1}g\|_{L_x^2L_T^\infty}.$$

The suffix of $|||\cdot|||_{\text{initial}}$ suggests that this norm is for the functions whose target space coincides with that of the initial data. The norm $|||\cdot|||_{\text{smooth}}$ causes the smoothing effects as in Lemma 3.1 and 3.3. $|||\cdot|||_{\text{lower}}$ is used for the lower order derivatives and $|||\cdot|||_{\text{maxim}}$ is for the maximal functions. For the nonlinear estimates, it suffices to see the following lemma.

Lemma 5.2. *There exists some $\beta > 0$ such that*

$$\begin{aligned} \|D_x^{s-1/2}(f\partial_x g)\|_{L_x^1L_T^2} \le{}& C\|\langle x\rangle^{\alpha_1}f\|_{L_x^2L_T^\infty}\|D_x^{s-1/2}\partial_x g\|_{L_x^\infty L_T^2} \\ &+ CT^\beta|||f|||_{Y_T}|||g|||_{Y_T}, \end{aligned} \tag{5.1}$$

$$\|D_x^{s-1/2}(\langle x\rangle^{1/2-\nu}f\partial_x g)\|_{L_x^2L_T^2} \le CT^\beta|||f|||_{Y_T}|||g|||_{Y_T}, \tag{5.2}$$

$$\|D_x^{s_1}(\langle x\rangle^{\alpha_1}f\partial_x g)\|_{L_x^2L_T^2} \le CT^\beta|||f|||_{Y_T}|||g|||_{Y_T}. \tag{5.3}$$

When we apply the third inequalities in Lemmas 4.1–4.3 to $G\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})$, the estimate of $\|\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})\|_{\mathrm{initial}}$ will appear. This quantity is estimated as follows.

Lemma 5.3. *We have*

$$(5.4)\quad \|\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})\|_{\mathrm{initial}} \le C_\varphi(1 + \|\vec{u}^{(2)}\|_{\mathrm{initial}} + \|v^{(\ell)}\|_{\mathrm{initial}})(\|\vec{u}^{(2)}\|_{\mathrm{initial}} + \|\langle D_x\rangle v^{(\ell)}\|_{\mathrm{initial}}),$$

$$(5.5)\quad \|\vec{f}^{(2)}(\varphi, \vec{u}_1^{(2)}, v_1^{(\ell)}) - \vec{f}^{(2)}(\varphi, \vec{u}_2^{(2)}, v_2^{(\ell)})\|_{\mathrm{initial}} \le C_\varphi\left(1 + \max_{j=1,2}(\|\vec{u}_j^{(2)}\|_{\mathrm{initial}} + \|v_{\ell,j}\|_{\mathrm{initial}})\right) \times (\|\vec{u}_1^{(2)} - \vec{u}_2^{(2)}\|_{\mathrm{initial}} + \|\langle D_x\rangle(v_1^{(\ell)} - v_2^{(\ell)})\|_{\mathrm{initial}}),$$

where $C_\varphi > 0$ is a constant which possibly diverges as $\varphi \to u_0$ in $H_x^{s,0} \cap H_x^{s_1,\alpha_1}$.

Proof of Lemma 5.2. We only prove (5.2) since the other inequalities likewise follow. Applying Leibniz's rule (Lemma 3.7), we see that

$$(5.6)\quad \|D_x^{s-1/2}(f\partial_x g)\|_{L_x^1L_T^2} \le \|fD_x^{s-1/2}\partial_x g\|_{L_x^1L_T^2} + \|(D_x^{s-1/2}f)\partial_x g\|_{L_x^1L_T^2} + C\|D_x^\sigma f\|_{L_x^{6/(6-\sigma)}L_T^{3/\sigma}}\|D_x^{s-1/2-\sigma}\partial_x g\|_{L_x^{6/\sigma}L_T^{6/(3-2\sigma)}}.$$

Note that the smoothing estimates (Lemma 3.3) is applicable to the last norm on the right hand side of (5.6). By the simple application of Hölder's inequality, the first term on the right hand side of (5.6) is estimated as

$$(5.7)\quad \|fD_x^{s-1/2}\partial_x g\|_{L_x^1L_T^2} \le \|f\|_{L_x^1L_T^\infty}\|D_x^{s-1/2}\partial_x g\|_{L_x^\infty L_T^2} \le C\|\langle x\rangle^{\alpha_1}f\|_{L_x^2L_T^\infty}\|D_x^{s-1/2}\partial_x g\|_{L_x^\infty L_T^2}.$$

We next estimate the second term of (5.6). Hölder's inequality and $D_x^{s-1/2} - \tilde{D}_x^{s-1/2} \in \mathscr{B}(L_x^pL_T^r)$ yield

$$\begin{aligned}(5.8)\quad &\|(D_x^{s-1/2}f)\partial_x g\|_{L_x^1L_T^2}\\ &\le \|(\tilde{D}_x^{s-1/2}f)\partial_x g\|_{L_x^1L_T^2} + \|((D_x^{s-1/2} - \tilde{D}_x^{s-1/2})f)\partial_x g\|_{L_x^1L_T^2}\\ &\le C\|(\langle x\rangle^{1/2-\nu}\tilde{D}_x^{s-1/2}f)(\langle x\rangle^{\nu'}\partial_x g)\|_{L_x^2L_T^2} + CT^{1/2}\|f\|_{L_x^2L_T^\infty}\|\partial_x g\|_{L_T^\infty L_x^2}\\ &\le CT^{1/4}\|\langle x\rangle^{1/2-\nu}\tilde{D}_x^{s-1/2}f\|_{L_T^4L_x^\infty}\|g\|_{L_T^\infty H_x^{1,\nu'}} + CT^{1/2}\|f\|_{\mathrm{maxim}}\|g\|_{\mathrm{initial}}\\ &\equiv CT^{1/4}I_1 \times I_2 + CT^{1/2}\|f\|_{\mathrm{maxim}}\|g\|_{\mathrm{initial}},\end{aligned}$$

