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On the Asymptotic Periodic Solutions of Abstract Functional Differential Equations

By

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Abstract. The paper is concerned with conditions for all mild solutions of abstract functional differential equations with finite delay in a Banach space to be periodic and asymptotic periodic, where forcing term is a continuous 1-periodic function. The obtained results extend various recent ones on the subject.

Key Words and Phrases. Abstract functional differential equation, Asymptotic periodic solutions.

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1. Introduction

This paper is concerned with equations of the form

$$\frac{du(t)}{dt} = Au(t) + Fu_t + f(t), \qquad t \in \boldsymbol{R}, \tag{1}$$

where A is the generator of a strongly continuous semigroup of bounded linear operators $(T(t))_{t \geq 0}$ on a given Banach space $\boldsymbol{X}$, F is a bounded linear operator from the phase space $\mathscr{C} := C([-r, 0]; \boldsymbol{X})$ to $\boldsymbol{X}$, u_t is an element of $\mathscr{C}$ which is defined as $u_t(\theta) = u(t + \theta)$ for $-r \leq \theta \leq 0$, and f is an $\boldsymbol{X}$-valued continuous 1-periodic function with Fourier coefficients:

$$\tilde{f}_k = \int_0^1 e^{-2ik\pi t} f(t)dt, \qquad k = 0, \pm 1, \pm 2, \ldots.$$

The main problem which we consider in this paper is to find conditions for all solutions of Eq. (1) to be periodic or asymptotic periodic. This problem has a long history and has been considered in part by many authors, see e.g. [4, 5, 11, 13, 16, 29, 30, 35, 39] and the references therein. On the other hand, it arises naturally from recent studies on the existence of (almost) periodic solutions of evolution equations (see e.g. [6, 7, 10, 17, 20, 22, 24, 25, 26, 27, 31, 32, 33, 38, 39, 40]). By the superposition principle, it is closely related to the conditions for the inhomogeneous equations to have at least one periodic solution, and for all solutions of the corresponding homogeneous equations to be (asymptotic) periodic solutions.

Our plan of the paper is first to prove a new criterion for Eq. (1) to have a periodic mild solution. Next, using this result we can apply known results on (almost) periodic C_0-semigroups to the homogeneous equations. By the superposition principle, the combination of these two steps allows us to study the inhomogeneous equation (1). The obtained results Theorems 3.4, 3.7, 3.15, 3.16, 3.18 extend the known ones in [15, 17, 22, 29, 30] and complement the ones in [4, 7, 10, 21, 27, 28, 31, 40, 41].

To prove the main results in this paper we will make use of the harmonic analysis of bounded functions (see [1, 3, 23, 38] and the references therein for more details). The applications of the method of sums of commuting operators into the study of almost periodic solutions of functional differential equations can be consulted in [27]. This method is based on a result by Arendt, Rabiger, Sourour [2] a summary of which is given in the next section. To study the homogeneous equations we will need the splitting theorem of Glicksberg and DeLeeuw. For the reader's convenience we summarize some notions and results in the Apprendix.

2. Preliminaries

2.1. Notation and Definitions. In this paper we use the following notations: $\boldsymbol{N}, \boldsymbol{Z}, \boldsymbol{R}, \boldsymbol{C}$ stand for the set of natural, integer, real, complex numbers, respectively; $\boldsymbol{X}$ will denote a given complex Banach space. If T is a linear operator on $\boldsymbol{X}$, then $D(T)$ stands for its domain. Given two Banach spaces $\boldsymbol{Y}, \boldsymbol{Z}$ by $L(\boldsymbol{Y}, \boldsymbol{Z})$ we will denote the space of all bounded linear operators from $\boldsymbol{Y}$ to $\boldsymbol{Z}$ and $L(\boldsymbol{X}, \boldsymbol{X}) := L(\boldsymbol{X})$. As usual, $\sigma(T), \rho(T), R(\lambda, T)$ are the notations of the spectrum, resolvent set and resolvent of the operator T. The notations $BUC(\boldsymbol{R}, \boldsymbol{X}), AP(\boldsymbol{X})$ will stand for the space of all $\boldsymbol{X}$-valued bounded uniformly continuous functions on $\boldsymbol{R}$ and its subspace of almost periodic functions in Bohr's sense (see, [23]); $AP(\boldsymbol{X})$ is a Banach space with supremum norm. We will denote by $\mathscr{B}$ the operater acting on $BUC(\boldsymbol{R}, \boldsymbol{X})$ defined by the formula $[\mathscr{B}u](t) := Fu_t$, $\forall u \in BUC(\boldsymbol{R}, \boldsymbol{X})$. We will denote by $S(t)$ the translation group on $BUC(\boldsymbol{R}, \boldsymbol{X})$, i.e., $S(t)v(s) := v(t+s)$, $\forall t, s \in \boldsymbol{R}$, $v \in BUC(\boldsymbol{R}, \boldsymbol{X})$ with infinitesimal generator $\mathscr{D} := d/dt$ which is defined on $D(\mathscr{D}) := BUC^1(\boldsymbol{R}, \boldsymbol{X})$. Let $\mathscr{M}$ be a subspace of $BUC(\boldsymbol{R}, \boldsymbol{X})$, A be a linear operator on $\boldsymbol{X}$. We shall denote by $\mathscr{A}_{\mathscr{M}}$ the operator $\mathscr{M} \ni f \mapsto Af(\cdot)$ with $D(\mathscr{A}_{\mathscr{M}}) = \{f \in \mathscr{M} \mid \forall t \in \boldsymbol{R}, f(t) \in D(A), Af(\cdot) \in \mathscr{M}\}$. When $\mathscr{M} = BUC(\boldsymbol{R}, \boldsymbol{X})$ we shall use the notation $\mathscr{A} := \mathscr{A}_{\mathscr{M}}$. For translation invariant subspaces $\mathscr{M} \subset BUC(\boldsymbol{R}, \boldsymbol{X})$ we will denote by $\mathscr{D}_{\mathscr{M}}$ the infinitesimal generator of the translation group $(S(t)|_{\mathscr{M}})_{t \in \boldsymbol{R}}$ in $\mathscr{M}$.

Definition 2.1. A function $f : \boldsymbol{R} \to \boldsymbol{X}$ is said to be *asymptotic periodic* if there exists a periodic function $f^1 : \boldsymbol{R} \to \boldsymbol{X}$ such that $\lim_{t\to\infty}(f(t) - f^1(t)) = 0$.

A C_0-semigroup $(T(t))_{t\geq 0}$ is called *compact* for $t > t_0$ if $T(t)$ is a compact operator for every $t > t_0$. $(T(t))_{t\geq 0}$ is called *compact* if $T(t)$ is compact for each $t > 0$.

2.2. Commuting Operators. In this subsection we recall the notion of two commuting operators and some related results on the spectral properties of their sum.

Definition 2.2. Let A and B be operators on a Banach space G with non-empty resolvent set. We say that *A and B commute* if one of the following equivalent conditions hold:

i) $R(\lambda, A)R(\mu, B) = R(\mu, B)R(\lambda, A)$ for some (all) $\lambda \in \rho(A)$, $\mu \in \rho(B)$,

ii) If $x \in D(A)$, $R(\mu, B)x \in D(A)$ and $AR(\mu, B)x = R(\mu, B)Ax$ for some (all) $\mu \in \rho(B)$.

