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On the Boussinesq Flow with Nondecaying Initial Data

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Abstract. This paper is concerned with the Boussinesq equations which describe the heat convection in a viscous incompressible fluid. Local existence and uniqueness theorems are established for the n-dimensional Boussinesq equations in the whole space with nondecaying initial data. In two dimensional case the solution can be extended globally in time without smallness of the initial data.

Key Words and Phrases. Boussinesq equations, Nondecaying initial data, Uniqueness.

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1. Introduction

In this paper, we consider the natural convection in a viscous incompressible fluid described by the Boussinesq equations:

$$
\text{(B)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = g\theta & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ \partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0 & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ \operatorname{div} u = 0 & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ u|_{t=0} = u_0, \ \theta|_{t=0} = \theta_0 & \text{in } x \in \boldsymbol{R}^n, \end{cases}
$$

where $u = (u^1(x,t), u^2(x,t), \dots, u^n(x,t))$, $\theta = \theta(x,t)$ and $p = p(x,t)$ denote the velocity vector field, the temperature and the pressure of the fluid at the point $(x,t) \in \boldsymbol{R}^n \times (0,\infty)$, respectively. (u_0, θ_0) is the given initial data and g is the given constant vector which denotes acceleration of gravity.

Many researchers have investigated the natural convection in various domains. See e.g. [6, 12, 13, 14, 15, 22, 23, 31, 32, 33, 34] and papers cited there. These results, however, are imposed the integrability condition on the initial data $u_0, \theta_0 \in L^q$ for some $q < \infty$. This condition implies that $u_0(x)$ and $\theta_0(x)$ decay at infinity in some sense, if the domain is the whole space $\boldsymbol{R}^n$ or an exterior domain.

In our physical model we assume that the motion of the fluid is not decaying at space infinity. In this situation the time-local existence of solution to the Navier-Stokes equations (N-S) is obtained by [7, 8, 18]:

$$
\text{(N-S)} \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = 0 & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ \operatorname{div} u = 0 & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ u|_{t=0} = u_0 & \text{in } x \in \boldsymbol{R}^n \end{cases}
$$

with nondecaying initial data $u_0 \in L^\infty(\boldsymbol{R}^n)$ for $n \geq 2$.

We sometimes consider the integral equation instead of (N-S). We call a solution of integral equation *a mild solution*. We should note that Cannone [8] proved the existence of mild solutions to (N-S) for more general initial data in Besov spaces. Giga-Inui-Matsui [18] obtained that the mild solutions in L^∞ (or BUC) actually satisfies (N-S) in strong sense. Koch-Tataru [27] recently showed the existence of mild solutions to (N-S) in BMO^{-1}; see also [28, 30, 36].

We next refer to the uniqueness of solutions to (N-S). It is well-known that the solution $(u, \nabla p)$ is uniquely determined under the hypothesis $u \in C_w([0,T); L^q(\boldsymbol{R}^n))$ for some $q \in (n, \infty)$ or $u \in C([0,T); L^n(\boldsymbol{R}^n))$; see e.g. [9]. Here, $C_w(I; X)$ denotes the space of all weakly continuous functions with value in X. In the critical case $q = \infty$, however, the uniqueness does not hold. Indeed, for an arbitrary constant vector C_0 we put

$$
(u(x,t), p(x,t)) = (C_0 t, -C_0 \cdot x),
$$

which is a solution to (N-S) with the initial velocity $u_0 = 0$; see [18]. Hence, we need some additional conditions for uniqueness of solution (u, p) with $u \in C_w([0,T); L^\infty(\boldsymbol{R}^n))$. Galdi-Maremonti [16] showed that if (u, p) is a smooth solution to (N-S) with

$$
u, \nabla u \in L^\infty(0,T; L^\infty) \qquad \text{and} \qquad |p(x,t)| \leq C\langle x \rangle^{1-\varepsilon}
$$

for some constants C and $\varepsilon > 0$ independent of (x,t), then $(u, \nabla p)$ is uniquely determined by u_0. Here, we put $\langle x \rangle = \sqrt{1 + |x|^2}$. Okamoto [35] showed that if

$$
|u(x,t)| \leq C\langle x \rangle^{1-\varepsilon}, \qquad |\nabla u(x,t)| \leq C, \qquad |p(x,t)| \leq C\langle x \rangle^{1-n/2}
$$

for some constants C and $\varepsilon > 0$ independent of (x,t), then $(u, \nabla p)$ is uniquely determined by u_0. Later, Chae-Kim [11] improved Okamoto's uniqueness theorem. They showed that the first condition $|u(x,t)| \leq C\langle x \rangle^{1-\varepsilon}$ is not needed for proving the uniqueness.

