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New Expressions for Discrete Painlevé Equations

By

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Abstract. We present a new expression for the elliptic-difference Painlevé equation. As in our construction all parameters in their equations appear in a symmetric way, the permutation symmetry of the equations is immediately apparent. We present expressions for other discrete Painlevé equations, obtained in a similar way.

Key Words and Phrases. Integrable systems, Difference equations, Painlevé-type functions.

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1. Introduction

Discrete Painlevé equations are studied from various points of view as integrable systems [2], [7]. They are discrete equations which are reduced to the Painlevé differential equations in a suitable limiting process, and moreover, which pass the singularity confinement test. Passing this test can be thought of as a difference version of the Painlevé property. The Painlevé differential equations were derived as second order ordinary differential equations whose solutions have no movable singularities other than poles. This property is called the Painlevé property. The singularity confinement test has been proposed by Grammaticos et al. as a criterion for the integrability of discrete dynamical systems [1]. It demands that singularities depending on particular initial values should disappear after a finite number of iteration steps, in which case the information about the initial values ought to be recovered.

H. Sakai constructed all difference Painlevé equations from the point of view as algebraic geometry [8]. Surfaces obtained by successive blow-ups of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ have been studied by means of connections between Weyl groups and groups of Cremona isometries of the Picard group of surfaces. The Picard group of a rational surface X is the group of isomorphism classes of invertible sheaves on X and is isomorphic to the group of linear equivalence classes of divisors on X. A Cremona isometry is an isomorphism of the Picard group that i) preserves the intersection number of any pair of divisors, ii) preserves the canonical divisor $\mathscr{K}_X$ and iii) leaves the set of effective classes of divisors invariant. In the case where eight points in general positions are blown up in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$, the group of Cremona isometries of X is isomorphic to an extension of the Weyl group of type $E_8^{(1)}$. Birational mappings on $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ are obtained by subsequent blow downs. Discrete Painlevé equations are recovered as the birational mappings corresponding to translations of affine Weyl groups.

Discrete Painlevé equations were classified on the basis of the types of rational surfaces connected to extended affine Weyl groups and some new equations were discovered in the process. See Table 1 [8].

Table 1: Classification of generalized Halphen surfaces with $\dim|-\mathscr{K}_X| = 0$

type	surface (symmetry)
Elliptic type	$A_0^{(1)}(E_8^{(1)})$
Multiplicative type	$A_0^{(1)*}(E_8^{(1)})$ $A_1^{(1)}(E_7^{(1)})$ $A_2^{(1)}(E_6^{(1)})$ $A_3^{(1)}(D_5^{(1)})$
	$A_4^{(1)}(A_4^{(1)})$ $A_5^{(1)}((A_2+A_1)^{(1)})$ $A_6^{(1)}((A_1 + \underset{\lvert\alpha\rvert^2=14}{A_1})^{(1)})$
	$A_7^{(1)}(\underset{\lvert\alpha\rvert^2=8}{A_1^{(1)}})$ $A_7^{(1)\prime}(A_1^{(1)})$ $A_8^{(1)}(A_0^{(1)})$
Additive type	$A_0^{(1)**}(E_8^{(1)})$ $A_1^{(1)*}(E_7^{(1)})$ $A_2^{(1)*}(E_6^{(1)})$
	$D_4^{(1)}(D_4^{(1)})$ $D_5^{(1)}(A_3^{(1)})$ $D_6^{(1)}((2A_1)^{(1)})$
	$D_7^{(1)}(\underset{\lvert\alpha\rvert^2=4}{A_1^{(1)}})$ $D_8^{(1)}(A_0^{(1)})$
	$E_6^{(1)}(A_2^{(1)})$ $E_7^{(1)}(A_1^{(1)})$ $E_8^{(1)}(A_0^{(1)})$

These equations are organized in a degeneration pattern obtained through coalescence. The $A_0^{(1)}$-surface discrete Painlevé equation ($dP(A_0^{(1)})$) is the most generic one among these, because the equation has Weyl group symmetry of type $E_8^{(1)}$ and any of the other discrete Painlevé equations can be obtained from this equation by limiting procedure. The form of this equation is however very complicated as its coefficients are described in terms of elliptic functions [5], [6].

Kajiwara et al. gave an explicit form of $dP(A_0^{(1)})$ based on a τ function formalism for the elliptic discrete Painlevé equation [4].

In this paper a new representation of $dP(A_0^{(1)})$ based on total transforms is presented. Kajiwara et al.'s form is described in terms of coordinates in $\boldsymbol{P}^2$, while our form is described by means of $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Both descriptions are related by a bitational transform [5]. However, as in our construction all eight points (that are blown up in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$) appear in a symmetric way, the $\mathfrak{S}_8$-symmetry of $dP(A_0^{(1)})$ is immediately apparent.

In Section 2, we present the representation of $dP(A_0^{(1)})$. In Section 3–8, we present expressions for other discrete Painlevé equations, obtained in a similar way. This is made possible by the fact that the limit process is a projective transformation of dependent variables.

Consequently, it can be seen that knowledge of the blow-up of eight points in $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ and of a trivial solution on a curve passing through these points allows one to construct a discrete Painlevé equation in explicit form.

2. $A_0^{(1)}$-surface

In this section, we give a new representation of the $A_0^{(1)}$-surface discrete Painlevé equation ($dP(A_0^{(1)})$). First we present some useful notation. Throughout this paper we follow the notation from [5] and [8].