For $\theta \in (0, \pi)$, $R > 0$ we denote $\Sigma(\theta, R) = \{z \in \boldsymbol{C} : |z| \geq R, |argz| \leq \theta\}$.

Definition 2.3. Let A and B be commuting operators. Then

i) A is said to be of class $\Sigma(\pi/2 + \theta, R)$ if there are positive constants θ, R such that $0 < \theta < \pi/2$, and

$$\Sigma(\pi/2 + \theta, R) \subset \rho(A) \quad \text{and} \quad \sup_{\lambda\in\Sigma(\pi/2+\theta, R)} \|\lambda R(\lambda, A)\| < \infty, \tag{2}$$

ii) A and B are said to satisfy *condition P* if there are positive constants $\theta, \theta', R, \theta' < \theta < \pi/2$, such that A and B are of class $\Sigma(\pi/2 + \theta, R)$, $\Sigma(\pi/2 - \theta', R)$, respectively.

If in addition, an operator A satisfying (i) in the above definition has dense domain, it generates an analytic (strongly continuous) semigroup. In this case A is said to be *sectorial.*

As usual, $A + B$ is defined by $(A + B)x = Ax + Bx$ with domain $D(A + B) = D(A) \cap D(B)$.

In this paper we will use the following norm, defined by A on the space $\boldsymbol{X}$, $\|x\|_{\mathscr{T}_A} := \|R(\lambda, A)x\|$, where $\lambda \in \rho(A)$. It is seen that different $\lambda \in \rho(A)$ yields equivalent norms. We say that an operator C on $\boldsymbol{X}$ is A-closed if its graph is closed with respect to the topology induced by $\mathscr{T}_A$ on the product $\boldsymbol{X} \times \boldsymbol{X}$. In this case, A-closure of C is denoted by $\overline{C}^A$.

Theorem 2.4. *Assume that A and B commute. Then the following assertions hold*:

i) *If one of the operators is bounded, then*

$$\sigma(A + B) \subset \sigma(A) + \sigma(B). \tag{3}$$

ii) *If A and B satisfy condition P, then $A+B$ is A-closable, and*

$$\sigma(\overline{(A+B)}^A) \subset \sigma(A) + \sigma(B). \tag{4}$$

In particular, if $D(A)$ is dense in $\boldsymbol{X}$, then $\overline{(A+B)}^A = \overline{A+B}$, where $\overline{A+B}$ denotes the usual closure of $A+B$.

Proof. For the proof we refer the reader to [2, Theorems 7.2, 7.3]. □

2.3. Functional differential equations.

Definition 2.5. Let A be a closed linear operator on $\boldsymbol{X}$. An $\boldsymbol{X}$-valued continuous function u on $\boldsymbol{R}$ is said to be a *mild solution* of Eq. (1) on $\boldsymbol{R}$ if for every s,

$$u(t) = u(s) + A\int_s^t u(\xi)d\xi + \int_s^t [Fu_\xi + f(\xi)]d\xi, \qquad \forall t \geq s.$$

If A is the generator of a C_0-semigroup, by [20, Lemma 2.11] this condition is equivalent to the condition that, for every s,

$$u(t) = T(t-s)u(s) + \int_s^t T(t-\xi)[Fu_\xi + f(\xi)]d\xi \qquad \forall t \geq s.$$

Consider the homogeneous equation of Eq. (1)

$$\frac{du(t)}{dt} = Au(t) + Fu_t. \tag{5}$$

One can define the solution semigroup $(V(t))_{t\geq 0}$ on $\mathscr{C}$ which is defined by $V(t)\phi := w_t$, $\phi \in \mathscr{C}$, where $w(\cdot)$ is the unique solution of the Cauchy problem

$$\begin{cases} w(t) = T(t-s)\phi(0) + \int_0^t T(t-\xi)[Fw_\xi]d\xi, \quad \forall t \geq 0, \\ w_0 = \phi. \end{cases} \tag{6}$$

Let $\mathscr{G}$ be the generator of $(V(t))_{t\geq 0}$. The characteristic operator $\Delta(\lambda)$ of Eq. (5) is defined by

$$\Delta(\lambda)x := (\lambda I - A - Fe^{\lambda\cdot})x, \qquad x \in D(A). \tag{7}$$

Moreover, we define the sets

$$\rho(\Delta) := \{\lambda \in \boldsymbol{C} : \exists \Delta^{-1}(\lambda) \in L(\boldsymbol{X})\},$$

$\sigma(\Delta) := \rho(\Delta)^c$ and $\sigma_i(\Delta) := \{\xi \in \boldsymbol{R} : i\xi \in \sigma(\Delta)\}$.

Lemma 2.6.

$$\sigma_i(\Delta) \subset \sigma_i(\mathscr{A} + \mathscr{B}) := \{\xi \in \boldsymbol{R} : i\xi \in \sigma(\mathscr{A} + \mathscr{B})\}.$$

Proof. We will follow the manner in the proof of [34, Proposition 3.6]. Let $i\lambda \in \rho(\mathscr{A} + \mathscr{B})$. Set $G = (i\lambda - \mathscr{A} - \mathscr{B})^{-1}$. For $f \in BUC(\boldsymbol{R}, \boldsymbol{X})$, we set $u_f = Gf$. Then

$$(i\lambda - \mathscr{A} - \mathscr{B})u_f = f.$$

Since for all $\xi \in \boldsymbol{R}$, $\mathscr{A}S(\xi) = S(\xi)\mathscr{A}$ and $\mathscr{B}S(\xi) = S(\xi)\mathscr{B}$, we have

$$(i\lambda - \mathscr{A} - \mathscr{B})S(\xi)u_f = S(\xi)(i\lambda - \mathscr{A} - \mathscr{B})u_f = S(\xi)f.$$

Therefore $S(\xi)Gf = GS(\xi)f$ for $\xi \in \boldsymbol{R}$, $f \in BUC(\boldsymbol{R}, \boldsymbol{X})$. On the other hand, for $f_\lambda := e^{i\lambda\cdot}x$, $(x \in \boldsymbol{X})$, we have $S(\xi)f_\lambda = e^{i\lambda\xi}f_\lambda$. Thus, we have

$$\begin{aligned}\frac{dGf_\lambda}{dt}(t) &= \lim_{h\to 0}\frac{S(h)Gf_\lambda(t) - Gf_\lambda(t)}{h} = \lim_{h\to 0} G\frac{S(h)f_\lambda(t) - f_\lambda(t)}{h}\\ &= G\lim_{h\to 0}\frac{S(h)f_\lambda(t) - f_\lambda(t)}{h} = G\lim_{h\to 0}\frac{e^{i\lambda h}f_\lambda(t) - f_\lambda(t)}{h} = i\lambda Gf_\lambda(t),\end{aligned}$$

that is, $Gf_\lambda(t) = e^{i\lambda t}y$ for some $y \in \boldsymbol{X}$. Since $Gf_\lambda \in D(\mathscr{A})$, $Gf_\lambda(t) = ye^{i\lambda t} \in D(A)$ that is $y \in D(A)$. From the definition of G it follows that $Gf_\lambda = e^{i\lambda\cdot}y$ satisfies $[(i\lambda - \mathscr{A} - \mathscr{B})e^{i\lambda\cdot}y](t) = e^{i\lambda t}x$, $t \in \boldsymbol{R}$. Hence we have

$$i\lambda y - Ay - F(e^{i\lambda\cdot}y) = x.$$

Thus $\Delta(i\lambda)$ is surjective. Let $x = 0$. Then by using the relation $e^{i\lambda t}y = Ge^{i\lambda t}0$, we get $y = 0$. Thus $\Delta(i\lambda)$ is injective. Consequently there exists $\Delta(i\lambda)^{-1} \in L(\boldsymbol{X})$, i.e., $i\lambda \in \rho(\Delta)$. □