On the other hand, there is another type of uniqueness theorem for bounded solutions to (N-S). Giga-Inui-Kato-Matsui [19] showed that if (u, p) satisfies (N-S) in weak sense and satisfies

$$
u \in C_w([0,T); L^\infty) \qquad \text{and} \qquad p \in \left\{ \sum_{i,j}^n R_i R_j \pi_{ij}; \pi_{ij} \in L^1_{loc}(0,T; L^\infty) \right\},
$$

then $(u, \nabla p)$ is uniquely determined by the initial velocity u_0. Here, R_k denotes the Riesz transform, i.e., $R_k = \partial_k(-\Delta)^{-1/2}$. Recently, J. Kato [24] successfully weakened this assumption to

$$u \in C_w([0,T); L^\infty) \qquad \text{and} \qquad p \in L^1_{loc}(0,T; BMO).$$

In this paper we improve his results by using Besov-type space and applying (B).

Concerning with the time-global solvability in two dimension, Giga-Matsui-first author of this paper [20] proved the global existence of solution to (N-S) with initial velocity $u_0 \in L^\infty(\boldsymbol{R}^2)$ without smallness and integrability condition on u_0. They utilized the maximal principle for the vorticity equation and established the *double* exponential estimate:

$$\|u(t)\|_{L^\infty} \le C_0 \exp\{C_0 \exp(C_0 t)\}. \tag{1.1}$$

for all $t > 0$, where the constant C_0 depends only on $\|u_0\|_{L^\infty}$ and $\|\mathrm{rot}\, u_0\|_{L^\infty}$.

In this paper we improve (1.1) to establish a *single* exponential estimate for solution u to (B):

$$\begin{aligned}\|u(t)\|_\infty \le C(&\|u_0\|_\infty + \|\mathrm{rot}\, u_0\|_\infty \\ &+ (t+1)|g|\,\|\theta_0\|_q) \exp\{C(1+t^{3/2})(\|\mathrm{rot}\, u_0\|_\infty + |g|\,\|\theta_0\|_q)\}\end{aligned}$$

for all $t > 0$ and $2 < q < \infty$. Here, $\|\cdot\|_q$ stands for the norm of L^q on $\boldsymbol{R}^n$ with $1 \le q \le \infty$.

We shall deal with slightly modified Boussinesq equations with the superfluous term f (which is a given vector valued function) in the first equations as follows:

$$(\mathrm{B}') \qquad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u + \nabla p = g\theta + f & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ \partial_t \theta - \Delta\theta + u \cdot \nabla\theta = 0 & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ \mathrm{div}\, u = 0 & \text{in } t > 0 \text{ and } x \in \boldsymbol{R}^n, \\ u|_{t=0} = u_0,\ \theta|_{t=0} = \theta_0 & \text{in } x \in \boldsymbol{R}^n, \end{cases}$$

where $f = (f^1(x,t), f^2(x,t), \dots, f^n(x,t))$ with $\mathrm{div}\, f = 0$. Of course, if $f \equiv 0$, (B$'$) coincides with (B).

Before closing this introduction, we shall sketch the way to construct the solution of (B$'$). Let $P = \{P_{kl}\}_{1 \le k,l \le n} = \{\delta_{kl} + R_k R_l\}_{1 \le k,l \le n}$, while P is called *Leray projection operator* by following Lemarié-Rieusset [30]. Formally, applying P to the first equations in (B$'$), so we have

$$\partial_t u - \Delta u + P(u \cdot \nabla u) = P(g\theta) + f, \tag{1.2}$$

$$\partial_t \theta - \Delta\theta + u \cdot \nabla\theta = 0. \tag{1.3}$$

To solve (1.2) and (1.3) we reduce to construct a solution (u, θ) to the integral equations:

$$(\mathrm{I.E.B})_1 \quad u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-\tau)\Delta}P(u\cdot\nabla u)(\tau)d\tau + \int_0^t e^{(t-\tau)\Delta}P(g\theta)(\tau)d\tau + \int_0^t e^{(t-\tau)\Delta}f(\tau)d\tau,$$

$$(\mathrm{I.E.B})_2 \quad \theta(t) = e^{t\Delta}\theta_0 - \int_0^t e^{(t-\tau)\Delta}(u\cdot\nabla\theta)(\tau)d\tau.$$

A solution (u, θ) of $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$ is called a *mild solution* to $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$.