where $0 < \nu < \nu'$. By Lemma 3.5, we have

$$(5.9) \qquad I_1 \le \|\tilde{D}_x^{s-1/2}\langle x\rangle^{1/2-\nu} f\|_{L_T^4 L_x^\infty} + \|[\langle x\rangle^{1/2-\nu}, \tilde{D}_x^{s-1/2}] f\|_{L_T^4 L_x^\infty}$$
$$\le \|\!|f|\!\|_{\text{lower}} + CT^{1/4}\|\!|f|\!\|_{\text{initial}}.$$

Also, by applying the interpolation $H_x^{s,0} \cap H_x^{s_1,\alpha_1} \subset H^{1,\nu'}$ with $1 = \theta s + (1-\theta)s_1$ and $\nu' = (1-\theta)\alpha_1$ for some $\theta \in [0,1]$, we see that

$$(5.10) \qquad I_2 \le C(\|g\|_{L_T^\infty H_x^{s,0}} + \|g\|_{L_T^\infty H_x^{s_1,\alpha_1}})$$
$$\le C\|\!|g|\!\|_{\text{initial}}.$$

Applying Lemma 3.5 to the third norm of (5.6), we have

$$(5.11) \qquad \|D_x^\sigma f\|_{L_x^{6/(6-\sigma)} L_T^{3/\sigma}} \le T^{\sigma/3}(\|\tilde{D}_x^\sigma f\|_{L_x^{6/(6-\sigma)} L_T^\infty} + \|f\|_{L_x^{6/(6-\sigma)} L_T^\infty})$$
$$\le CT^{\sigma/3}(\|\langle x\rangle^{\alpha_1}\tilde{D}_x^\sigma f\|_{L_x^2 L_T^\infty} + \|\!|f|\!\|_{\text{maxim}})$$
$$\le CT^{\sigma/3}\|\!|f|\!\|_{\text{maxim}}.$$

Hence, by (5.6)–(5.11), we obtain Lemma 5.2. □

Proof of Lemma 5.3. The proof easily follows from the definition of $\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})$ (see section 2) and Lemma 3.6. □

6. Contraction mapping principle (proof of Theorem 1.1—existence)

We consider the integral equations:

$$(6.1) \qquad \begin{cases} \vec{u}^{(2)} = \Phi(\vec{u}^{(2)}, v^{(\ell)}) \\ \qquad \equiv U(t)\vec{u}_0^{(2)} - G\{A(u-\varphi)\partial_x \vec{u}^{(2)} - i\vec{f}^{(2)}(\varphi, \vec{u}^{(2)})\}, \\ v^{(\ell)} = \Psi(\vec{u}^{(2)}, v^{(\ell)}) \\ \qquad \equiv v_{0,\ell} + \int_0^t \partial_x \mathscr{F}^{-1}\eta\mathscr{F}(\partial_x^2 u - u - u(\partial_x w + v^{(\ell)})), \end{cases}$$

where the initial data is given by $\vec{u}_0^{(2)} = K(\varphi)QR(u_0, \partial_x^{-1}\mathscr{F}^{-1}(1-\eta)\mathscr{F}v_0)^t$, $v_{0,\ell} = \mathscr{F}^{-1}\eta\mathscr{F}v_0$ and $(u,w) = R^{-1}Q^{-1}K(\varphi)^{-1}\vec{u}^{(2)}$. We show that the map (Φ, Ψ) is a contraction on $S_{u_0,v_0,\rho}$ defined by

$$S_{u_0,v_0,\rho} = \left\{ (\vec{u}^{(2)}, v^{(\ell)});\ \begin{array}{l} \|\!|\vec{u}^{(2)}|\!\|_{Y_T} + \|\!|\langle D_x\rangle v^{(\ell)}|\!\|_{\text{initial}} \le 2C(u_0, v_0), \\ \|\langle x\rangle^{\alpha_1}(u-\varphi)\|_{L_x^2 L_T^\infty} \le \rho \text{ and } \|D_x^{s-1/2}\partial_x \vec{u}^{(2)}\|_{L_x^\infty L_T^2} \le \rho \end{array} \right\},$$

with the metric $\|\!|(\vec{u}^{(2)}, v^{(\ell)})|\!\|_{Y_T'} \equiv \|\!|\vec{u}^{(2)}|\!\|_{Y_T} + \|\!|\langle D_x\rangle v^{(\ell)}|\!\|_{\text{initial}}$ where $C(u_0, v_0) > 0$ is a constant depending on the size of initial data. We note that $S_{u_0,v_0,\rho} \neq \phi$, if

φ is sufficiently close to u_0 in $H_x^{s,0} \cap H_x^{s_1,\alpha_1}$ and $\rho > 0$ is small enough. The existence of the solution to (1.1) is the direct consequence of the proposition given below.