We construct the $A_0^{(1)}$-surface by blowing up $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at eight points p_i $(i = 1, \dots, 8)$. There exists an elliptic curve that passes through generic eight points. We parametrize these eight points and the curve as follows:

$$(2.1)\qquad (f_1 g_0 + f_0 g_1 + \wp(2t) f_0 g_0)(4\wp(2t) f_1 g_1 - g_3 f_0 g_0) = \left(f_1 g_1 + \wp(2t)(f_1 g_0 + f_0 g_1) + \frac{g_2}{4} f_0 g_0\right)^2,$$

$$(2.2)\qquad p_i : (f_{0i} : f_{1i}, g_{0i} : g_{1i}) = (1 : \wp(b_i + t), 1 : \wp(t - b_i)) \qquad (i = 1, \dots, 8),$$

where g_2 and g_3 are parameters of Weierstrass $\wp$-function. Here the eight points p_i depend on a variable t:

$$(f_{0i} : f_{1i}, g_{0i} : g_{1i}) = (f_{0i}(t) : f_{1i}(t), g_{0i}(t) : g_{1i}(t)) \qquad (i = 1, \dots, 8).$$

We then define the following points:

$$(\overline{f_{0i}} : \overline{f_{1i}}, \overline{g_{0i}} : \overline{g_{1i}}) = (f_{0i}(\bar{t}) : f_{1i}(\bar{t}), g_{0i}(\bar{t}) : g_{1i}(\bar{t})) \qquad (i = 1, \dots, 8),$$

$$(\underline{f_{0i}} : \underline{f_{1i}}, \underline{g_{0i}} : \underline{g_{1i}}) = (f_{0i}(\underline{t}) : f_{1i}(\underline{t}), g_{0i}(\underline{t}) : g_{1i}(\underline{t})) \qquad (i = 1, \dots, 8),$$

where $\bar{t} = t + \lambda$, $\underline{t} = t - \lambda$, $\lambda = \frac{1}{2}\sum_{i=1}^{8} b_i$.

$dP(A_0^{(1)})$ has the following trivial solution moving on the elliptic curve (2.1) [5]:

$$(2.3)\qquad (f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (1 : \wp(q + 2t^2/\lambda + t), 1 : \wp(t - q - 2t^2/\lambda)),$$

where q is a constant determined by the initial condition. We define the following points:

$$(\overline{f_{0c}} : \overline{f_{1c}}, \overline{g_{0c}} : \overline{g_{1c}}) = (f_{0c}(\bar{t}) : f_{1c}(\bar{t}), g_{0c}(\bar{t}) : g_{1c}(\bar{t})),$$

$$(\underline{f_{0c}} : \underline{f_{1c}}, \underline{g_{0c}} : \underline{g_{1c}}) = (f_{0c}(\underline{t}) : f_{1c}(\underline{t}), g_{0c}(\underline{t}) : g_{1c}(\underline{t})).$$

We now introduce the vectors

$$w_{4,1}(f_0 : f_1, g_0 : g_1) = {}^t(f_1^4 g_1, f_0 f_1^3 g_1, f_0^2 f_1^2 g_1, f_0^3 f_1 g_1, f_0^4 g_1, f_1^4 g_0, f_0 f_1^3 g_0, f_0^2 f_1^2 g_0, f_0^3 f_1 g_0, f_0^4 g_0),$$

$$v = w_{4,1}(f_0 : f_1, g_0 : g_1), \qquad \check{v} = w_{4,1}(f_0 : f_1, \overline{g_0} : \overline{g_1}),$$

$$u = w_{4,1}(g_0 : g_1, f_0 : f_1), \qquad \hat{u} = w_{4,1}(g_0 : g_1, \underline{f_0} : \underline{f_1}),$$

$$v_i = w_{4,1}(f_{0i} : f_{1i}, g_{0i} : g_{1i}), \qquad \check{v}_i = w_{4,1}(f_{0i} : f_{1i}, \overline{g_{0i}} : \overline{g_{1i}}),$$

$$u_i = w_{4,1}(g_{0i} : g_{1i}, f_{0i} : f_{1i}), \qquad \hat{u}_i = w_{4,1}(g_{0i} : g_{1i}, \underline{f_{0i}} : \underline{f_{1i}}) \qquad (i = 1, \dots, 8),$$

$$v_c = w_{4,1}(f_{0c} : f_{1c}, g_{0c} : g_{1c}), \qquad \check{v}_c = w_{4,1}(f_{0c} : f_{1c}, \overline{g_{0c}} : \overline{g_{1c}}),$$

$$u_c = w_{4,1}(g_{0c} : g_{1c}, f_{0c} : f_{1c}), \qquad \hat{u}_c = w_{4,1}(g_{0c} : g_{1c}, \underline{f_{0c}} : \underline{f_{1c}}).$$

$$w_{3,1}(f_0 : f_1, g_0 : g_1) = {}^t(f_1^3 g_1, f_0 f_1^2 g_1, f_0^2 f_1 g_1, f_0^3 g_1, f_1^3 g_0, f_0 f_1^2 g_0, f_0^2 f_1 g_0, f_0^3 g_0),$$

$$\phi_i = w_{3,1}(f_{0i} : f_{1i}, g_{0i} : g_{1i}), \qquad \check{\phi}_i = w_{3,1}(f_{0i} : f_{1i}, \overline{g_{0i}} : \overline{g_{1i}}),$$

$$\psi_i = w_{3,1}(g_{0i} : g_{1i}, f_{0i} : f_{1i}), \qquad \hat{\psi}_i = w_{3,1}(g_{0i} : g_{1i}, \underline{f_{0i}} : \underline{f_{1i}}) \qquad (i = 1, \dots, 8).$$

We describe the following theorem by using these vectors.