We consider (B′) with nondecaying initial data (u_0, θ_0). In order to use Leray projection operator P, we have to handle the temperature in the homogeneous Besov space or L^q space for some $q \in (1, \infty)$, because the projection P is not a bounded operator in L^∞. The homogeneous Besov space which we treat contains some nondecaying functions, for example, smooth periodic functions. Moreover, it contains some aperiodic and almost periodic functions; see Remark 1-(ii).

We first construct the local solution to (B′) by using the usual iteration method. We next consider the uniqueness of the solution to (B′). The space which we treat is bigger than *BMO*, then our uniqueness result is an improvement in Kato's [24]. Finally, we obtain the global existence of it for $n = 2$. We show that if the initial temperature θ_0 belongs to L^q for some $q \in (1, \infty)$, then there exists a unique global solution to (B′) with nondecaying initial velocity $u_0 \in L^\infty(\boldsymbol{R}^n)$. The *Uniqueness* and *Global solvability* are our main themes.

This paper is organized as follows. In Section 2 we introduce the homogeneous Besov space. In Section 3 we shall describe main results on this paper. In Section 4 we shall prepare some lemmas. In Section 5 we give the proofs of main theorems. In Appendix we shall show that the mild solution is actually a solution to (B′) in the strong sense.

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Throughout this paper we denote positive constants by C the value of which may differ from one occasion to another. The indices in the constant $C(\cdot, \cdot, \dots, \cdot)$ indicate the dependence of the parameters denoted as in bracket. We do not distinguish the space of vector-valued and scalar functions.

2. Function spaces

We first recall the definition of the homogeneous Besov space (cf., [5], [39]). For integer j, let φ_j be the Littlewood-Paley decomposition satisfying $\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi) \in C_0^\infty(\boldsymbol{R}^n)$, $\operatorname{supp} \hat{\varphi}_0 \subset \{1/2 < |\xi| < 2\}$ and $\sum_{j=-\infty}^{\infty} \hat{\varphi}_j(\xi) = 1$ excepting $\xi = 0$. Here, $\mathscr{F}f = \hat{f}$ denotes the Fourier transform and $\mathscr{F}^{-1}$ is its inverse. Let $\mathscr{S}'(\boldsymbol{R}^n)$ be the space of all tempered distributions, i.e., the topological dual of $\mathscr{S}(\boldsymbol{R}^n)$ which is the space of rapidly decreasing functions in the sense of L. Schwartz.

Definition. The homogeneous Besov space $\dot{\boldsymbol{B}}^s_{p,q} = \dot{\boldsymbol{B}}^s_{p,q}(\boldsymbol{R}^n)$ is introduced by

$$\dot{\boldsymbol{B}}^s_{p,q} \equiv \{f \in \mathscr{Z}'; \|f\|_{\dot{\boldsymbol{B}}^s_{p,q}} < \infty\}$$

for $s \in \boldsymbol{R}$, $1 \le p, q \le \infty$, where

$$\|f\|_{\dot{\boldsymbol{B}}^s_{p,q}} \equiv \begin{cases} \left[\sum_{j=-\infty}^{\infty} (2^{js}\|\varphi_j * f\|_p)^q\right]^{1/q} & \text{if } q < \infty, \\ \sup_{-\infty<j<\infty} 2^{js}\|\varphi_j * f\|_p & \text{if } q = \infty. \end{cases}$$

Here, $\mathscr{Z}'$ is the topological dual space of $\mathscr{Z}$ which is the space of

$$\mathscr{Z} \equiv \{f \in \mathscr{S}; D^\alpha \hat{f}(0) = 0, \forall\alpha \text{ is multi-index}\}.$$

We sometimes suppress the domain $\boldsymbol{R}^n$ in the notation. It should be noted that $\dot{\boldsymbol{B}}^s_{p,q}$ is a Banach space. However, there is some difficulty in $\dot{\boldsymbol{B}}^s_{p,q}$ and $\mathscr{Z}'$. Let $\mathscr{P}$ be the set of polynomials. Since

$$f \equiv f + g \quad \text{in } \mathscr{Z}' \text{ for } f \in \mathscr{S}', g \in \mathscr{P}, \qquad \text{and}$$

$$\dot{\boldsymbol{B}}^s_{p,q} \cong \{f \in \mathscr{S}'/\mathscr{P}; \|f\|_{\dot{\boldsymbol{B}}^s_{p,q}} < \infty\}$$