Proposition 6.1. *Let $\varphi(x) \in C_0^\infty(\boldsymbol{R})$ sufficiently close to u_0 in $H_x^{s,0} \cap H_x^{s_1,\alpha_1}$ and $\rho, T > 0$ sufficiently small. Then, (Φ, Ψ) is a contraction map on $S_{u_0,v_0,\rho}$.*

To prove Proposition 6.1, we require several lemmas.

Lemma 6.2. *There exist positive constants $C(u_0, v_0), C, C_\varphi$ and β such that*

$$(6.2)\quad \begin{aligned} &|||(\Phi(\vec{u}^{(2)}, v^{(\ell)}), \Psi(\vec{u}^{(2)}, v^{(\ell)}))|||_{Y_T'} \\ &\quad \le C(u_0, v_0) + C\|\langle x\rangle^{\alpha_1}(u - \varphi)\|_{L_x^2 L_T^\infty} |||(\vec{u}^{(2)}, v^{(\ell)})|||_{Y_T'} \\ &\qquad + C_\varphi T^\beta (1 + |||(\vec{u}^{(2)}, v^{(\ell)})|||_{Y_T'}) |||(\vec{u}^{(2)}, v^{(\ell)})|||_{Y_T'}, \end{aligned}$$

$$(6.3)\quad \begin{aligned} &|||(\Phi(\vec{u}_1^{(2)}, v_1^{(\ell)}), \Psi(\vec{u}_1^{(2)}, v_1^{(\ell)})) - (\Phi(\vec{u}_2^{(2)}, v_2^{(\ell)}), \Psi(\vec{u}_2^{(2)}, v_2^{(\ell)}))|||_{Y_T'} \\ &\quad \le C(\|\langle x\rangle^{\alpha_1}(u_1 - \varphi)\|_{L_x^2 L_T^\infty} \\ &\qquad + \|D_x^{s-1/2}\partial_x \vec{u}_2^{(2)}\|_{L_x^\infty L_T^2}) |||(\vec{u}_1^{(2)}, v_1^{(\ell)}) - (\vec{u}_2^{(2)}, v_2^{(\ell)})|||_{Y_T'} \\ &\qquad + C_\varphi T^\beta \left(1 + \max_{j=1,2} |||(\vec{u}_j^{(2)}, v_j^{(\ell)})|||_{Y_T'}\right) |||(\vec{u}_1^{(2)}, v_1^{(\ell)}) - (\vec{u}_2^{(2)}, v_2^{(\ell)})|||_{Y_T'}, \end{aligned}$$

where C_φ may diverge as $\varphi \to u_0$ in $H_x^{s,0} \cap H_x^{s_1,\alpha_1}$.

Lemma 6.3. *Let $(\vec{u}^{(2)}, v^{(\ell)}) \in S_{u_0,v_0,\rho}$ with $\rho > 0$ sufficiently small. Then, there exist $\varphi \in C_0^\infty(\boldsymbol{R})$ and $T > 0$ such that*

$$(6.4)\quad \|\langle x\rangle^{\alpha_1}(P_1 R^{-1} Q^{-1} K(\varphi)^{-1} \Phi(\vec{u}^{(2)}, v^{(\ell)}) - \varphi)\|_{L_x^2 L_T^\infty} \le \rho,$$

$$(6.5)\quad \|D_x^{s-1/2}\partial_x \Phi(\vec{u}^{(2)}, v^{(\ell)})\|_{L_x^\infty L_T^2} \le \rho.$$

Proof of Lemma 6.2. We only prove (6.2) since (6.3) likewise follows. By Lemma 3.1 and 4.1, we see that

$$(6.6)\quad \begin{aligned} |||\Phi(\vec{u}^{(2)}, v^{(\ell)})|||_{\mathrm{initial}} \le C(&\|\vec{u}_0^{(2)}\|_{H_x^{s,0} \cap H_x^{s_1,\alpha_1}} + \|D_x^{s-1/2} A(u - \varphi)\partial_x \vec{u}^{(2)}\|_{L_x^1 L_T^2} \\ &+ \|A(u - \varphi)\partial_x \vec{u}^{(2)}\|_{L_T^2 H_x^{s_1,\alpha_1}} \\ &+ T |||\vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})|||_{\mathrm{initial}}). \end{aligned}$$

Note that Lemma 5.2 yields

$$(6.7)\qquad \|D_x^{s-1/2}A(u-\varphi)\partial_x\vec{u}^{(2)}\|_{L_x^1L_T^2} \le C\|\langle x\rangle^{\alpha_1}(u-\varphi)\|_{L_x^2L_T^\infty}|||\vec{u}^{(2)}|||_{Y_T} + C_\varphi T^\beta(1+|||\vec{u}^{(2)}|||_{Y_T})|||\vec{u}^{(2)}|||_{Y_T}$$

and

$$(6.8)\qquad \|A(u-\varphi)\partial_x\vec{u}^{(2)}\|_{L_T^2H_x^{s_1,\alpha_1}} \le C_\varphi T^\beta(1+|||\vec{u}^{(2)}|||_{Y_T})|||\vec{u}^{(2)}|||_{Y_T}.$$