Theorem 1. *$dP(A_0^{(1)})$ can be written as*

$$\begin{aligned}
\text{(2.4a)} \quad & \det(v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_c) \det(\check{v}, \check{v}_1, \check{v}_2, \check{v}_3, \check{v}_4, \check{v}_5, \check{v}_6, \check{v}_7, \check{v}_8, \check{v}_c) \\
&= \det(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8) \det(\check{\phi}_1, \check{\phi}_2, \check{\phi}_3, \check{\phi}_4, \check{\phi}_5, \check{\phi}_6, \check{\phi}_7, \check{\phi}_8) \\
&\quad \times \prod_{i=1}^{8} (f_{0c} f_{1i} - f_{1c} f_{0i}) \times (g_{0c} g_1 - g_{1c} g_0)(\overline{g_{0c}}\,\overline{g_1} - \overline{g_{1c}}\,\overline{g_0}) \\
&\quad \times \prod_{i=1}^{8} (f_{0i} f_1 - f_{1i} f_0),
\end{aligned}$$

$$\begin{aligned}
\text{(2.4b)} \quad & \det(u, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_c) \det(\hat{u}, \hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5, \hat{u}_6, \hat{u}_7, \hat{u}_8, \hat{u}_c) \\
&= \det(\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6, \psi_7, \psi_8) \det(\hat{\psi}_1, \hat{\psi}_2, \hat{\psi}_3, \hat{\psi}_4, \hat{\psi}_5, \hat{\psi}_6, \hat{\psi}_7, \hat{\psi}_8) \\
&\quad \times \prod_{i=1}^{8} (g_{0c} g_{1i} - g_{1c} g_{0i}) \times (f_{0c} f_1 - f_{1c} f_0)(\underline{f_{0c}}\,\underline{f_1} - \underline{f_{1c}}\,\underline{f_0}) \\
&\quad \times \prod_{i=1}^{8} (g_{0i} g_1 - g_{1i} g_0).
\end{aligned}$$

For later use, we calculate the determinants A and B given by:

(2.5a) $$A = \det(\phi_1,\phi_2,\phi_3,\phi_4,\phi_5,\phi_6,\phi_7,\phi_8)\det(\check{\phi}_1,\check{\phi}_2,\check{\phi}_3,\check{\phi}_4,\check{\phi}_5,\check{\phi}_6,\check{\phi}_7,\check{\phi}_8) \times \prod_{i=1}^{8}(f_{0c}f_{1i}-f_{1c}f_{0i}),$$

(2.5b) $$B = \det(\psi_1,\psi_2,\psi_3,\psi_4,\psi_5,\psi_6,\psi_7,\psi_8)\det(\hat{\psi}_1,\hat{\psi}_2,\hat{\psi}_3,\hat{\psi}_4,\hat{\psi}_5,\hat{\psi}_6,\hat{\psi}_7,\hat{\psi}_8) \times \prod_{i=1}^{8}(g_{0c}g_{1i}-g_{1c}g_{0i}),$$

for which we find:

(2.6a) $$A = \frac{\sigma(4t)^4\sigma(4t+2\lambda)^4\prod_{\substack{i,j=1,\dots,8\\ i<j}}\sigma(b_i-b_j)^2}{\sigma(q+2t^2/\lambda+t)^{16}} \times \prod_{i=1}^{8}\frac{\sigma(q+2t^2/\lambda-b_i)\sigma(q+2t^2/\lambda+b_i+2t)}{\sigma(b_i+t)^{14}\sigma(t-b_i)^2\sigma(t+\lambda-b_i)^2},$$

(2.6b) $$B = \frac{\sigma(4t)^4\sigma(4t-2\lambda)^4\prod_{\substack{i,j=1,\dots,8\\ i<j}}\sigma(b_i-b_j)^2}{\sigma(t-q-2t^2/\lambda)^{16}} \times \prod_{i=1}^{8}\frac{\sigma(q+2t^2/\lambda-b_i)\sigma(2t-q-2t^2/\lambda-b_i)}{\sigma(t-b_i)^{14}\sigma(b_i+t)^2\sigma(b_i+t-\lambda)^2}.$$

$dP(A_0^{(1)})$ is a birational action of a translation contained in $W(E_8^{(1)})$. After we show the action on total transforms, we will construct the birational action.

We can parametrize an isomorphism class of surfaces by using the period mapping. The period mapping maps the elements of the second homology to $\boldsymbol{C}$. Let ω be a meromorphic 2-form on X which has a divisor D with $\mathrm{div}(\omega) = -D$. Then ω determines a period mapping $\hat{\chi}: H_2(X-D,\boldsymbol{Z}) \to \boldsymbol{C}$ which sends $\Gamma \in H_2(X-D,\boldsymbol{Z})$ to $\int_\Gamma \omega$. Now, there exists a short exact sequence:

$$0 \to H_1(D,\boldsymbol{Z}) \to H_2(X-D,\boldsymbol{Z}) \to Q(E_8^{(1)}) \to 0,$$

where $Q(E_8^{(1)}) = \sum_{i=0}^{8}\boldsymbol{Z}\alpha_i$ is the root lattice of type $E_8^{(1)}$. So we obtain the mapping

$$\chi: Q(E_8^{(1)}) \to \boldsymbol{C} \mod \hat{\chi}(H_1(D,\boldsymbol{Z}))$$

through the period mapping $\hat{\chi}$. In this case, the parametrization is

$$\chi(\alpha_1) = -4t,$$

$$\chi(\alpha_2) = b_1 + b_2 + 2t,$$

$$\chi(\alpha_i) = b_i - b_{i-1} \qquad (i = 3, \dots, 7),$$

$$\chi(\alpha_8) = b_2 - b_1,$$

$$\chi(\alpha_0) = b_8 - b_7.$$

Here $Q(E_8^{(1)})$ is realized in $\mathrm{Pic}(X) = H_2(X, \boldsymbol{Z})$. And the root basis is represented by elements of the Picard group as follows:

$$\alpha_1 = H_1 - H_0,$$

$$\alpha_2 = H_0 - E_1 - E_2,$$

$$\alpha_i = E_{i-1} - E_i \qquad (i = 3, \dots, 7),$$

$$\alpha_8 = E_1 - E_2,$$

$$\alpha_0 = E_7 - E_8.$$

We denote the total transform of $c_0 f_1 - c_1 f_0 = 0$ where $(c_0 : c_1)$ is constant, (or $d_0 g_1 - d_1 g_0 = 0$ where $(d_0 : d_1)$ is constant) on X by H_0 (or H_1 respectively) and the total transform of the point p_i by E_i. The Picard group $\mathrm{Pic}(X)$ and canonical divisor $\mathscr{K}_X$ are

$$\mathrm{Pic}(X) = \boldsymbol{Z}H_0 + \boldsymbol{Z}H_1 + \sum_{i=1}^{8} \boldsymbol{Z}E_i, \qquad \mathscr{K}_X = -2H_0 - 2H_1 + \sum_{i=1}^{8} E_i,$$

where the intersection numbers of pairs of base elements are

$$H_i \cdot H_j = 1 - \delta_{i,j}, \quad E_i \cdot E_j = -\delta_{i,j}, \quad H_i \cdot E_j = 0, \qquad \text{where } \delta_{i,j} = \begin{cases} 1 & i = j, \\ 0 & i \neq j. \end{cases}$$

The generators of the affine Weyl group $W(E_8^{(1)}) = \langle w_i \ (i = 0, 1, \dots, 8) \rangle$ will act on the total transforms. We give a representation of these actions that will enable us to construct $dP(A_0^{(1)})$.

$$w_1 : (H_0, H_1) \mapsto (H_1, H_0),$$

$$w_2 : (H_1, E_1, E_2) \mapsto (H_1 + H_0 - E_1 - E_2, H_0 - E_2, H_0 - E_1),$$

$$w_i : (E_{i-1}, E_i) \mapsto (E_i, E_{i-1}) \qquad (i = 3, \dots, 7),$$

$$w_8 : (E_1, E_2) \mapsto (E_2, E_1),$$

$$w_0 : (E_7, E_8) \mapsto (E_8, E_7).$$

By taking a translation contained in $W(E_8^{(1)})$, we obtain a nonlinear difference equation. The translation can be described by a product of simple reflections w_i:

$$dP(A_0^{(1)}) = w_1 \circ r^{-1} \circ w_1 \circ r : (b_i, t, f_0 : f_1, g_0 : g_1) \tag{2.7}$$
$$\mapsto (b_i, t+\lambda, \overline{f_0} : \overline{f_1}, \overline{g_0} : \overline{g_1}) \qquad (i = 1, \dots, 8),$$
$$\lambda = \frac{1}{2}\sum_{i=1}^{8} b_i,$$

where

$$r = w_2 \circ w_3 \circ w_4 \circ w_5 \circ w_6 \circ w_7 \circ w_0 \circ w_8 \circ w_3 \circ w_4 \circ w_5 \circ w_6 \circ w_7 \circ w_2 \tag{2.8}$$
$$\circ\, w_3 \circ w_4 \circ w_5 \circ w_6 \circ w_8 \circ w_3 \circ w_4 \circ w_5 \circ w_2 \circ w_3 \circ w_4 \circ w_8 \circ w_3 \circ w_2.$$

Elements r and $(w_1 \circ r^{-1} \circ w_1)^{-1} = w_1 \circ r \circ w_1$ of the affine Weyl group act on these total transforms as

$$r : (H_0, H_1, E_i) \mapsto \left(H_0, H_1 + 4H_0 - \sum_{i=1}^{8} E_i, H_0 - E_{9-i}\right), \tag{2.9a}$$

$$w_1 \circ r \circ w_1 : (H_0, H_1, E_i) \mapsto \left(H_0 + 4H_1 - \sum_{i=1}^{8} E_i, H_1, H_1 - E_{9-i}\right). \tag{2.9b}$$

Equations (2.4) are uniquely determined by the condition that the equation is a realization of the map (2.9) and that if we restrict coordinates $(f_0 : f_1, g_0 : g_1)$ on the elliptic curve (2.1) the transformation of this coordinates coincides the trivial solution (2.3). The following gives a presentation of the actions of r and $w_1 \circ r \circ w_1$ on coordinates and parameters:

$$r : (f_0 : f_1, g_0 : g_1, f_{0i} : f_{1i}, g_{0i} : g_{1i}) \tag{2.10a}$$
$$\mapsto (f_0 : f_1, \overline{g_0} : \overline{g_1}, f_{0(9-i)} : f_{1(9-i)}, \overline{g_{0(9-i)}} : \overline{g_{1(9-i)}}),$$

$$w_1 \circ r \circ w_1 : (f_0 : f_1, g_0 : g_1, f_{0i} : f_{1i}, g_{0i} : g_{1i}) \tag{2.10b}$$
$$\mapsto (\underline{f_0} : \underline{f_1}, g_0 : g_1, \underline{f_{0(9-i)}} : \underline{f_{1(9-i)}}, g_{0(9-i)} : g_{1(9-i)}).$$