Lemma 2.7. *Let A be the generator of a compact semigroup. Then, $i\sigma_i(\Delta) = \sigma(\mathscr{G}) \cap i\boldsymbol{R}$, which is a finite set.*

Proof. For the proof we refer the reader to [42, Lemma 5.5] and [19, Proposition 3.2]. □

2.4. Spectrum of a function. We denote by $\mathscr{F}$ the Fourier transform, i.e.

$$(\mathscr{F}f)(s) := \int_{-\infty}^{+\infty} e^{-ist}f(t)dt$$

$(s \in \boldsymbol{R},\ f \in L^1(\boldsymbol{R}))$. The *Beurling spectrum* of $u \in BUC(\boldsymbol{R}, \boldsymbol{X})$ is defined to be the following set

(8) $\mathrm{sp}(u) := \{\xi \in \boldsymbol{R} : \forall\varepsilon > 0\ \exists f \in L^1(\boldsymbol{R}), \mathrm{supp}\,\mathscr{F}f \subset (\xi - \varepsilon, \xi + \varepsilon), f * u \neq 0\},$

where $f * u(s) := \int_{-\infty}^{+\infty} f(s - t)u(t)dt$.

Example 2.8. If $f(t)$ is a 1-periodic function with the corresponding Fourier series $f \sim \sum_{k \in \boldsymbol{Z}} \tilde{f}_k e^{2ik\pi t}$, then $\mathrm{sp}(f) = \{2k\pi : \tilde{f}_k \neq 0\}$.

Theorem 2.9. *Under the notation as above,* $\mathrm{sp}(u)$ *coincides with the set consisting of* $\xi \in \boldsymbol{R}$ *such that the Fourier-Carleman transform of* u

$$\hat{u}(\lambda) = \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt, & (\mathrm{Re}\,\lambda > 0) \\ -\int_0^\infty e^{\lambda t} u(-t)dt, & (\mathrm{Re}\,\lambda < 0) \end{cases} \tag{9}$$

has no holomorphic extension to any neighborhood of $i\xi$.

Proof. For the proof we refer the reader to [38, Proposition 0.5, p. 22]. □

We list some properties of the spectrum of a function, which we will need in the sequel.

Theorem 2.10. *Let* $f, g_n \in BUC(\boldsymbol{R}, \boldsymbol{X})$, $n \in \boldsymbol{N}$ *such that* $g_n \to f$ *as* $n \to \infty$. *Then*

i) $\mathrm{sp}(f)$ *is closed,*
ii) $\mathrm{sp}(f(\cdot + h)) = \mathrm{sp}(f)$,
iii) *If* $\alpha \in \boldsymbol{C} \backslash \{0\}$, $\mathrm{sp}(\alpha f) = \mathrm{sp}(f)$,
iv) *If* $\mathrm{sp}(g_n) \subset \Lambda$ *for all* $n \in \boldsymbol{N}$, $\mathrm{sp}(f) \subset \overline{\Lambda}$,
v) *If* A *is a closed linear operator,* $f(t) \in D(A)$ $\forall t \in \boldsymbol{R}$ *and* $Af(\cdot) \in BUC(\boldsymbol{R}, \boldsymbol{X})$, *then* $\mathrm{sp}(\mathscr{A}f) \subset \mathrm{sp}(f)$,
vi) $\mathrm{sp}(\psi * f) \subset \mathrm{sp}(f) \cap \mathrm{supp}\,\mathscr{F}\psi$, $\forall \psi \in L^1(\boldsymbol{R})$.

Proof. For the proof we refer the reader to [38, p. 20–21]. □

As an immediate consequence of the above theorem we have the following:

Corollary 2.11. *Let* Λ *be a closed subset of* $\boldsymbol{R}$. *Then the set*

$$\Lambda(\boldsymbol{X}) := \{g \in AP(\boldsymbol{X}) : \mathrm{sp}(g) \subset \Lambda\}$$

is a closed subspace of $AP(\boldsymbol{X})$ *which is invariant under translations.*

The following theorem is very important to derive main results in the next section.

Theorem 2.12. *A function* f *is* 1*-periodic if and only if* $\mathrm{sp}(f) \subset 2\pi\boldsymbol{Z}$.

Proof. For the proof we refer the reader to [3, Theorem 4.8.8]. □

3. Main results

3.1. Conditions for all solutions to be periodic. We begin this subsection by proving a necessary and sufficient condition for the existence of 1-periodic solutions to the inhomogeneous equation (1). We will extend the following theorem for ordinary differential equations, which is derived instantly by [17, Theorem 1.2].

Proposition 3.1. *Let L be a $d \times d$ matrix and $f(t)$ is a $\boldsymbol{C}^d$-valued,* 1-*periodic continuous function. Then the equation $\dot{z}(t) = Lz(t) + f(t)$ has a* 1-*periodic solution if and only if, for every $k \in \boldsymbol{Z}$, the equation*

$$(2ik\pi - L)x = \tilde{f}_k$$

has a solution $x \in \boldsymbol{C}^d$.

To this purpose we will use the following two lemmas.

Let $S(\boldsymbol{R})$ be the family of rapidly decreasing functions on $\boldsymbol{R}$.

Lemma 3.2. *Let A be a closed linear operator and $\phi \in S(\boldsymbol{R})$. If u is a bounded mild solution of Eq.* (1) *on $\boldsymbol{R}$, then $\phi * u$ is a classical solution to Eq.* (1) *with forcing term $\phi * f$.*

Proof. This lemma is proved in the similar manner in the proof of [20, Lemma 2.12]. In fact, let us define

$$U(t) := \int_0^t u(s)ds, \qquad E(t) := \int_0^t [Fu_s + f(s)]ds, \qquad t \in \boldsymbol{R}.$$

Then, by definition, we have

$$u(t) = u(0) + AU(t) + E(t), \qquad t \in \boldsymbol{R}.$$

From the closedness of A, we have

$$(\phi * u)(t) = \int_{-\infty}^{\infty} \phi(\xi)d\xi u(0) + A(\phi * U(t)) + (\phi * E)(t), \qquad t \in \boldsymbol{R}.$$

Since ϕ is a rapidly decreasing function, all convolutions above are infinitely differentiable. From the closedness of A, we have that $d(\phi * U)/dt(t) \in D(A)$, $t \in \boldsymbol{R}$,

$$\begin{aligned}\frac{d}{dt}A((\phi * U)(t)) &= A\left(\frac{d(\phi * U)}{dt}(t)\right)\\ &= A((\phi * u)(t)),\end{aligned}$$

and

$$\frac{d}{dt}(\phi * u)(t) = A((\phi * u)(t)) + (\phi * (\mathscr{B}u))(t) + (\phi * f)(t), \qquad t \in \boldsymbol{R}.$$

By definition of $\mathscr{B}u$, we have

$$(\phi * \mathscr{B}u)(t) = \int_{-\infty}^{\infty} \phi(s) F u_{t-s}\, ds$$
$$= \int_{-\infty}^{\infty} F(\phi(s) u_{t-s}) ds$$
$$= F \int_{-\infty}^{\infty} \phi(s) u_{t-s}\, ds.$$