Combining (6.6)–(6.8) and Lemma 5.2, we have

$$(6.9)\qquad |||\Phi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{initial}} \le C|||(u_0,v_0)|||_{X^s} + C\|\langle x\rangle^{\alpha_1}(u-\varphi)\|_{L_x^2}|||\vec{u}^{(2)}|||_{Y_T} + C_\varphi T^\beta(1+|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'})|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'}.$$

It is easy to see that

$$(6.10)\qquad |||\langle D_x\rangle\Psi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{initial}} \le C|||(u_0,v_0)|||_{X^s} + CT(1+|||\vec{u}^{(2)}|||_{Y_T} + |||v^{(\ell)}|||_{\text{initial}})|||\vec{u}^{(2)}|||_{Y_T}.$$

We next estimate $|||\Phi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{smooth}}$. Note that, by Lemma 3.1(3.1) and 3.3(3.8),

$$\begin{aligned}|||G\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)})|||_{\text{smooth}} &\le \int_0^T |||U(\cdot)U(-t')\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)})|||_{\text{smooth}}dt' \\ &\le C\int_0^T \|\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)})\|_{H_x^{s,0}}dt' \\ &\le CT\|\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)})\|_{L_T^\infty H_x^{s,0}}.\end{aligned}$$

Then, Lemma 3.1, 3.3, 5.3 and (6.7) give

$$\begin{aligned}(6.11)\qquad |||\Phi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{smooth}} &\le C(\|\vec{u}_0{}^{(2)}\|_{H_x^{s,0}} + \|D_x^{s-1/2}A(u-\varphi)\partial_x\vec{u}^{(2)}\|_{L_x^1L_T^2} \\ &\quad + T\|\vec{f}^{(2)}(\varphi,\vec{u}^{(2)},v^{(\ell)})\|_{L_T^\infty H_x^{s,0}}) \\ &\le C\|(u_0,v_0)\|_{X^s} + C\|\langle x\rangle^{\alpha_1}(u-\varphi)\|_{L_x^2L_T^\infty}|||\vec{u}^{(2)}|||_{Y_T} \\ &\quad + C_\varphi T^\beta(1+|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'})|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'}.\end{aligned}$$

Finally, by the simple application of Lemma 4.1–4.2 to $|||\Phi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{lower}}$ and Lemma 4.3 to $|||\Phi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{maxim}}$, we have

$$(6.12)\qquad |||\Phi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{lower}} + |||\Phi(\vec{u}^{(2)},v^{(\ell)})|||_{\text{maxim}} \le C\|(u_0,v_0)\|_{X^s} + C_\varphi T^\beta(1+|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'})|||(\vec{u}^{(2)},v^{(\ell)})|||_{Y_T'}.$$

Hence, by combining (6.9)–(6.12), we obtain Lemma 6.2. □

Proof of Lemma 6.3. Let $L(\varphi) = P_1 R^{-1} Q^{-1} K(\varphi)^{-1}$. Lemma 4.3 and 5.2 yield

$$\begin{aligned}(6.13) \qquad & \|\langle x\rangle^{\alpha_1}(L(\varphi)\Phi(\vec{u}^{(2)}, v^{(\ell)}) - \varphi)\|_{L_x^2 L_T^\infty} \\ & \le \|\langle x\rangle^{\alpha_1}(L(\varphi)U(t)\vec{u}_0^{(2)} - \varphi)\|_{L_x^2 L_T^\infty} \\ & \quad + C_\varphi T^\beta (1 + |||(\vec{u}^{(2)}, v^{(\ell)})|||_{Y_T'})|||(\vec{u}^{(2)}, v^{(\ell)})|||_{Y_T'}.\end{aligned}$$

The first term on the right hand side of (6.13) is estimated as

$$\begin{aligned}\|\langle x\rangle^{\alpha_1}(L(\varphi)U(t)\vec{u}_0^{(2)} - \varphi)\|_{L_x^2 L_T^\infty} & \le \|\langle x\rangle^{\alpha_1} L(\varphi)U(t)(\vec{u}_0^{(2)} - \vec{\varphi}^{(2)})\|_{L_x^2 L_T^\infty} \\ & \quad + \|\langle x\rangle^{\alpha_1} L(\varphi)(U(t) - I)\vec{\varphi}^{(2)}\|_{L_x^2 L_T^\infty} \\ & \equiv I_1 + I_2,\end{aligned}$$

where $\vec{\varphi}^{(2)} = K(\varphi)QR(\varphi, \partial_x^{-1}\mathscr{F}^{-1}(1-\eta)\mathscr{F}\psi)^t$ with $\psi \in C_0^\infty(\boldsymbol{R})$ sufficiently close to v_0 in $H_x^{s-1,0} \cap H_x^{s_1-1,\alpha_1}$. Since Lemma 4.3 gives

$$I_1 \le C\|\vec{u}_0^{(2)} - \vec{\varphi}^{(2)}\|_{H_x^{s,0} \cap H_x^{s_1,\alpha_1}},$$

we can take $(\varphi, \psi) \in C_0^\infty \times C_0^\infty$ close to (u_0, v_0) so that $I_1 < \rho/3$. Let us fix (φ, ψ) hereafter. For the estimate of I_2, we observe that

$$\begin{aligned}I_2 & \le C\|(1+x^2)(U(t) - I)\vec{\varphi}^{(2)}\|_{L_T^\infty L_x^\infty} \\ & \le CT^\beta \|\vec{\varphi}^{(2)}\|_{H_x^{\sigma,\alpha}}\end{aligned}$$

with $\sigma > 0$ and $\alpha > 0$ so large. Thus, for small $T > 0$, it is possible to see that $I_2 < \rho/3$. Plugging these inequalities into (6.13) and taking $T > 0$ further small, we have

$$\begin{aligned}\|\langle x\rangle^{\alpha_1}(L(\varphi)\Phi(\vec{u}^{(2)}, v^{(\ell)}) - \varphi)\|_{L_x^2 L_T^\infty} & \le 2\rho/3 + C_\varphi(1 + C(u_0, v_0))C(u_0, v_0)T^\beta \\ & \le \rho.\end{aligned}$$

Hence, we obtain (6.4).