In order to help us to understand Theorem 1, we show a presentation of the action of w_2 which we understand more easily on coordinates and some parameters:

$$\begin{aligned} w_2\colon\quad (f_0 : f_1, g_0 : g_1) &\mapsto (f_0 : f_1, \tilde{g}_0 : \tilde{g}_1),\\ (f_{0i} : f_{1i}, g_{0i} : g_{1i}) &\mapsto (f_{0(3-i)} : f_{1(3-i)}, \tilde{g}_{0(3-i)} : \tilde{g}_{1(3-i)})\\ &= (1 : \wp(b_{3-i} + t), 1 : \wp(2t - (b_i - b_{3-i})/2) \qquad (i = 1, 2),\\ (f_{0c} : f_{1c}, g_{0c} : g_{1c}) &\mapsto (f_{0c} : f_{1c}, \tilde{g}_{0c} : \tilde{g}_{1c})\\ &= (1 : \wp(q + 2t^2/\lambda + t), 1 : \wp(-(b_1 + b_2)/2 - q - 2t^2/\lambda)). \end{aligned}$$

The coordinates $(\tilde{g}_0 : \tilde{g}_1)$ satisfies

$$\begin{vmatrix} f_1g_1 & f_0g_1 & f_1g_0 & f_0g_0\\ f_{11}g_{11} & f_{01}g_{11} & f_{11}g_{01} & f_{01}g_{01}\\ f_{12}g_{12} & f_{02}g_{12} & f_{12}g_{02} & f_{02}g_{02}\\ f_{1c}g_{1c} & f_{0c}g_{1c} & f_{1c}g_{0c} & f_{0c}g_{0c} \end{vmatrix} \begin{vmatrix} f_1\tilde{g}_1 & f_0\tilde{g}_1 & f_1\tilde{g}_0 & f_0\tilde{g}_0\\ f_{11}\tilde{g}_{11} & f_{01}\tilde{g}_{11} & f_{11}\tilde{g}_{01} & f_{01}\tilde{g}_{01}\\ f_{12}\tilde{g}_{12} & f_{02}\tilde{g}_{12} & f_{12}\tilde{g}_{02} & f_{02}\tilde{g}_{02}\\ f_{1c}\tilde{g}_{1c} & f_{0c}\tilde{g}_{1c} & f_{1c}\tilde{g}_{0c} & f_{0c}\tilde{g}_{0c} \end{vmatrix}$$

$$\begin{aligned} &= (g_{02}g_{11} - g_{12}g_{01})(\tilde{g}_{02}\tilde{g}_{11} - \tilde{g}_{12}\tilde{g}_{01})(f_{0c}f_{11} - f_{1c}f_{01})(f_{0c}f_{12} - f_{1c}f_{02})\\ &\quad \times (g_{0c}g_1 - g_{1c}g_0)(\tilde{g}_{0c}\tilde{g}_1 - \tilde{g}_{1c}\tilde{g}_0)(f_{01}f_1 - f_{11}f_0)(f_{02}f_1 - f_{12}f_0). \end{aligned}$$

Similarly we obtain equations (2.4).

We can check that arbitrary trivial solutions satisfy the equation (2.4). If we take a trivial solution

$$(f_{0d} : f_{1d}, g_{0d} : g_{1d}) = (1 : \wp(r + 2t^2/\lambda + t), 1 : \wp(t - r - 2t^2/\lambda))$$

and vectors

$$v_d = w_{4,1}(f_{0d} : f_{1d}, g_{0d} : g_{1d}), \qquad \check{v}_d = w_{4,1}(f_{0d} : f_{1d}, \overline{g_{1d}} : \overline{g_{1d}}),$$

then

$$\det(v_d, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_c)\,\det(\check{v}_d, \check{v}_1, \check{v}_2, \check{v}_3, \check{v}_4, \check{v}_5, \check{v}_6, \check{v}_7, \check{v}_8, \check{v}_c)$$

and

$$\begin{aligned} &\det(\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_8)\,\det(\check{\phi}_1, \check{\phi}_2, \check{\phi}_3, \check{\phi}_4, \check{\phi}_5, \check{\phi}_6, \check{\phi}_7, \check{\phi}_8)\\ &\qquad \times \prod_{i=1}^{8}(f_{0c}f_{1i} - f_{1c}f_{0i})\\ &\qquad \times (g_{0c}g_{1d} - g_{1c}g_{0d})(\overline{g_{0c}}\,\overline{g_{1d}} - \overline{g_{1c}}\,\overline{g_{1d}}) \prod_{i=1}^{8}(f_{0i}f_{1d} - f_{1i}f_{0d}) \end{aligned}$$

equal to

$$\frac{\sigma(4t)^4\sigma(4t+2\lambda)^4\prod_{\substack{i,j=1,\dots,8\\ i<j}}\sigma(b_i-b_j)^2}{\sigma(q+2t^2/\lambda+t)^{16}\sigma(t-q-2t^2/\lambda)^2\sigma(q+2t^2/\lambda+3t+\lambda)^2}$$

$$\times\prod_{i=1}^{8}\frac{\sigma(q+2t^2/\lambda-b_i)\sigma(q+2t^2/\lambda+b_i+2t)}{\sigma(b_i+t)^{16}\sigma(t-b_i)^2\sigma(t+\lambda-b_i)^2}$$

$$\times\frac{\sigma(r-q)^2\sigma(r+q+4t^2/\lambda+6t+2\lambda)\sigma(r+q+4t^2/\lambda-2t)}{\sigma(r+2t^2/\lambda+t)^{16}\sigma(t-r-2t^2/\lambda)^2\sigma(r+2t^2/\lambda+3t+\lambda)^2}$$

$$\times\prod_{i=1}^{8}\sigma(r+2t^2/\lambda-b_i)\sigma(r+2t^2/\lambda+b_i+2t).$$

We can also check that the trivial solution satisfies (2.4b).