Since $u_{t-s}(\theta) = u(t-s+\theta)$, $\theta \in [-r, 0]$, by the definition of a Riemann integral, it follows that

$$\left[\int_{-\infty}^{\infty} \phi(s) u_{t-s}\, ds\right](\theta) = \int_{-\infty}^{\infty} \phi(s) u(t-s+\theta) ds$$
$$= (\phi * u)(t+\theta).$$

Hence, $(\phi * (\mathscr{B}u))(t) = F((\phi * u)_t)$, and

$$\frac{d}{dt}(\phi * u)(t) = A(\phi * u)(t) + F((\phi * u)_t) + (\phi * f)(t). \qquad \square$$

Lemma 3.3. *If Eq.* (1) *has a* 1*-periodic mild solution u, then, for every* $k \in \boldsymbol{Z}$,

$$\Delta(2ik\pi)\tilde{u}_k = \tilde{f}_k.$$

Proof. Let u be a 1-periodic mild solution to Eq. (1). If u is a classical solution, it is easy to see that $\Delta(2ik\pi)\tilde{u}_k = \tilde{f}_k$. If u is not a classical solution, we set $w = u * \phi$, $g = f * \phi$, where ϕ is a rapidly decreasing smooth scalar function ϕ such that the Fourier transform $\mathscr{F}\phi$ has support concentrated on $(2k\pi - \varepsilon, 2k\pi + \varepsilon)$ and is equal to 1 on a neighborhood of $2k\pi$. Then by Lemma 3.2, w is a classical solution to Eq. (1) with f replaced by g; hence $\Delta(2ik\pi)\tilde{w}_k = \tilde{g}_k$. Moreover we have

$$\tilde{g}_k = \mathscr{F}\phi(2k\pi)\tilde{f}_k = \tilde{f}_k,$$

and $\tilde{w}_k = \tilde{u}_k$ similarly. Consequently we have $\Delta(2ik\pi)\tilde{u}_k = \tilde{f}_k$. $\square$

Theorem 3.4. *Let A be the generator of an analytic semigroup. Then, Eq.* (1) *has a* 1*-periodic mild solution if and only if for every* $k \in \boldsymbol{Z}$, *the equation*

$$\Delta(2ik\pi)x = \tilde{f}_k \tag{10}$$

has solutions $x \in \boldsymbol{X}$. *If* x_k *is a solution of Eq.* (10) *for* $k \in \boldsymbol{Z}$, *then* $\sum_{k=-\infty}^{\infty} x_k e^{2ik\pi t}$ *is the Fourier series of a* 1*-periodic mild solution of Eq.* (1).

Proof. It is sufficient to prove the sufficiency. To this end, let us consider the operator $\mathscr{D} - \mathscr{A} - \mathscr{B}$ as a sum of commuting operators $\mathscr{D}$ and $-\mathscr{A} - \mathscr{B}$ (see [34, Lemma 3.1]). By [20, Lemma 2.8], $\mathscr{A}$ is sectorial, and $\mathscr{B}$ is a bounded linear operator. Hence by [36, Corollary 2.2], $\mathscr{A} + \mathscr{B}$ is sectorial, so $\sigma_i(\mathscr{A} + \mathscr{B})$ is a bounded subset of $\boldsymbol{R}$. Meanwhile, if Λ is a closed subset of the real line, then $\sigma(\mathscr{D}_{\Lambda(\boldsymbol{X})}) = i\Lambda$ by [20, Lemma 2.6]. Moreover by [20, Theorem 2.8], it is seen that if $\sigma(\mathscr{D}_{\Lambda(\boldsymbol{X})}) \cap \sigma(\mathscr{A} + \mathscr{B}) = \emptyset$, for every $f \in \Lambda(\boldsymbol{X})$, Eq. (1) has a unique solution $u \in \Lambda(\boldsymbol{X})$. Let N be a natural sufficiently large number such that

$$\sigma_i(\mathscr{A} + \mathscr{B}) \subset [-N, N]. \tag{11}$$

Thus, if $\mathrm{sp}(f) \subset \boldsymbol{R} \backslash [-N, N]$, then Eq. (1) has a unique solution u with $\mathrm{sp}(u) \subset \mathrm{sp}(f)$.

Therefore, we decompose $f = f_1 + f_2$ as follows:

$$f_1(t) := \sum_{k=-N}^{N} \tilde{f}_k e^{2ik\pi t},$$

$$f_2(t) := f(t) - f_1(t).$$

By the above remark, Eq. (1) with f replaced by f_2 has a unique 1-periodic mild solution u_2 by Theorem 2.12. On the other hand, for every $-N \le k \le N$ there exists an $\tilde{x}_k$ such that $\Delta(2ik\pi)\tilde{x}_k = \tilde{f}_k$ by the assumption of this theorem. Thus Eq. (1) with f replaced by $\tilde{f}_k e^{2ik\pi t}$ has at least one 1-periodic solution $\tilde{x}_k e^{2ik\pi t}$. Consequently, Eq. (1) with f replaced by f_1 has at least one 1-periodic solution $u_1(t) = \sum_{k=-N}^{N} \tilde{x}_k e^{2ik\pi t}$. By the superposition principle, Eq. (1) has at least one 1-periodic mild solution $u = u_1 + u_2$.

Let x_k be a solution of Eq. (10), $k \in \boldsymbol{Z}$, and $u(t)$ the 1-periodic mild solution for Eq. (1) in the above. Since the relation (11) holds, $\sigma_i(\Delta) \subset [-N, N]$ by Lemma 2.6; hence, if $|k| > N$ $x_k = \tilde{u}_k$. Set

$$v(t) = \sum_{|k| \le N} (x_k - \tilde{u}_k) e^{2ik\pi t}.$$

Since $\Delta(2ik\pi)(x_k - \tilde{u}_k) = 0$, $v(t)$ is a solution of the homogeneous equation of Eq. (1). Thus, $w(t) := u(t) + v(t)$ is a 1-periodic mild solution of Eq. (1). If $|k| > N$, we have that $\tilde{w}_k = \tilde{u}_k = x_k$. If $|k| \le N$, we have that

$$\tilde{w}_k = \tilde{u}_k + \tilde{v}_k = \tilde{u}_k + x_k - \tilde{u}_k = x_k.$$

Hence, $\tilde{w}_k = x_k$ for every $k \in \boldsymbol{Z}$. □

Remark 3.5. Since $\sigma_i(\Delta)$ is bounded, Eq. (10) should have solutions at most at finitely many $k \in \boldsymbol{Z}$, $|k| \le N$, where N depends only of A, F.

Remark 3.6. By the same argument we can prove the above theorem for equations of more general form:

$$\dot{x}(t) = Ax(t) + \int_{-\infty}^{+\infty} dB(\eta)x(t+\eta) + f(t), \qquad t \in \boldsymbol{R}, \tag{12}$$

where A is the generator of an analytic semigroup. This result extends a main result of [17] and [22] to the infinite dimensional case. The analyticity of the semigroup generated by A cannot be dropped due to the failure of the spectral mapping theorem for linear semigroups in the infinite dimensional case (see e.g. [8], [36]). This theorem can be generalized to cover the general case of functional equations discussed in [27]. For periodic functional equations with infinite delay we refer the reader to [40] for a general criterion for the existence of periodic solutions.

In the case that instead of an analytic semigroup the operator A generates a compact semigroup, all conclusions of Theorem 3.4 hold true from the decomposition of the variation of constants formula.