We next prove (6.5). By Lemma 3.1, 5.2 and 5.3, we see that

$$\begin{aligned}(6.14) \qquad \|D_x^{s-1/2}\partial_x \Phi(\vec{u}^{(2)}, v^{(\ell)})\|_{L_x^\infty L_T^2} & \le \|D_x^{s-1/2}\partial_x U(t)\vec{u}_0^{(2)}\|_{L_x^\infty L_T^2} \\ & \quad + C\|\langle x\rangle^{\alpha_1}(u - \varphi)\|_{L_x^2 L_T^\infty}\|D_x^{s-1/2}\partial_x \vec{u}^{(2)}\|_{L_x^\infty L_T^2} \\ & \quad + C_\varphi T^\beta(1 + |||(\vec{u}^{(2)}, v^{(\ell)})|||_{Y_T'})|||(\vec{u}^{(2)}, v^{(\ell)})|||_{Y_T'}.\end{aligned}$$

We note that $\|D_x^{s-1/2}\partial_x U(t)\vec{u}_0^{(2)}\|_{L_x^\infty L_T^2} < \rho/3$ for small $T > 0$. Indeed, by Lemma 3.1,

$$\begin{aligned}
&\|D_x^{s-1/2}\partial_x U(t)\vec{u}_0^{(2)}\|_{L_x^\infty L_T^2} \\
&\quad \le \|D_x^{s-1/2}\partial_x U(t)(\vec{u}_0^{(2)} - \vec{\varphi}^{(2)})\|_{L_x^\infty L_T^2} + \|D_x^{s-1/2}\partial_x U(t)\vec{\varphi}^{(2)}\|_{L_x^\infty L_T^2} \\
&\quad \le C\|\vec{u}_0^{(2)} - \vec{\varphi}^{(2)}\|_{H_x^{s,0}} + CT^{1/2}\|\vec{\varphi}^{(2)}\|_{H_x^{3,0}} \\
&\quad \le \rho/3.
\end{aligned}$$

Hence, (6.14) implies that

$$\begin{aligned}
\|D_x^{s-1/2}\partial_x \Phi(\vec{u}^{(2)}, v^{(\ell)})\|_{L_x^\infty L_T^2} &\le \rho/3 + C\rho^2 + C_\varphi(1 + C(u_0, v_0))C(u_0, v_0)T^\beta \\
&\le \rho. \quad \square
\end{aligned}$$

Proof of Proposition 6.1. Let $\vec{u}^{(2)} \in S_{u_0, v_0, \rho}$. Then, by using Lemma 6.2 and taking φ sufficiently close to u_0 and $\rho, T > 0$ sufficiently small, we see that

$$\begin{aligned}
&|||(\Phi(\vec{u}^{(2)}, v^{(\ell)}), \Psi(\vec{u}^{(2)}, v^{(\ell)}))|||_{Y_T'} \\
&\quad \le C(u_0, v_0) + C\rho C(u_0, v_0) + C_\varphi T^\beta(1 + C(u_0, v_0))C(u_0, v_0) \\
&\quad \le 2C(u_0, v_0), \\
&|||(\Phi(\vec{u}_1^{(2)}, v_1^{(\ell)}), \Psi(\vec{u}_1^{(2)}, v_1^{(\ell)})) - (\Phi(\vec{u}_2^{(2)}, v_2^{(\ell)}), \Psi(\vec{u}_2^{(2)}, v_2^{(\ell)}))|||_{Y_T'} \\
&\quad \le (C\rho + C_\varphi T^\beta(1 + C(u_0, v_0)))|||(\vec{u}_1^{(2)}, v_1^{(\ell)}) - (\vec{u}_2^{(2)}, v_2^{(\ell)})|||_{Y_T'} \\
&\quad \le \frac{1}{2}|||(\vec{u}_1^{(2)}, v_1^{(\ell)}) - (\vec{u}_2^{(2)}, v_2^{(\ell)})|||_{Y_T'}.
\end{aligned}$$

Hence, (Φ, Ψ) is a contraction map on $S_{u_0, v_0, \rho}$. $\square$

7. Remark on uniqueness

Of course, the uniqueness result in Y_T' easily follows. This function space is, however, so strict and complicated that we want to relax it for the simple statement of Theorem 1.1. Let (u_j, v_j) $(j = 1, 2)$ be the solutions to (1.1) in $C([0, T]; X^s)$ with $\langle x\rangle^{\alpha_1} u_j \in L_x^2 L_T^\infty$. According to the transformation as in section 2, the uniqueness problem of (1.1) is reduced into that of the integral equations (6.1). We remark that $\varphi \in C_0^\infty(\boldsymbol{R})$ is not always close to u_0 but it is taken as an approximation of $u(T^*)$ for some $T^* \in [0, T]$. Therefore, we replace φ by $\tilde{\varphi}$ at the present argument. Let $T^* = \sup\{T'; (\vec{u}_1^{(2)}(t), v_1^{(\ell)}(t)) = (\vec{u}_2^{(2)}(t), v_2^{(\ell)}(t))$ for $0 \le t \le T'\}$, where $(\vec{u}_j^{(2)}, v_j^{(\ell)})$ $(j = 1, 2)$ are the solutions to