Remark 2.1. We presented a expression of $dP(A_0^{(1)})$ in [5]. The equations (2.4) are equivalent to the expression. This system is also equivalent to ell.P derived in [8].

3. $A_0^{(1)*}$-surface

We construct the $A_0^{(1)*}$-surface by blowing up $\boldsymbol{P}^1\times\boldsymbol{P}^1$ at eight points. These eight points should be chosen so that the curve passing through them degenerates into a rational curve with a node. We parametrize such curve and points as follows:

$$f_1^2g_0^2+f_0^2g_1^2-\left(t^2+\frac{1}{t^2}\right)f_0f_1g_0g_1+\left(t^2-\frac{1}{t^2}\right)^2f_0^2g_0^2=0, \tag{3.1}$$

$$p_i:(f_{0i}:f_{1i},g_{0i}:g_{1i})=\left(1:b_it+\frac{1}{b_it},1:\frac{t}{b_i}+\frac{b_i}{t}\right)\qquad(i=1,\dots,8). \tag{3.2}$$

The $A_0^{(1)*}$-surface discrete Painlevé equation ($dP(A_0^{(1)*})$) has the following trivial solution moving on the curve (3.1) [5]:

$$\begin{aligned}(f_{0c}:f_{1c},g_{0c}:g_{1c})\\ =\left(1:tq\exp\left(\frac{2(\log t)^2}{\log\lambda}\right)+\frac{1}{tq\exp\left(\frac{2(\log t)^2}{\log\lambda}\right)},\right.\\ \left.1:\frac{t}{q\exp\left(\frac{2(\log t)^2}{\log\lambda}\right)}+\frac{q\exp\left(\frac{2(\log t)^2}{\log\lambda}\right)}{t}\right),\end{aligned} \tag{3.3}$$

where q is a constant determined by the initial condition.

Using these expressions we formulate the theorem:

Theorem 2. *$dP(A_0^{(1)*})$ can be written in the form* (2.4).

Let $q = 0$, then the trivial solution (3.3) is

$$(f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (0 : 1, 0 : 1), \tag{3.4}$$

and A, B (2.5) are

$$A = \left(t^2 - \frac{1}{t^2}\right)^4 \left(t\bar{t} - \frac{1}{t\bar{t}}\right)^4 \prod_{\substack{i,j=1,\dots,8 \\ i<j}} (b_i - b_j)^2 \Big/ \prod_{i=1}^{8} b_i^7, \tag{3.5a}$$

$$B = \left(t^2 - \frac{1}{t^2}\right)^4 \left(t\underline{t} - \frac{1}{t\underline{t}}\right)^4 \prod_{\substack{i,j=1,\dots,8 \\ i<j}} (b_i - b_j)^2 \Big/ \prod_{i=1}^{8} b_i^7. \tag{3.5b}$$

4. $A_0^{(1)**}$-surface

We construct the $A_0^{(1)**}$-surface by blowing up $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at eight points. These eight points should be chosen so that the curve on which they lie degenerates into a rational curve with a cusp. We parametrize such curve and points as follows:

$$(f_1 g_0 - f_0 g_1)^2 - 8t^2(f_0 f_1 g_0^2 + f_0^2 g_0 g_1) + 16t^4 f_0^2 g_0^2 = 0, \tag{4.1}$$

$$p_i : (f_{0i} : f_{1i}, g_{0i} : g_{1i}) = (1 : (b_i + t)^2, 1 : (t - b_i)^2) \qquad (i = 1, \dots, 8), \tag{4.2}$$

The $A_0^{(1)**}$-surface discrete Painlevé equation ($dP(A_0^{(1)**})$) has the following trivial solution moving on the curve (4.1) [5]:

$$(f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (1 : (q + 2t^2/\lambda + t)^2, 1 : (t - q - 2t^2/\lambda)^2), \tag{4.3}$$

where q is a constant determined by initial condition.

Theorem 3. *$dP(A_0^{(1)**})$ can be written in the form* (2.4).

If $q = \infty$ the trivial solution (4.3) is

$$(f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (0 : 1, 0 : 1) \tag{4.4}$$

and A, B (2.5) are

$$A = 4096 t^4 (t + \bar{t})^4 \prod_{\substack{i,j=1,\dots,8 \\ i<j}} (b_i - b_j)^2, \tag{4.5a}$$

$$B = 4096 t^4 (t + \underline{t})^4 \prod_{\substack{i,j=1,\dots,8 \\ i<j}} (b_i - b_j)^2. \tag{4.5b}$$

5. $A_1^{(1)}$-surface

We construct the $A_1^{(1)}$-surface by blowing up $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at eight points. This surface can be obtained by the limit process from the $A_0^{(1)*}$-surface. These eight points and a curve through these points are

$$(f_1g_1 - t^2 f_0g_0)(f_1g_1 - f_0g_0) = 0, \tag{5.1}$$

$$p_i : (f_{0i} : f_{1i}, g_{0i} : g_{1i}) = \begin{cases} \left(1 : b_i t, 1 : \frac{t}{b_i}\right) & (i = 1, \ldots, 4), \\ \left(1 : b_i, 1 : \frac{1}{b_i}\right) & (i = 5, \ldots, 8). \end{cases} \tag{5.2}$$

The $A_1^{(1)}$-surface discrete Painlevé equation ($dP(A_1^{(1)})$) has the following trivial solution.

$$(f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (0 : 1, 1 : 0).$$

Theorem 4. *$dP(A_1^{(1)})$ can be written in the form* (2.4).