(see [39]), $\mathscr{Z}'$ seems to be non-appropriate as a function space where equations are treated. Fortunately, if the exponents satisfy the following condition:

$$\text{(2.1)} \qquad \text{either } s < n/p \qquad \text{or} \qquad s = n/p \text{ and } q = 1,$$

then $\dot{\boldsymbol{B}}^s_{p,q}$ can be regarded as a subspace of $\mathscr{S}'$. To be precise, there holds

$$\text{(2.2)} \qquad \dot{\boldsymbol{B}}^s_{p,q} \cong \left\{ f \in \mathscr{S}'; \|f\|_{\dot{\boldsymbol{B}}^s_{p,q}} < \infty \text{ and } f = \sum_{j=-\infty}^{\infty} \varphi_j * f \text{ in } \mathscr{S}' \right\},$$

if s, p and q satisfy (2.1). For the details one can see e.g. Kozono-Yamazaki [29, Proposition 2.10].

In what follows, we deal with (2.2) as the definition of $\dot{\boldsymbol{B}}^s_{p,q}$, when s, p and q satisfy (2.1). We note that any polynomial excepting 0 no longer belongs to $\dot{\boldsymbol{B}}^s_{p,q}$, when s, p and q satisfy (2.1).

Now we introduce the following spaces which are slightly different from the homogeneous Besov spaces;

Definition. If the exponents $s \in \boldsymbol{R}$, $1 \le p, q \le \infty$ satisfy

$$\text{(2.3)} \qquad \text{either } s < n/p + 1 \quad \text{or} \quad s = n/p + 1 \text{ and } q = 1,$$

then the associated homogeneous Besov space $\dot{\mathscr{A}}^s_{p,q}$ is introduced by

$$\text{(2.4)} \qquad \dot{\mathscr{A}}^s_{p,q} \equiv \left\{ f \in \mathscr{S}'; \|f\|_{\dot{B}^s_{p,q}} < \infty \text{ and } \nabla f = \sum_{j=-\infty}^{\infty} \nabla \varphi_j * f \text{ in } \mathscr{S}' \right\}.$$

We note that $\dot{\mathscr{A}}^s_{p,q}$ has the semi-norm $\|f\|_{\dot{\mathscr{A}}^s_{p,q}} = \|f\|_{\dot{B}^s_{p,q}}$. For $f \in \dot{\mathscr{A}}^s_{p,q}$, $\|f\|_{\dot{B}^s_{p,q}} = 0$ if and only if f is a constant. We have

$$\dot{\mathscr{A}}^s_{p,q}/\boldsymbol{R} \cong \dot{B}^s_{p,q},$$

when the exponents satisfy (2.3). See e.g. [5, p163, ex12]. In particular, we thus have

$$\dot{\mathscr{A}}^1_{\infty,1}/\boldsymbol{R} \cong \dot{B}^1_{\infty,1} \quad \text{and} \quad \dot{\mathscr{A}}^0_{\infty,\infty}/\boldsymbol{R} \cong \dot{B}^0_{\infty,\infty}.$$

By the interpolation theory we also note that

$$BMO/\boldsymbol{R} \subset \dot{\mathscr{A}}^0_{\infty,\infty}/\boldsymbol{R} \subset \dot{B}^{-\kappa}_{\infty,\infty} + \dot{\mathscr{A}}^1_{\infty,1}/\boldsymbol{R}$$

for $\kappa > 0$. For the definition of BMO we refer to [38]. It is easy to see that

$$\|f\|_\infty \le \|f\|_{\dot{B}^0_{\infty,1}} \qquad \text{for all } f \in \dot{B}^0_{\infty,1}.$$