(6.1). Note that $(\vec{u}_j^{(2)}, v_j^{(\ell)})$ belongs to Z_T' defined by

$$Z_T' = \{(\vec{u}^{(2)}, v^{(\ell)}); \|\!|(\vec{u}^{(2)}, v^{(\ell)})\|\!|_{Z_T'} < \infty\},$$

where

$$\|\!|(\vec{u}^{(2)}, v^{(\ell)})\|\!|_{Z_T'} = \|\!|\vec{u}^{(2)}\|\!|_{\mathrm{initial}} + \|\!|\langle D_x\rangle v^{(\ell)}\|\!|_{\mathrm{initial}} + \|\langle x\rangle^{\alpha_1} P_1 R^{-1} Q^{-1} K(\tilde{\varphi})^{-1}\vec{u}^{(2)}\|_{L_x^2 L_T^\infty}.$$

It is easy to see that $Y_T' \subset Z_T'$.

Our main concern in this section is to show that $T^* = T$ by saying contradiction. For this purpose, we first assume that $T^* < T$. Then there exists some $\tau > 0$ such that $I = [T^*, T^* + \tau] \subset [0, T]$. We try to measure $\vec{u}_1^{(2)} - \vec{u}_2^{(2)}$ and $v_1^{(\ell)} - v_2^{(\ell)}$ with the metric of Z_I'' defined by

$$Z_I'' = \{(\vec{u}^{(2)}, v^{(\ell)}); \|\!|(\vec{u}^{(2)}, v^{(\ell)})\|\!|_{Z_I''} < \infty\},$$

where

$$\|\!|(\vec{u}^{(2)}, v^{(\ell)})\|\!|_{Z_I''} = \|\vec{u}^{(2)}\|_{L_I^\infty(H_x^{1/2,0}\cap H_x^{0,1/2-\nu})} + \|v^{(\ell)}\|_{L_I^\infty(H_x^{3/2,0}\cap H_x^{1,1/2-\nu})} + \|\partial_x \vec{u}^{(2)}\|_{L_x^\infty L_I^2}$$
$$+ \|\langle x\rangle^{1/2-\nu}\vec{u}^{(2)}\|_{L_I^4 L_x^\infty} \quad \text{with } \nu > 0$$

defined in Definition 5.1. For the brief notation, we often write $L^r(I; X)$ (resp. $L_x^p(\boldsymbol{R}; L^r(I))$) as $L_I^r X$ (resp. $L_x^p L_I^r$). The readers can check $\vec{u}_j^{(2)} \in L_I^\infty(H_x^{1/2,0} \cap H_x^{0,1/2-\nu})$, $\langle x\rangle^{1/2-\nu}\vec{u}_j^{(2)} \in L_I^4 L_x^\infty$ and, in particular, $\partial_x \vec{u}_j^{(2)} \in L_x^\infty L_I^2$. Indeed, by applying Lemma 3.1 to the integral equation (6.1), we see that

$$\begin{aligned} \|\partial_x \vec{u}_j^{(2)}\|_{L_x^\infty L_I^2} &\le \|\partial_x \vec{u}_j^{(2)}\|_{L_x^\infty L_T^2} \\ &\le C\|\vec{u}_{0,j}^{(2)}\|_{H_x^{1/2,0}} + C\|A(u_j - \tilde{\varphi})\partial_x \vec{u}_j^{(2)}\|_{L_x^1 L_T^2} \\ &\quad + CT\|\vec{f}^{(2)}(\tilde{\varphi}, \vec{u}_j^{(2)}, v_j^{(\ell)})\|_{L_T^\infty H_x^{1/2,0}}. \end{aligned} \tag{7.1}$$

Since

$$\|A(u_j - \tilde{\varphi})\partial_x \vec{u}_j^{(2)}\|_{L_x^1 L_T^2} \le T^{1/2}\|(u_j - \tilde{\varphi})\|_{L_T^\infty H_x^{s_1,\alpha_1}} \|\vec{u}_j^{(2)}\|_{L_T^\infty H_x^{s,0}} < \infty$$

and

$$\|\vec{f}^{(2)}(\tilde{\varphi}, \vec{u}_j^{(2)}, v_j^{(\ell)})\|_{L_T^\infty H_x^{1/2,0}} \le C_{\tilde{\varphi}}(1 + \|\!|(\vec{u}_j^{(2)}, v_j^{(\ell)})\|\!|_{Z_T'})\|\!|(\vec{u}_j^{(2)}, v_j^{(\ell)})\|\!|_{Z_T'} < \infty,$$

it follows from (7.1) that $\|\partial_x \vec{u}_j^{(2)}\|_{L_x^\infty L_I^2} < \infty$. Now we are ready to prove the uniqueness result in Theorem 1.1.