A, B (2.5) are

$$A = t^8(1-t^2)^4(1-t\bar{t})^4 \prod_{\substack{i,j=1,\ldots,4 \\ i<j}} (b_i - b_j)^2 \prod_{\substack{i,j=5,\ldots,8 \\ i<j}} (b_i - b_j)^2 \Big/ \prod_{i=1}^{8} b_i, \tag{5.3a}$$

$$B = t^{12}(1-t^2)^4(1-t\underline{t})^4 \prod_{\substack{i,j=1,\ldots,4 \\ i<j}} (b_i - b_j)^2 \prod_{\substack{i,j=5,\ldots,8 \\ i<j}} (b_i - b_j)^2 \Big/ \prod_{i=1}^{8} b_i^6. \tag{5.3b}$$

6. $A_1^{(1)*}$-surface

We construct the $A_1^{(1)*}$-surface by blowing up $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at eight points. This surface can be obtained through coalescence from the $A_1^{(1)}$-surface. These eight points and a curve passing through these points are

$$(f_1g_0 + f_0g_1 - 2tf_0g_0)(f_1g_0 + f_0g_1) = 0, \tag{6.1}$$

$$p_i : (f_{0i} : f_{1i}, g_{0i} : g_{1i}) = \begin{cases} (1 : b_i + t, 1 : t - b_i) & (i = 1, \ldots, 4), \\ (1 : b_i, 1 : -b_i) & (i = 5, \ldots, 8). \end{cases} \tag{6.2}$$

The $A_1^{(1)*}$-surface discrete Painlevé equation ($dP(A_1^{(1)*})$) has the following trivial solution.

$$(f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (0 : 1, 0 : 1)$$

and we have the following theorem:

Theorem 5. *$dP(A_1^{(1)*})$ can be written in the form* (2.4).

A, B (2.5) are

$$A = 16t^4(t+\bar{t})^4 \prod_{\substack{i,j=1,\dots,4\\ i<j}} (b_i - b_j)^2 \prod_{\substack{i,j=5,\dots,8\\ i<j}} (b_i - b_j)^2, \tag{6.3a}$$

$$B = 16t^4(t+\underline{t})^4 \prod_{\substack{i,j=1,\dots,4\\ i<j}} (b_i - b_j)^2 \prod_{\substack{i,j=5,\dots,8\\ i<j}} (b_i - b_j)^2. \tag{6.3b}$$

7. $A_2^{(1)}$-surface

We construct $A_2^{(1)}$-surface by blowing up $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at eight points. This surface can be obtained as a coalescence limit from the $A_1^{(1)}$-surface. These eight points and a curve on which these points lie are

$$f_0 g_0 (f_1 g_1 - f_0 g_0) = 0, \tag{7.1}$$

$$p_i : (f_{0i} : f_{1i}, g_{0i} : g_{1i}) = \begin{cases} (1 : b_i t, 0 : 1) & (i = 1, 2), \\ \left(0 : 1, 1 : \frac{t}{b_i}\right) & (i = 3, 4), \\ \left(1 : b_i, 1 : \frac{1}{b_i}\right) & (i = 5, \dots, 8). \end{cases} \tag{7.2}$$

The $A_2^{(1)}$-surface discrete Painlevé equation ($dP(A_2^{(1)})$) has the trivial solution

$$(f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (0 : 1, 1 : 0).$$

If we write $dP(A_2^{(1)})$ in the form (2.4), then both sides of (2.4a) are 0. However we can derive the following expression from the form of $dP(A_1^{(1)})$.

Theorem 6. *$dP(A_2^{(1)})$ can be written as*

$$\begin{aligned} &\det(v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, \mathrm{v_c}) \det(\check{v}, \check{v}_1, \check{v}_2, \check{v}_3, \check{v}_4, \check{v}_5, \check{v}_6, \check{v}_7, \check{v}_8, \check{\mathrm{v}}_\mathrm{c}) \\ &\quad = t^3 \bar{t}^3 (b_1 - b_2)^2 (b_3 - b_4)^2 \prod_{\substack{i,j=5,\dots,8\\ i<j}} (b_i - b_j)^2 / b_3^3 b_4^3 \\ &\quad\quad \times (g_{0c} g_1 - g_{1c} g_0)(\overline{g_{0c}}\,\overline{g_1} - \overline{g_{1c}}\,\overline{g_0}) \prod_{i=1}^{8} (f_{0i} f_1 - f_{1i} f_0), \end{aligned} \tag{7.3a}$$

(7.3b) $$\det(u, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_c) \det(\hat{u}, \hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5, \hat{u}_6, \hat{u}_7, \hat{u}_8, \hat{u}_c)$$
$$= t^9 \underline{t}(b_1 - b_2)^2 (b_3 - b_4)^2 \prod_{\substack{i,j=5,\dots,8 \\ i<j}} (b_i - b_j)^2 \Big/ \prod_{i=3}^{8} b_i^5$$
$$\times (f_{0c} f_1 - f_{1c} f_0)(\underline{f_{0c}}\,\underline{f_1} - \underline{f_{1c}}\,\underline{f_0}) \prod_{i=1}^{8} (g_{0i} g_1 - g_{1i} g_0),$$

where

$$\mathrm{v}_c = \check{\mathrm{v}}_c = {}^t(0, 1, 0, 0, 0, 0, 0, 0, 0, 0).$$

8. $A_3^{(1)}$-surface

We construct the $A_3^{(1)}$-surface by blowing up $\boldsymbol{P}^1 \times \boldsymbol{P}^1$ at eight points. This surface is obtained through coalescence from the $A_2^{(1)}$-surface. These eight points and a curve through them are