For $f \in \dot{B}^0_{\infty,1}$, $\sum_{|j|<N} \varphi_j * f \in W^{1,\infty} \subset BUC$ and $\sum_{|j|<N} \varphi_j * f \to f$ in $\dot{B}^0_{\infty,1}$ as $N \to \infty$. Then it is clear that $\dot{B}^0_{\infty,1} \subset BUC$, where BUC is the space of bounded and uniformly continuous functions; see also [36]. Therefore, the relationships among these spaces are as follows:

$$\dot{B}^0_{\infty,1} \subset BUC/\boldsymbol{R} \subset L^\infty/\boldsymbol{R} \subset BMO/\boldsymbol{R} \subset \dot{\mathscr{A}}^0_{\infty,\infty}/\boldsymbol{R} \subset \dot{B}^{-\kappa}_{\infty,\infty} + \dot{\mathscr{A}}^1_{\infty,1}/\boldsymbol{R}$$

for $\kappa > 0$. In particular, $BMO \subset \dot{B}^{-\kappa}_{\infty,\infty} + \dot{\mathscr{A}}^1_{\infty,1}$ for $\kappa > 0$.

The advantage to use the homogeneous Besov spaces consists of the fact that the Riesz transforms are bounded in these spaces, but not in L^∞. We recall that

$$f = \sum_{j=-\infty}^{\infty} \varphi_j * f = \sum_{j=-\infty}^{\infty} \tilde{\varphi}_j * \varphi_j * f \qquad \text{in } \mathscr{S}' \text{ for } \dot{B}^s_{p,q},$$

when the exponents s, p and q satisfy (2.1). Here, $\tilde{\varphi}_j \equiv \varphi_{j-1} + \varphi_j + \varphi_{j+1}$. We can thus define the Riesz transform R_k for $1 \le k \le n$ in $\dot{B}^s_{p,q}$ as follows:

$$R_k f \equiv \sum_{j=-\infty}^{\infty} (R_k \tilde{\varphi}_j) * \varphi_j * f \qquad \text{in } \mathscr{S}' \text{ for } f \in \dot{B}^s_{p,q}, \tag{2.5}$$

if the exponents s, p and q satisfy (2.1). Here, R_k in the right hand side of (2.5) is the usual Riesz transform on $\mathscr{Z}$ with symbol $\sqrt{-1}\xi_k/|\xi|$. Since for any integer j there exists a positive constant C_0 (independent of k and j) such that $\|R_k \tilde{\varphi}_j\|_1 \le C_0$, we see that the Riesz transform is bounded in the homogeneous Besov space as a subspace of $\mathscr{S}'$ when the exponents satisfy (2.1).

3. Main results

In this section, we shall state main theorems on this paper.

We first describe the theorem of the time-local existence of the mild solution to the Boussinesq equations in general dimension.

Theorem 1 (Local Existence). *Assume that the initial data* $(u_0, \theta_0) \in L^\infty(\boldsymbol{R}^n) \times \dot{B}^0_{\infty,1}(\boldsymbol{R}^n)$ *with* $\operatorname{div} u_0 = 0$ (*in distribution sense*), *and that* $f \in L^\infty_{loc}([0,\infty); L^\infty(\boldsymbol{R}^n))$ *with* $\operatorname{div} f = 0$. *Then there exist* $T > 0$ *and a unique mild solution*

$$(u, \theta) \in C_w([0,T); L^\infty(\boldsymbol{R}^n)) \times C([0,T); \dot{B}^0_{\infty,1}(\boldsymbol{R}^n)) \tag{3.1}$$

to (I.E.B)$_1$ *and* (I.E.B)$_2$. *Here, there exists* C *depending only on* n *such that*

$$T > \frac{C}{\left(\|u_0\|_\infty + \|\theta_0\|_{\dot{B}^0_{\infty,1}} + \|f\|_{L^\infty(0,1;L^\infty)}\right)^2 + |g| + 1}.$$

Remark 1. (i) If we assume that $u_0 \in BUC(\boldsymbol{R}^n)$, then we easily see that the solution is continuous up to initial time, i.e.,

$$(u, \theta) \in C([0,T); BUC(\boldsymbol{R}^n)) \times C([0,T); \dot{B}^0_{\infty,1}(\boldsymbol{R}^n)).$$