Proof of Theorem 1.1 (Uniqueness). Note that $\vec{u}_d^{(2)} = \vec{u}_1^{(2)} - \vec{u}_2^{(2)}$ and $v_d^{(\ell)} = v_1^{(\ell)} - v_2^{(\ell)}$ satisfy

$$\begin{aligned} \vec{u}_d^{(2)} = -\int_{T^*}^t U(t-t')[&A(u_1 - \tilde{\varphi})\partial_x \vec{u}_d^{(2)} + A(u_1 - u_2)\partial_x \vec{u}_2^{(2)} \\ &- i(\vec{f}^{(2)}(\tilde{\varphi}, \vec{u}_1^{(2)}, v_1^{(\ell)}) - \vec{f}^{(2)}(\tilde{\varphi}, \vec{u}_2^{(2)}, v_2^{(\ell)}))]dt'. \end{aligned}$$

Then, by Lemma 3.1, we have

$$\begin{aligned}
(7.2)\qquad & \|D_x^{1/2}\vec{u}_d^{(2)}\|_{L_I^\infty L_x^2} + \|\partial_x \vec{u}_d^{(2)}\|_{L_x^\infty L_I^2} \\
&\le C\|A(u_1-\tilde{\varphi})\partial_x \vec{u}_d^{(2)}\|_{L_x^1 L_I^2} + C\|A(u_1-u_2)\partial_x \vec{u}_2^{(2)}\|_{L_x^1 L_I^2} \\
&\quad + C\tau\|\vec{f}^{(2)}(\tilde{\varphi}, \vec{u}_1^{(2)}, v_1^{(\ell)}) - \vec{f}^{(2)}(\tilde{\varphi}, \vec{u}_2^{(2)}, v_2^{(\ell)})\|_{L_I^\infty H_x^{1/2,0}} \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Hölder's inequality and the interpolation ($H_x^{\sigma_0,\ell_0} \cap H^{\sigma_1,\ell_1} \subset H_x^{\sigma,\ell}$ with $\sigma = \theta\sigma_1 + (1-\theta)\sigma_0$, $\ell = \theta\ell_1 + (1-\theta)\ell_0$ for $\theta \in [0,1]$) yield

$$\begin{aligned}
I_1 &\le C\|\langle x\rangle^{\alpha_1}(u_1-\tilde{\varphi})\|_{L_x^2 L_I^\infty}\|\partial_x \vec{u}_d^{(2)}\|_{L_x^\infty L_I^2}, \\
I_2 &\le C\tau^{1/4}\|\langle x\rangle^{1/2-\nu}\vec{u}_d^{(2)}\|_{L_I^4 L_x^\infty}\|\langle x\rangle^{\nu'}\partial_x \vec{u}_2^{(2)}\|_{L_I^\infty L_x^2} \\
&\le C\tau^{1/4}\|\langle x\rangle^{1/2-\nu}\vec{u}_d^{(2)}\|_{L_I^4 L_x^\infty}\|\!|\vec{u}_2^{(2)}|\!\|_{Z_T'} \qquad (\text{with } \nu < \nu').
\end{aligned}$$

By using the representation like $u_1\partial_x w_1 - u_2\partial_x w_2 = (u_1-u_2)\partial_x w_1 + \partial_x(u_2(w_1-w_2)) - (\partial_x u_2)(w_1-w_2)$, we see that

$$I_3 \le C_{\tilde{\varphi}}\left(1 + \max_{j=1,2}\|\!|(\vec{u}_j^{(2)}, v_j^{(\ell)})|\!\|_{Z_T'}\right)(\|\vec{u}_d^{(2)}\|_{L_I^\infty H_x^{1/2,0}} + \|v_d^{(\ell)}\|_{L_I^\infty H_x^{3/2,0}}).$$

Plugging the estimates of I_1, I_2 and I_3 into (7.2), we have

$$\begin{aligned}
(7.3)\qquad & \|D_x^{1/2}\vec{u}_d^{(2)}\|_{L_I^\infty L_x^2} + \|\partial_x \vec{u}_d^{(2)}\|_{L_x^\infty L_I^2} \\
&\le C\|\langle x\rangle^{\alpha_1}(u_1-\tilde{\varphi})\|_{L_x^2 L_I^\infty}\|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''} \\
&\quad + C_{\tilde{\varphi}}\tau^\beta\left(1 + \max_{j=1,2}\|\!|(\vec{u}_j^{(2)}, v_j^{(\ell)})|\!\|_{Z_T'}\right)\|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''}.
\end{aligned}$$

On the other hand, by Lemma 4.2, we observe that

$$(7.4)\quad \|\langle x\rangle^{1/2-\nu}\vec{u}_d^{(2)}\|_{L_I^\infty L_x^2 \cap L_I^4 L_x^\infty} \le C_{\tilde{\varphi}}\tau^\beta\left(1 + \max_{j=1,2}\|\!|(\vec{u}_j^{(2)}, v_j^{(\ell)})|\!\|_{Z_T'}\right)\|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''}.$$

From (7.3) and (7.4), it follows that

$$\begin{aligned}
(7.5)\quad \|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''} &\le C\|\langle x\rangle^{\alpha_1}(u_1-\tilde{\varphi})\|_{L_x^2 L_I^\infty}\|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''} \\
&\quad + C_{\tilde{\varphi}}\tau^\beta\left(1 + \max_{j=1,2}\|\!|(\vec{u}_j^{(2)}, v_j^{(\ell)})|\!\|_{Z_T'}\right)\|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''}.
\end{aligned}$$