Theorem 3.7. *Let A be the generator of a compact semigroup. Then, Eq.* (1) *has a* 1*-periodic mild solution if and only if for every $k \in \boldsymbol{Z}$, the equation* (10) *has solutions $x \in \boldsymbol{X}$. If x_k is a solution of Eq.* (10) *for $k \in \boldsymbol{Z}$, then $\sum_{k=-\infty}^{\infty} x_k e^{2i\pi kt}$ is the Fourier series of a* 1*-periodic mild solution of Eq.* (1).

Proof. It suffices to prove the sufficiency. To this end, we use the results in the paper [19]. The space $\mathscr{C}$ is decomposed as

$$C = S \oplus U, \qquad V(t)S \subset S, \qquad V(t)U \subset U,$$

where S is a stable subspace for $V(t)$ and U is finite dimensional. Let $u(t)$ be a mild solution on $\boldsymbol{R}$ of Eq. (1), and $\Pi^S : \mathscr{C} \mapsto S$ and $\Pi^U : \mathscr{C} \mapsto U$ be projections corresponding to the decomposition. The solution $u(t)$ is a 1-periodic solution if and only if $\Pi^S u_t$ and $\Pi^U u_t$ are 1-periodic. Let $y(t)$ be the S-valued function defined in [21, p. 346]. Then $y(t)$ is 1-periodic, and $\Pi^S u_t$ is 1-periodic if and only if $\Pi^S u_t = y(t)$ by [21, Propostition 4.1]. Hence $u(t)$ is a 1-periodic mild solution if and only if $\Pi^U u_t$ is 1-periodic. Let $\dim U = d$, and $\Phi = (\phi_1, \dots, \phi_d)$ be a basis vector of U. Then $\Pi^U u_t = \Phi z(t)$ by a vector $z(t) \in \boldsymbol{C}^d$.

By [21, Proposition 4.2] there is a d-column vector $x^* = col(x_1^*, \dots, x_d^*)$, $x_i^* \in X^*$, $i = 1, \dots, d$, such that $z(t)$ is a solution of the ordinary differential equation:

$$\dot{z}(t) = Lz(t) + \langle x^*, f(t) \rangle, \tag{13}$$

where L is a $d \times d$ matrix. Let N be a positive integer such that, if $|k| > N$, then $2ik\pi \notin \sigma(L)$. Set

$$f_1(t) := \sum_{k=-N}^{N} \tilde{f}_k e^{2ik\pi t}, \qquad f_2(t) := f(t) - f_1(t).$$

Consider Eq. (13) with $f(t)$ replaced by $f_2(t)$. Set $g(t) = \langle x^*, f_2(t) \rangle$. Then

$$\tilde{g}_k = \begin{cases} 0, & |k| \le N, \\ \langle x^*, \tilde{f}_k \rangle, & |k| > N. \end{cases}$$

Hence for every $k \in \boldsymbol{Z}$, the equation $(2ik\pi - L)x = \tilde{g}_k$ has a solution $x \in \boldsymbol{C}^d$. By Proposition 3.1, Eq. (13) with $f(t)$ replaced by $f_2(t)$ has a 1-periodic solution. Thus, Eq. (1) with forcing term f_2 has a 1-periodic mild solution.

By repeating the argument of the proof of Theorem 3.4, Eq. (1) has at least one 1-periodic mild solution. Hence Eq. (1) has at least one 1-periodic mild solution.

Since $\sigma_i(\Delta)$ is bounded by Lemma 2.7, the rest part is proved by repeating the argument in the proof of Theorem 3.4. □

Remark 3.8. In the paper [21] the variation of constants formula is proved under the following condition: F is represented as

$$F\phi = \int_{-r}^{0} dB(\eta)\phi(\eta), \qquad \forall \phi \in \mathscr{C}, \tag{14}$$

where $B : [-r, 0] \to L(\boldsymbol{X})$ is of bounded variation with a given positive real $r > 0$. Note that this result has been improved by dropping this condition in the recent paper [28].

Let us consider conditions for all mild solutions of Eq. (1) to be 1-periodic. We first consider conditions for all mild solutions of the equations without delay to be 1-periodic. For this purpose we shall prove the following lemma.

Lemma 3.9. *Let A be the generator of a C_0-semigroup $(T(t))_{t \ge 0}$, and f be 1-periodic. If all mild solutions of*

$$u(t) = T(t)u(0) + \int_0^t T(t-\xi)f(\xi)d\xi, \qquad \forall t \geq 0, \tag{15}$$

are 1*-periodic, all mild solutions of Eq.* (15) *on* $(-\infty, \infty)$ *are also* 1*-periodic.*

Proof. Let u be a mild solution of Eq. (15) on $(-\infty, \infty)$. Since u is a mild solution of Eq. (15) on $[0, \infty)$, $u(t) = u(t+1)$ for $t \geq 0$ from the assumption of this lemma. Take a $t_0 < 0$ arbitrary. We can choose $n_0 \in \boldsymbol{N}$ such that $t_0 + n_0 \geq 0$. Define a function $v(t) := u(t - n_0)$. Since $f(t)$ is 1-periodic, the translated function $v(t)$ is also a mild solution of Eq. (15) on $(-\infty, \infty)$. Hence, $v(t+1) = v(t)$ for $t \geq 0$. In particularly $v(t_0 + n_0 + 1) = v(t_0 + n_0)$, which implies $u(t_0 + 1) = u(t_0)$. Therefore $u(t)$ is also a 1-periodic for $t < 0$. □

Theorem 3.10. *Let A be the generator of an analytic semigroup. Then, it is necessary and sufficient for all mild solutions of Eq.* (15) *on* $[0, \infty)$ *to be* 1*-periodic that the following conditions are satisfied*:

i) *For every $k \in \boldsymbol{Z}$ the equation*

$$(2ik\pi - A)x = \tilde{f}_k \tag{16}$$

has a solution $x \in X$,

ii) $\sigma(A) = \sigma_p(A) \subset 2i\pi\boldsymbol{Z}$, *and the corresponding eigenvectors spans a dense subspace in $\boldsymbol{X}$.*

Proof. Necessity: Since all mild solutions of Eq. (15) on $[0, \infty)$ are 1-periodic, by Lemma 3.9 all mild solutions of Eq. (15) on $(-\infty, \infty)$ are also 1-periodic. By Lemma 3.3, for every $k \in \boldsymbol{Z}$, Eq. (16) is solvable. The superposition principle yields that all mild solutions of $\dot{u}(t) = Au(t)$ are 1-periodic, i.e., $(T(t))_{t \geq 0}$ is 1-periodic. Hence, by [8, Theorem 2.26] $\sigma(A) = \sigma_p(A) \subset 2i\pi\boldsymbol{Z}$ and the set of all eigenvectors of A span a dense subset in $\boldsymbol{X}$.