(ii) We note that $\dot{B}^0_{\infty,1}$ is a Banach space. By the definition (2.2) we see that non-zero polynomial does not belong to $\dot{B}^0_{\infty,1}$ and hence $\|f\|_{\dot{B}^0_{\infty,1}} = 0$ if and only if $f = 0$. Moreover, $\dot{B}^0_{\infty,1}$ contains some nondecaying function as well as L^∞. For example, $\sin(a \cdot x)$ and $\cos(b \cdot x)$ are contained by $\dot{B}^0_{\infty,1}$, where a and b are constant vectors and $b \neq 0$; it is easy to see that the $\dot{B}^0_{\infty,1}$-norms of those are independent of a and b. Hence $\dot{B}^0_{\infty,1}$ contains some almost periodic functions, for example, $\sum_{j=1}^\infty \alpha_j \exp\{\sqrt{-1}\lambda_j \cdot x\} \in \dot{B}^0_{\infty,1}$, provided that $\{\alpha_j\}_j \in l^1$ and $\lambda_j \neq 0$ for all j. Furthermore, if $f \in W^{2,\infty}$, then $\nabla f \in \dot{B}^0_{\infty,1}(\boldsymbol{R}^n)$. Indeed, since $W^{2,\infty} \subset L^\infty$ and $W^{2,\infty} \subset \dot{B}^2_{\infty,\infty}$, we have

$$\|\nabla f\|_{\dot{B}^0_{\infty,1}} = \sum_{j<0} \|\nabla\varphi_j * f\|_\infty + \sum_{j\geq 0} \|\varphi_j * \nabla f\|_\infty$$
$$\lesssim \sum_{j<0} 2^j \|f\|_\infty + \sum_{j\geq 0} 2^{-j} \|f\|_{\dot{B}^2_{\infty,\infty}} \lesssim \|f\|_{W^{2,\infty}},$$

where $a \lesssim b$ denotes that $a \leq Cb$ for some constant $C \geq 0$. We also note that

$$\nabla f = \sum_{j=-\infty}^{\infty} \varphi_j * \nabla f \qquad \text{in } \mathscr{S}', \tag{3.2}$$

since $\nabla f = \psi_{-N} * \nabla f + \sum_{j=-N+1}^{\infty} \varphi_j * \nabla f$ in $\mathscr{S}'$ and $\psi_{-N} * \nabla f = -2^{-N}(\nabla\psi)_{-N} * f \to 0$ in $\mathscr{S}'$ as $N \to \infty$. Here, we define that $\psi = \mathscr{F}^{-1}(1 - \sum_{j=1}^{\infty} \hat{\varphi}_j)$, and ψ_{-N} denotes $2^{-nN}\psi(2^{-N}\cdot)$. Furthermore, it is important that periodic function $f \in W^{1,\infty}(T)$ with $\int_T f\,dx = 0$ belongs to $\dot{B}^0_{\infty,1}(\boldsymbol{R}^n)$. We can observe this fact by the Fourier series. Hence, Theorem 1 can deal with smooth periodic initial data.

We next get the strong solution to the Boussinesq equations.

Theorem 2 (Strong solution). *Assume that u_0, θ_0 and f satisfy the hypotheses of Theorem 1, in addition, let us assume that $f \in C((0,T); W^{1,\infty}(\boldsymbol{R}^n))$. Then the solution (u,θ) given in Theorem 1 satisfies*

$$u \in C_w([0,T); L^\infty) \cap C^1((0,T); L^\infty) \cap C((0,T); W^{2,\infty}), \tag{3.3}$$

$$\theta \in C([0,T); \dot{B}^0_{\infty,1}) \cap C^1((0,T); L^\infty) \cap C((0,T); W^{2,\infty}), \tag{3.4}$$

$$u \cdot \nabla u, \qquad g\theta \in C((0,T); \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1}), \tag{3.5}$$

$$\partial_t u - \Delta u + P(u\cdot\nabla u) = P(g\theta) + f \qquad \text{in } L^\infty, \tag{3.6}$$

$$\partial_t \theta - \Delta\theta + u\cdot\nabla\theta = 0 \qquad \text{in } L^\infty. \tag{3.7}$$

Furthermore, put

$$p(x,t) = -\int_0^x \{(1-P)(u\cdot\nabla u)(y,t) - (1-P)(g\theta)(y,t)\}ds_y + p_0(t),$$

where this integral is curvilinear integral whose contour is from 0 to x, and $p_0 \in C((0,T))$ is an arbitrary function depending only on time. Then $p \in C((0,T); \dot{\mathscr{A}}^1_{\infty,1})$ and (u,θ,p) actually satisfies (B$'$) *on* $(0,T)$.

Remark 2. Since $u\cdot\nabla u \in \dot{B}^0_{\infty,1}$, we observe that $P(u\cdot\nabla u) \in \dot{B}^0_{\infty,1} \subset BUC$. Moreover, we easily observe that $\mathrm{rot}[(1-P)(u\cdot\nabla u) - (1-P)(g\theta)] = 0$ and that the definition of p is independent of the contour.