We take $\tilde{\varphi} \in C_0^\infty(\boldsymbol{R})$ close to $u_1(T^*)$ in $H_x^{s,0} \cap H_x^{s_1,\alpha_1}$ so that $C\|\langle x\rangle^{\alpha_1}(u_1-\tilde{\varphi})\|_{L_x^2 L_I^\infty} < 1/4$ for small $\tau > 0$. Then, by fixing $\tilde{\varphi}$ and letting τ further small in (7.5), we see that $\|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''} \le (1/2)\|\!|(\vec{u}_d^{(2)}, v_d^{(\ell)})|\!\|_{Z_I''}$ and, hence, $(\vec{u}_1^{(2)}(t), v_1^{(\ell)}(t)) = (\vec{u}_2^{(2)}(t), v_2^{(\ell)}(t))$ for $t \in [T^*, T^* + \tau]$. This contradicts to the definition of T^*. $\square$

8. Lipschitz' continuity of data-solution map

Let $(u_0', v_0') \in X^s$ be the initial data of Boussinesq system with $(u_0', v_0') \in B_\delta(u_0, v_0) \equiv \{(u_0', v_0'); \|\|(u_0', v_0') - (u_0, v_0)\|\|_{X^s} < \delta\}$. Then, the existence of the time local solution to this system follows from the same argument as in section 6. We denote this solution by $(u'(t), v'(t))$ for $t \in [0, T']$. We transform (u', v') as in section 2, i.e.,

$$\vec{u}'^{(2)} = K(\varphi)QR(u', \partial_x^{-1}\mathscr{F}^{-1}(1-\eta)\mathscr{F}v') \qquad \text{and} \qquad v'^{(\ell)} = \mathscr{F}^{-1}\eta\mathscr{F}v'.$$

Note that, in this transformation, $\varphi \in C_0^\infty(\boldsymbol{R})$ is not an approximation of u_0' but is close to u_0 in $H_x^{s,0} \cap H_x^{s_1,\alpha_1}$. Then, $\vec{u}'^{(2)}$ satisfies the integral equation:

$$\vec{u}'^{(2)} = U(t)\vec{u}_0'^{(2)} - G\{A(u' - \varphi)\partial_x\vec{u}'^{(2)} - i\vec{f}^{(2)}(\varphi, \vec{u}'^{(2)}, v'^{(\ell)})\},$$

where $\vec{u}_0'^{(2)}(x) = \vec{u}'^{(2)}(0, x)$. Taking the subtraction with the integral equation of $\vec{u}^{(2)}$, we see that $\vec{u}_d^{(2)} \equiv \vec{u}'^{(2)} - \vec{u}^{(2)}$ satisfy

$$\begin{aligned}(8.1) \qquad \vec{u}_d^{(2)} = U(t)(\vec{u}_0'^{(2)} - \vec{u}_0^{(2)}) &- G\{A(u - \varphi)\partial_x\vec{u}_d^{(2)} + A(u' - u)\partial_x\vec{u}'^{(2)}\} \\ &+ iG\{\vec{f}^{(2)}(\varphi, \vec{u}'^{(2)}, v'^{(\ell)}) - \vec{f}^{(2)}(\varphi, \vec{u}^{(2)}, v^{(\ell)})\}.\end{aligned}$$

Let us prove the stability of the solution in Theorem 1.1.

Proof of Theorem 1.1 (Continuous dependence on the initial data). Applying Lemmas 3.1, 4.1–4.3 to the first term on the right hand side of (8.1) and employing the nonlinear estimate similar to Lemma 6.2, we observe

$$\begin{aligned}\|\|(\vec{u}_d^{(2)}, v_d^{(\ell)})\|\|_{Y_{T'}'} \le\ & C\|\|(u_0' - u_0, v_0' - v_0)\|\|_{X^s} \\ & + C(\|\langle x\rangle^{\alpha_1}(u - \varphi)\|_{L_x^2 L_{T'}^\infty} + \|\partial_x\vec{u}'^{(2)}\|_{L_x^\infty L_{T'}^2})\|\|(\vec{u}_d^{(2)}, v_d^{(\ell)})\|\|_{Y_{T'}'} \\ & + C_\varphi T'^\beta(1 + \|\|(\vec{u}'^{(2)}, v'^{(\ell)})\|\|_{Y_{T'}'} \\ & + \|\|(\vec{u}^{(2)}, v^{(\ell)})\|\|_{Y_{T'}'})\|\|(\vec{u}_d^{(2)}, v_d^{(\ell)})\|\|_{Y_{T'}'},\end{aligned}$$

where $v_d^{(\ell)} = v'^{(\ell)} - v^{(\ell)}$. Since, by the proof of Lemma 6.3, there exists $T' > 0$ such that $C\|\partial_x\vec{u}'^{(2)}\|_{L_x^\infty L_{T'}^2} < 1/6$ for $(u_0', v_0') \in B_\delta(u_0, v_0)$ if $\delta > 0$ is sufficiently small, we have

$$\|\|(\vec{u}_d^{(2)}, v_d^{(\ell)})\|\|_{Y_{T'}'} \le C\|\|(u_0' - u_0, v_0' - v_0)\|\|_{X^s} + \frac{1}{2}\|\|(\vec{u}_d^{(2)}, v_d^{(\ell)})\|\|_{Y_{T'}'}.$$

This implies the stability of the solution. $\square$

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