Sufficiency: Since for every $k \in \boldsymbol{Z}$ Eq. (16) is solvable, by Theorem 3.4, Eq. (15) has at least one 1-periodic mild solution on the whole line. On the other hand, since $\sigma(A) = \sigma_p(A) \subset 2i\pi\boldsymbol{Z}$ and the set of all eigenvectors of A is a dense subset in $\boldsymbol{X}$, by [8, Theorem 2.26], $(T(t))_{t \geq 0}$ is 1-periodic, i.e., all mild solutions of $\dot{u}(t) = Au(t)$ are 1-periodic. Hence, the superposition principle yields that all mild solutions of Eq. (15) on $[0, \infty)$ are 1-periodic. □

Remark 3.11. However, if $1 < r$, it is impossible that all mild solutions of Eq. (1) are 1-periodic. In fact, if for $\phi \in \mathscr{C}$, $V(t)\phi = V(t+1)\phi$, $\forall t \geq 0$, then we have $\phi = V(1)\phi$. Then since $1 < r$, we have $(V(1)\phi)(-1) = \phi(-1)$. Since $V(1)\phi := w_1$, where $w(\cdot)$ is the mild solution of Eq. (6), we have $(V(1)\phi)(-1) = w(0) = \phi(0)$. Thus we have $\phi(0) = \phi(-1)$. In other words, if $\phi(0) \neq \phi(-1)$, then $V(1)\phi \neq \phi$. Hence $(V(t))_{t\geq 0}$ is not 1-periodic, i.e., there should be some mild solutions which are not 1-periodic.

Remark 3.12. If A is the generator of a compact semigroup, then from the well known knowledge of abstract functional differential equations it follows that the solution semigroup $V(t)$ is compact for $t > r$ (see [42, Proposition 2.4]). Consequently, since $\dim \mathscr{C} = \infty$, the identity $V(k)\phi = \phi$, $\forall \phi \in \mathscr{C}$, for some $k \in \boldsymbol{N}$ never holds, i.e., there should be some mild solutions which are not 1-periodic. However, there may happen that all mild solutions of Eq. (1) are asymptotic periodic as shown in the next subsection.

3.2. Conditions for all solutions to be asymptotic periodic.

3.2.1. Necessary conditions. We have the following necessary conditions for all mild solutions to be asymptotic 1-periodic. To this purpose we will use the following proposition.

Proposition 3.13. *Let f be a* 1*-periodic function, and u an asymtotic* 1*-periodic mild solution on $[0, \infty)$ of Eq.* (1)*. If $u(t)$ is decomposed as $u(t) = u^0(t) + u^1(t)$ for $t \geq -r$ such that $\lim_{t\to\infty} u^0(t) = 0$ and $u^1(t)$ is* 1*-periodic on $(-\infty, \infty)$, then $u^0(t)$ is a mild solution on $[0, \infty)$ of Eq.* (5) *and $u^1(t)$ is a mild solution on $(-\infty, \infty)$ of Eq.* (1) *respectively.*

Proof. We will follow the manner in the proof of [1, Proposition 3.4]. Since u is a mild solution on $[0, \infty)$ of Eq. (1), $\int_0^t u(\xi)d\xi \in D(A)$, $\forall t \in [0, \infty)$ and

$$u(t) = u(0) + A\int_0^t u(\xi)d\xi + \int_0^t [Fu_\xi + f(\xi)]d\xi.$$

Take an $n \in \boldsymbol{N}$ such that $n > r$. Then $u^0_{\xi+n} \in \mathscr{C}$ for $\xi > 0$, and we have

$$\begin{aligned}
u^0(t+n) + u^1(t) &= u^0(t+n) + u^1(t+n) = u(t+n) \\
&= u^0(n) + u^1(n) + A\left(\int_n^{t+n} u^0(\xi)d\xi + \int_n^{t+n} u^1(\xi)d\xi\right) \\
&\quad + \int_n^{t+n} Fu^0_\xi \, d\xi + \int_n^{t+n} Fu^1_\xi \, d\xi + \int_n^{t+n} f(\xi)d\xi \\
&= u^0(n) + u^1(0) + A\left(\int_0^t u^0(\xi+n)d\xi + \int_0^t u^1(\xi)d\xi\right) \\
&\quad + \int_0^t Fu^0_{\xi+n} \, d\xi + \int_0^t Fu^1_\xi \, d\xi + \int_0^t f(\xi)d\xi.
\end{aligned}$$

Since A is closed and F is bounded, by taking the limit as $n \to \infty$ we have that $\int_0^t u^1(\xi)d\xi \in D(A)$ and

$$u^1(t) = u^1(0) + A\int_0^t u^1(\xi)d\xi + \int_0^t Fu^1_\xi \, d\xi + \int_0^t f(\xi)d\xi, \qquad t \geq 0.$$

Therefore, u^1 is a mild solution on $[0, \infty)$ of Eq. (1). However, since u^1 and f are 1-periodic functions, u^1 is a mild solution of Eq. (1) on $(-\infty, \infty)$. By linearity, $u^0(t) = u(t) - u^1(t)$, $t \geq -r$, is a mild solution on $[0, \infty)$ of Eq. (5). □

Using Proposition 3.13 and the superposition principle, we derive the following lemma.

Lemma 3.14. *All mild solutions of Eq.* (1) *are asymptotic* 1*-periodic if and only if the following conditions satisfied*:
i) *Eq.* (1) *has a* 1*-periodic mild solution,*
ii) *All mild solutions of Eq.* (5) *are asymptotic* 1*-periodic.*

Proposition 3.15. *Let A be the generator of a C_0-semigroup. If all mild solutions of Eq.* (1) *on* $[0, +\infty)$ *are asymptotic* 1*-periodic, the following conditions hold*:
i) *For every $k \in \boldsymbol{Z}$, Eq.* (10) *has a solution $x \in \boldsymbol{X}$,*
ii) *The solution semigroup $(V(t))_{t \geq 0}$ is uniformly bounded,*
iii) $\sigma_p(\mathscr{G}) \cap i\boldsymbol{R} \subset 2i\pi\boldsymbol{Z}$.

Proof. By Proposition 3.13 there exists a 1-periodic mild solution u on $\boldsymbol{R}$. Hence by Lemma 3.3, Condition (i) is satisfied.

Moreover by Lemma 3.14, all mild solutions of Eq. (5) are asymptotic 1-periodic. For a given $\phi \in \mathscr{C}$, we denote by $w(t, \phi)$ the solution of Eq. (5) such that $w_0 = \phi$. Then $w(t, \phi)$ is decomposed uniquely as $w(t, \phi) = w^0(t, \phi) + w^1(t, \phi)$, for $t \geq -r$, such that $\lim_{t \to \infty} w^0(t, \phi) = 0$ and $w^1(t, \phi)$ is a 1-periodic function on $\boldsymbol{R}$. Then by Proposition 3.13, we have

$$w^i(t, \phi) = w^i(0, \phi) + A\int_0^t w^i(\xi, \phi)d\xi + \int_0^t Fw^i_\xi(\phi)d\xi, \qquad t \geq 0,\ i = 0, 1.$$

We set

$$D_0 = \left\{\phi \in \mathscr{C} : \lim_{s \to \infty} V(s)\phi = 0\right\}, \qquad D_1 = \{\phi \in \mathscr{C} : V(1)\phi = \phi\}.$$

Then D_0 and D_1 are subspaces of $\mathscr{C}$ and $D_0 \cap D_1 = \{0\}$. Moreover it is clear that $V(t)D_0 \subset D_0$ and $V(t)D_1 \subset D_1$. For any $\phi \in \mathscr{C}$, we set $\phi^0 = w^0_0$

and $\phi^1 = w_0^1$. Then we have $\lim_{s\to\infty} V(s)\phi^0 = 0$ and $V(t)\phi^1 = w_t^1 = w_{t+1}^1 = V(t+1)\phi^1$. Hence, $\phi = \phi^0 + \phi^1$, $\phi^0 \in D_0$, $\phi^1 \in D_1$.