Although a mild solution (u, θ) to $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$ in the class (3.1) is uniquely determined for given initial data u_0, θ_0 and f, a solution $(u, \theta, \nabla p)$ to (B') in the class (3.3) and (3.4) is not unique. This fact was shown by Giga-Inui-Matsui [18] for (N-S) for constant initial velocities. We also see that for any given initial data u_0, θ_0 and f as in Theorem 1, there exist the infinite number of solutions in the class (3.3) and (3.4). Indeed, from Theorem 2 we can obtain the following:

Corollary 1. *Let $C_0(t) \in C[0, T)$ be an arbitrary vector valued function of t (independent of space variable x). Under the same hypotheses of Theorem 2, there exist $T' \in (0, T)$ and (u, θ) belonging to (3.3) and (3.4) on $(0, T')$ such that*

$$\partial_t u - \Delta u + P(u \cdot \nabla u) = P(g\theta) + f + C_0 \qquad \text{in } L^\infty \text{ with } u(0) = u_0, \tag{3.8}$$

$$\partial_t \theta - \Delta \theta + u \cdot \nabla \theta = 0 \qquad \text{in } L^\infty \text{ with } \theta(0) = \theta_0. \tag{3.9}$$

Furthermore, put

$$p(x, t) = -C_0(t) \cdot x - \int_0^x \{(1 - P)(u \cdot \nabla u)(y, t) - (1 - P)(g\theta)(y, t)\} ds_y + p_0(t).$$

Then $\nabla p = -C_0 - (1 - P)u \cdot \nabla u + (1 - P)(g\theta)$ and this implies that (u, θ, p) is also a solution to (B') with the same initial data (u_0, θ_0, f) as in Theorem 2.

Remark 3. Assume that $u_0 = 0$, $\theta_0 = 0$ and $f = 0$ and let $C_0 = g$. Then the solution given in Corollary 1 is $(u, \theta, \nabla p) = (gt, 0, -g)$. On the other hand, under the same assumptions as above, the solution in Theorem 2 is $(u, \theta, \nabla p) = (0, 0, 0)$.

This corollary suggests that we need to impose some condition on pressure p to derive the uniqueness other than (3.3) and (3.4). Recently, J. Kato [24] proved the uniqueness of bounded solution u to (N-S) under the assumption

$$p \in L^1_{loc}(0, T; BMO).$$

In this paper we improve J. Kato's result on the uniqueness of bounded solution to (N-S), this means that our space is bigger than BMO. In addition, this method can be extended to the Boussinesq equations.

Theorem 3 (Uniqueness). *Let the initial data (u_0, θ_0, f) satisfy the hypotheses of Theorem 1. Assume that (u, θ, p) satisfies (B') in the distribution sense on $(0, T) \times \boldsymbol{R}^n$ and (u, θ) belongs to (3.1). If*

$$p \in L^1_{loc}(0, T; \dot{B}^{-\kappa}_{\infty,\infty} + \dot{\mathscr{A}}^1_{\infty,1}) \qquad \text{for some } \kappa > 0, \tag{3.10}$$

then $(u, \theta, \nabla p)$ is unique.

Remark 4. (i) Theorem 3 implies that the solution $(u, \theta, \nabla p)$ given in Theorem 2 is unique one in the class (3.1) and (3.10).

(ii) Since $BMO \subset \dot{\mathscr{A}}^0_{\infty,\infty} \subset \dot{B}^{-\kappa}_{\infty,\infty} + \dot{\mathscr{A}}^1_{\infty,1}$ for $\kappa > 0$, Theorem 3 is an improvement of J. Kato's result. We note that $(|x|^2+1)^{(1-\varepsilon)/2} \in \dot{\mathscr{A}}^1_{\infty,1}$, but $\notin BMO$ for $0 < \varepsilon < 1$.

(iii) Galdi-Maremonti [16] proved the uniqueness of bounded solution u to the Navier-Stokes equations under the assumptions; for $\varepsilon > 0$

$$|p(x,t)| \le C(|x|^{1-\varepsilon}+1) \qquad \text{and} \qquad u, \nabla u \in L^\infty(\boldsymbol{R}^n \times (0,T)).$$

There is no inclusion between Theorem 3 and the result of Galdi-Maremonti.