(8.1) $$f_0 f_1 g_0 g_1 = 0,$$

(8.2) $$p_i : (f_{0i} : f_{1i}, g_{0i} : g_{1i}) = \begin{cases} (1 : b_i, 0 : 1) & (i = 1, 2), \\ \left(0 : 1, 1 : \frac{1}{b_i}\right) & (i = 3, 4), \\ (1 : b_i t, 1 : 0) & (i = 5, 6), \\ \left(1 : 0, 1 : \frac{t}{b_i}\right) & (i = 7, 8). \end{cases}$$

The $A_3^{(1)}$-surface discrete Painlevé equation ($dP(A_3^{(1)})$) has the trivial solution.

$$(f_{0c} : f_{1c}, g_{0c} : g_{1c}) = (0 : 1, 1 : 0),$$

for which we have

Theorem 7. *$dP(A_3^{(1)})$ can be written as*

(8.3a) $$\det(v, v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, \mathrm{v}_c) \det(\check{v}, \check{v}_1, \check{v}_2, \check{v}_3, \check{v}_4, \check{v}_5, \check{v}_6, \check{v}_7, \check{v}_8, \check{\mathrm{v}}_c)$$
$$= \frac{1}{(b_1 - b_2)^2 (b_3 - b_4)^2 (b_5 - b_6)^2 (b_7 - b_8)^2 t^7 \bar{t}} \left(\frac{b_3 b_4 b_7 b_8}{b_1 b_2 b_5 b_6} \right)^2$$
$$\times g_1 \overline{g_1} (f_1 - b_1 f_0)(f_1 - b_2 f_0) f_0^2 (f_1 - b_5 t f_0)(f_1 - b_6 t f_0) f_1^2,$$

$$\text{(8.3b)}\quad \det(u, u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, \mathrm{u_c}) \det(\hat{u}, \hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4, \hat{u}_5, \hat{u}_6, \hat{u}_7, \hat{u}_8, \hat{\mathrm{u}}_\mathrm{c})$$
$$= \frac{b_3^4 b_4^4 b_7^6 b_8^6}{(b_1 - b_2)^2 (b_3 - b_4)^2 (b_5 - b_6)^2 (b_7 - b_8)^2 t^{11} \bar{t}}$$
$$\times f_0 \underline{f_0} g_0^2 (g_1 - 1/b_3 g_0)(g_1 - 1/b_4 g_0) g_1^2 (g_1 - t/b_7 g_0)(g_1 - t/b_8 g_0),$$

where

$$\mathrm{u_c} = \hat{\mathrm{u}}_\mathrm{c} = {}^t(0,0,0,0,0,0,0,0,1,0).$$

If we expand the determinant in (8.3), we obtain

$$\text{(8.4a)}\qquad \frac{g_1 \overline{g_1}}{g_0 \overline{g_0}} = \frac{(f_1 - b_5 t f_0)(f_1 - b_6 t f_0)}{(f_1 - b_1 f_0)(f_1 - b_2 f_0)},$$

$$\text{(8.4b)}\qquad \frac{f_1 \underline{f_1}}{f_0 \underline{f_0}} = \frac{(g_1 - t/b_7 g_0)(g_1 - t/b_8 g_0)}{(g_1 - 1/b_3 g_0)(g_1 - 1/b_4 g_0)}.$$

This is exactly q-$\mathrm{P_{VI}}$ [3].

9. Discussion

In this paper we presented a new representation of discrete Painlevé equations. Up to now, the complexity of some of the difference Painlevé equations prevented us from studying their properties. It is our hope that these new forms of these equations, especially for $dP(A_0^{(1)}), dP(A_0^{(1)*})$, and $dP(A_0^{(1)**})$ which have Weyl groups of type $E_8^{(1)}$, will prove useful in such analysis.

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References

[1] Grammaticos, B., Ramani, A., and Papageorgiou, V. G., Do integrable mappings have the Painlevé property?, Phys. Rev. Lett., **67** (1991), 1825–1828.

[2] Grammaticos, B., Ohta, Y., Ramani, A., and Sakai, H., Degeneration through coalescence of the q-Painlevé VI equation, J. Phys. A: Math. Gen., **31** (1998), 3545–3558.

[3] Jimbo, M. and Sakai, H., A q-analog of the sixth Painlevé equation, Lett. Math. Phys., **38** (1996), 145–154.

[4] Kajiwara, K., Masuda, T., Noumi, M., Ohta, Y., and Yamada, Y., ${}_{10}E_9$ solution to the elliptic Painlevé equation, J. Phys. A: Math. Gen., **36** (2003), L263–L272.

[5] Murata, M., Sakai, H., and Yoneda, J., Riccati solutions of discrete Painlevé equations with Weyl group symmetry of type $E_8^{(1)}$, J. Math. Phys., **44** (2003), 1396–1414.

[6] Ohta, Y., Ramani, A., and Grammaticos, B., An affine Weyl group approach to the eight-parameter discrete Painlevé equation, J. Phys. A: Math. Gen., **34** (2001), 10523–10532.
[7] Ramani, A., Grammaticos, B., and Hietarinta, J., Discrete versions of the Painlevé equations, Phys. Rev. Lett., **67** (1991), 1829–1832.
[8] Sakai, H., Rational Surfaces Associated with Affine Root Systems and Geometry of the Painlevé Equations, Comm. Math. Phys., **220** (2001), 165–229.

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