If $n \in \boldsymbol{N}$, then

$$\lim_{n\to\infty} V(n)\phi = \lim_{n\to\infty} V(n)\phi^0 + \lim_{n\to\infty} V(n)\phi^1 = \phi^1$$

for every $\phi \in \mathscr{C}$; hence $M := \sup_{n\in\boldsymbol{N}} \|V(n)\|$ is bounded. For $t \geq 0$, we have $V(t) = V([t])V(t-[t])$. Since $0 \leq t - [t] < 1$, it follows that $\|V(t)\| \leq \|V([t])\| \, \|V(t-[t])\| \leq M \sup_{0\leq s<1} \|V(s)\|$. Thus Condition (ii) is satisfied.

Take an $i\xi \in \sigma_p(\mathscr{G}) \cap i\boldsymbol{R}$. Then, we have $\mathscr{G}\phi = i\xi\phi$, for some $\phi \in \mathscr{C}$, $\phi \neq 0$. Thus

$$V(t)\phi = e^{i\xi t}\phi = e^{i\xi t}\phi^0 + e^{i\xi t}\phi^1.$$

Since D_0 and D_1 are subspaces of $\mathscr{C}$, we have $e^{i\xi t}\phi^0 \in D_0$ and $e^{i\xi t}\phi^1 \in D_1$. On the other hand, we can rewrite $V(t)\phi = V(t)\phi^0 + V(t)\phi^1$, where $V(t)\phi^0 \in D^0$ and $V(t)\phi^1 \in D^1$. Therefore, $V(t)\phi^0 = e^{i\xi t}\phi^0$ and $V(t)\phi^1 = e^{i\xi t}\phi^1$. Since $\phi^0 = \lim_{s\to\infty} e^{-i\xi s} V(s)\phi^0 = 0$, we have $\phi = \phi^1$. Consequently,

$$\phi = \phi^1 = V(1)\phi^1 = e^{i\xi}\phi^1 = e^{i\xi}\phi.$$

Since $\phi \neq 0$, we have $i\xi \in 2i\pi\boldsymbol{Z}$; Condition (iii) is satisfied. □

3.2.2. Sufficient conditions. For sufficient conditions for all mild solutions to be asymptotic 1-periodic we have the following results.

For compact semigroups $(T(t))_{t\geq 0}$, the conditions turn out to be simple as follows.

Proposition 3.16. *Let A be the generator of a compact semigroup. Assume further that the following conditions are satisfied:*

i) *For every $k \in \boldsymbol{Z}$, Eq.* (10) *has a solution $x \in \boldsymbol{X}$,*

ii) *The solution semigroup $(V(t))_{t\geq 0}$ is uniformly bounded,*

iii) $\sigma_p(\mathscr{G}) \cap i\boldsymbol{R} \subset 2i\pi\boldsymbol{Z}$.

Then, all mild solutions of Eq. (1) *on* $[0, +\infty)$ *are asymptotic* 1*-periodic.*

Proof. Condition (i) guarantees the existence of a 1-periodic mild solution on the whole line. Since A is the generator of a compact semigroup, the operator $V(t)$ is compact for $t > r$. Hence, $\sigma(\mathscr{G}) \cap \{\lambda : \operatorname{Re}\lambda \geq 0\}$ consists of finite number of normal eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_p$. Moreover, $\mathscr{C}$ is decomposed as follows:

$$\mathscr{C} = S \oplus U, \qquad U = U_1 \oplus U_2 \oplus \cdots \oplus U_p,$$

where $U_j = N((\mathscr{G} - \lambda_j I)^{m_j})$, $j = 1, 2, \ldots, p$, and S is the stable subspace of

$V(t)$. Condition (ii) implies $V^{U_j}(t)$ is uniformly bounded for $j = 1, 2, \dots, p$; thus, $\operatorname{Re}\lambda_j = 0$. Since, for $\phi \in U_j$,

$$\|V^{U_j}(t)\phi\| = \left\| \sum_{k=0}^{m_j-1} \frac{t^k}{k!} (\mathscr{G} - \lambda_j)^k \phi \right\|,$$

$\|V^{U_j}(t)\phi\|$ is bounded if and only if $(\mathscr{G} - \lambda_j)\phi = 0$; that is $m_j = 1$, and $V^{U_j}(t)\phi = e^{\lambda_j t}\phi$. Since $\lambda_j \in 2i\pi\boldsymbol{Z}$ by Condition (iii), $(V^{U_j}(t))_{t\geq 0}$ is 1-periodic. Hence, $V(t)\phi = V^S(t)\Pi^S\phi + V^U(t)\Pi^U\phi$ is asymptotic 1-periodic. Therefore, by Lemma 3.14, this shows that all mild solutions of Eq. (1) are asymptotic 1-periodic. □

If $(T(t))_{t\geq 0}$ is an analytic semigroup, we need an additional condition. To this purpose, we now prove the following lemma.

Lemma 3.17. *Let z be a bounded mild solution of $\dot{z}(t) = \mathscr{G}z(t)$. Then we have* $\operatorname{sp}(z) \subset \sigma_i(\mathscr{G}) := \{\xi \in \boldsymbol{R} : i\xi \in \sigma(\mathscr{G})\}$.

Proof. We will follow the manner in the proof of [20, Lemma 2.21]. Since z is a mild solution of $\dot{z}(t) = \mathscr{G}z(t)$, the Fourier-Carleman transform of z satisfies

$$(\lambda I - \mathscr{G})\hat{z}(\lambda) = -z(0),$$

where $\operatorname{Re}\lambda \neq 0$. Assume $i\xi \in \rho(\mathscr{G})$. Then $(\lambda I - \mathscr{G})^{-1}$ is holomorphic in a neighborhood of $i\xi$. Hence $\hat{z}(\lambda)$ has a holomorphic extension on a neighborhood of $i\xi$, i.e., $\xi \notin \operatorname{sp}(z)$. □

Theorem 3.18. *Let A be the generator of an analytic semigroup. Assume further that the following conditions are satisfied:*

i) *For every $k \in \boldsymbol{Z}$, Eq.* (10) *has a solution $x \in \boldsymbol{X}$,*
ii) *The solution semigroup $(V(t))_{t\geq 0}$ is uniformly bounded,*
iii) $\sigma_i(\mathscr{G}) \subset 2\pi\boldsymbol{Z}$,
iv) *For every $\omega \in \sigma_i(\mathscr{G})$ the limit $\lim_{t\to\infty} t^{-1}\int_0^t e^{-i\omega s}V(s)\phi\,ds$ exists for every $\phi \in \mathscr{C}$; or $R(\mathscr{G} - i\omega) + N(\mathscr{G} - i\omega)$ is dense in $\mathscr{C}$ for all $\omega \in \sigma_i(\mathscr{G})$.*

Then, all mild solutions of Eq. (1) *on* $[0, \infty)$ *are asymptotic* 1*-periodic.*

Proof. First, the existence of a 1-periodic mild solution $u(\cdot)$ to Eq. (1) is guaranteed by Condition (i). On the other hand, Condition (ii), (iii) and (iv) imply that the semigroup $(V(t))_{t\geq 0}$ is asymptotic almost periodic by Theorem 3.21 in Appendix. By Theorem 3.22 in Appendix there exists a decomposition of the space $\mathscr{C} = \mathscr{C}_0 \oplus \mathscr{C}_1$ such that for every $\phi \in \mathscr{C}_0$ $\lim_{t\to\infty} V(t)\phi = 0$ and $\mathscr{C}_1$ is $(V(t))_{t\geq 0}$-invariant and $(V(t)|_{\mathscr{C}_1})_{t\geq 0}$ can be extended to a bounded group. For every $\phi \in \mathscr{C}_1$ denote $z(t) := V(t)|_{\mathscr{C}_1}\phi$. Then, z is bounded and uniformly continuous on $\boldsymbol{R}$. By Lemma 3.17, we have $\mathrm{sp}(z) \subset \sigma_i(\mathcal{G})$. Hence, by Condition (iii) it is 1-periodic. Consequentry $(V(t))_{t\geq 0}$ is asymptotic 1-periodic. By Lemma 3.14, this shows that all mild solutions of Eq. (1) should be the sum of the 1-periodic mild solution $u(\cdot)$ and asymptotic 1-periodic mild solution of Eq. (5). □

To illustrate the above abstract results we will give below an example in which the conditions (ii) and (iii) in Prroposition 3.16 can be verified.

Example 3.19.
Consider the equation

$$(17)\qquad \begin{cases} w_t(x,t) = w_{xx}(x,t) + w(x,t) - (\pi/2)w(x,t-1) + f(x,t), \\ \qquad 0 \leq \forall x \leq \pi,\ \forall t \geq 0, \\ w(0,t) = w(\pi,t) = 0, \quad \forall t > 0, \end{cases}$$

where $w(x,t), f(x,t)$ are scalar-valued functions. We define the space $\boldsymbol{X} := L^2[0,\pi]$ and $A_T : \boldsymbol{X} \to \boldsymbol{X}$ by the formula

$$(18)\qquad \begin{cases} A_T y = y'' + y, \\ D(A_T) = \{y \in \boldsymbol{X} : y, y' \text{ are absolutely continuous, } y'' \in \boldsymbol{X}, \\ \qquad\qquad y(0) = y(\pi) = 0\}. \end{cases}$$

We define $F : \mathscr{C} \to \boldsymbol{X}$ by the formula $F(\varphi) = -(\pi/2)\varphi(-1)$. In this case the evolution equation we are concerned with is the following

$$(19)\qquad \frac{du(t)}{dt} = A_T u(t) + Fu_t + f(t), \qquad u(t) \in \boldsymbol{X},$$

where A_T is the infinitesimal generator of a compact and analytic semigroup $(T(t))_{t\geq 0}$ in $\boldsymbol{X}$ (see [42, p. 414]). Moreover, the eigenvalues of A_T are $1 - n^2$, $n = 1, 2, \ldots$, and since the set $\sigma_i(\Delta)$ is determined from the set of imaginary solutions of the equation

$$(20)\qquad \lambda + \pi/2e^{-\lambda} = 1 - n^2, \qquad n = 1, 2, \ldots,$$

a simple computation shows $\sigma_i(\Delta) = \{-\pi/2, \pi/2\} =: \Lambda$. As is shown in [42, Lemma 5.8], $(\mathcal{G} - \lambda I)^{-1}$ has simple poles at Λ, where $\mathcal{G}$ is the generator of the solution semigroup $(V(t))_{t\geq 0}$. The space $\mathscr{C}$ is decomposed as $\mathscr{C} = N(\mathcal{G} - i\pi/2) \oplus N(\mathcal{G} + i\pi/2) \oplus Q_\Lambda$, where $Q_\Lambda = R(\mathcal{G} - i\pi/2) \cap R(\mathcal{G} + i\pi/2)$. There exsist positive K and ω such that $\|V(t)\phi\| \leq Ke^{-\omega t}\|\phi\|$ for $\phi \in Q_\Lambda$; $V(t)\phi = e^{\pm i\pi t/2}\phi$ for $\phi \in N(\mathcal{G} - (\pm i\pi/2))$. Hence, $(V(t))_{t\geq 0}$ is asymptotic 4-periodic. Let $f(x,t)$ be 4-periodic. Then $\mathrm{sp}(f) \subset \pi\boldsymbol{Z}/2$ (here we consider f as the function $t \mapsto f(t) := f(\cdot, t) \in \boldsymbol{X}$). By our theory, it is necessary and sufficient for all mild solutions of Eq. (19) to be asymptotic 4-periodic that the following equations are solvable

$$\begin{cases} \Delta(i\pi/2)u = \frac{1}{4}\int_0^4 e^{-i\pi t/2} f(t)dt, \\ \Delta(-i\pi/2)u = \frac{1}{4}\int_0^4 e^{i\pi t/2} f(t)dt. \end{cases}$$

Moreover, if u_1, u_{-1} are solutions of equations in the above, respectively, then

$$u_1 e^{i\pi t/2} + u_{-1} e^{-i\pi t/2} + \frac{1}{4} \sum_{k \neq \pm 1} e^{ik\pi t/2} \Delta^{-1}(ik\pi/2) \int_0^4 e^{-ik\pi t/2} f(t)dt$$

is the Fourier series of a 4-periodic mild solution of Eq. (19).

Appendix

In this Appendix for the reader's convenience, we collect some known notions and results on asymptotic almost periodic semigroups and the splitting Theorem of Glicksberg and DeLeeuw which we have used above (more details can be found in [35, Chap. 5, §7]).

Definition 3.20. A C_0-semigroup $(T(t))_{t \geq 0}$ on $\boldsymbol{X}$ is said to be *asymptotic almost periodic* if for each $x \in \boldsymbol{X}$ the set $\{T(t)x, t \in [0, +\infty)\}$ is relatively compact in $\boldsymbol{X}$. (Originally, in [35, Chap. 5] this notion is referred to as the notion of almost periodic semigroups. To distiguish this notion from our mentioned one we refer to it as the notion of asymptotic almost periodic semigroups.)

Theorem 3.21 ([35, Theorem 5.7.10]). *Let $(T(t))_{t \geq 0}$ be a uniformly bounded C_0-semigroup on a Banach space $\boldsymbol{X}$, with generator A, and assume that $\sigma(A) \cap i\boldsymbol{R}$ is countable. Then the following assertions are equivalent:*

i) *$(T(t))_{t \geq 0}$ is asymptotic almost periodic,*

ii) *For every $i\omega \in \sigma(A) \cap i\boldsymbol{R}$ the limit $\lim_{t \to \infty} t^{-1} \int_0^t e^{-i\omega s} T(s)x \, ds$ exists for every $x \in \boldsymbol{X}$,*

iii) *For every $i\omega \in \sigma(A) \cap i\boldsymbol{R}$, $R(A - i\omega) + N(A - i\omega)$ is dense in $\boldsymbol{X}$.*

The following is referred to as the splitting Theorem of Glicksberg and DeLeeuw.

Theorem 3.22 ([35], Theorem 5.7.7]). *Let $(T(t))_{t \geq 0}$ be an asymptotic almost periodic C_0-semigroup on a Banach space $\boldsymbol{X}$. Then there exists a direct sum decomposition $\boldsymbol{X} = X_0 \oplus X_1$ of $(T(t))_{t \geq 0}$-invariant subspaces, where*

$$X_0 = \left\{ x \in \boldsymbol{X} : \lim_{t \to \infty} \|T(t)x\| = 0 \right\}$$

and X_1 is the closed linear span of all eigenvectors of the generator A with purely imaginary eigenvalues. Moreover, the restriction of $(T(t))_{t\geq 0}$ to X_1 extends to an almost periodic C_0-group on X_1. If $(T(t))_{t\geq 0}$ is contractive, this group is isometric.

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