(iv) There holds Theorem 3 with the assumption $\theta \in C([0,T); \dot{B}^0_{\infty,1})$ replaced by $\theta \in C([0,T); L^q)$ for some $q \in (1,\infty)$. This proof is parallel to that of Theorem 3.

In two dimension we get the global solution to (B′) as well as the Navier-Stokes equations. In our situation we can choose the initial velocity $u_0 \in L^\infty$, however, by technical reason we have to restrict the initial temperature $\theta_0 \in L^q$ $(q < \infty)$, then θ_0 decays at space infinity.

Theorem 4 (global existence). *Let* $f \in L^\infty_{loc}([0,\infty); W^{1,\infty}(\boldsymbol{R}^2))$ *and let the initial data* $(u_0, \theta_0) \in L^\infty(\boldsymbol{R}^2) \times L^q(\boldsymbol{R}^2)$ *for some* $q \in (1,\infty)$ *with* $\operatorname{div} u_0 = 0$. *Then there exists a unique mild solution* $(u,\theta) \in C_w([0,\infty); L^\infty(\boldsymbol{R}^2)) \times C([0,\infty); L^q(\boldsymbol{R}^2))$ *to* (B′).

Remark 5. (i) In the proof of Theorem 4 we establish the single exponential estimate (5.33), which is sharper than the double exponential estimate (1.1) given in [20].

(ii) Let $f \in C((0,\infty); W^{1,\infty})$. Then we easily show that (u,θ) given in Theorem 4 satisfies (3.3), $\theta \in C([0,\infty); L^q) \cap C^1((0,\infty); L^\infty) \cap C((0,\infty); W^{2,\infty}) \cap C((0,\infty); \dot{B}^0_{\infty,1})$ and satisfies (3.6) and (3.7) on $(0,\infty)$ in strong sense. Moreover, when we define p as in Theorem 2, from Remark 4-(iv) we observe that $(u,\theta,\nabla p)$ is a unique solution to (B′) with $p \in L^1_{loc}(0,\infty; \dot{B}^{-\kappa}_{\infty,\infty} + \dot{\mathscr{A}}^1_{\infty,1})$ for some $\kappa > 0$.

4. Preliminary

In this section we shall present the several lemmas for the proofs of Theorems.

We first recall the fundamental estimate for the heat semigroup $e^{t\Delta}$ in Besov spaces.

Lemma 4.1. (i) *There holds for all* $f \in L^\infty(\boldsymbol{R}^n)$

$$\|e^{t\Delta}f\|_{\dot{B}^s_{\infty,1}} \le C(n,s)t^{-s/2}\|f\|_\infty \qquad \textit{for } t>0 \textit{ and } s>0. \tag{4.1}$$

(ii) *There holds for all* $f \in \dot{B}^s_{p,q}$

$$\|e^{t\Delta}f\|_{\dot{B}^{s+\alpha}_{p,q}} \le C(\alpha,n)t^{-\alpha/2}\|f\|_{\dot{B}^s_{p,q}} \qquad \textit{for } t>0,\ s\in \boldsymbol{R} \textit{ and } \alpha \ge 0. \tag{4.2}$$

We easily see that $\|e^{\Delta}f\|_{\dot{B}^s_{\infty,1}} \le C\|f\|_\infty$ for $s>0$. Combining this inequality with scaling argument yields (4.1). The proof of (4.2) is very easy, so we omit it.

We note that (4.1) yields

$$\|\nabla Pe^{t\Delta}f\|_\infty \le C(n)t^{-1/2}\|f\|_\infty \qquad \text{for } t>0, \tag{4.3}$$

since $\|Ph\|_\infty \le C\|h\|_{\dot{B}^0_{\infty,1}}$. See also [18].

Lemma 4.2. (a) *Let* $s\in\boldsymbol{R}$ *and* $f\in \dot{B}^s_{\infty,1}\cap BUC$. *Then*

$$\|e^{t\Delta}f - f\|_{\dot{B}^s_{\infty,1}\cap BUC} \to 0 \qquad \textit{as } t\downarrow 0. \tag{4.4}$$

(b) *Let* $s\in\boldsymbol{R}$ *and* $f\in L^\infty$. *Then*

$$\|e^{t'\Delta}f - e^{t\Delta}f\|_{\dot{B}^s_{\infty,1}} \le C(\alpha,n)(t'-t)^\alpha\|e^{t\Delta}f\|_{\dot{B}^{s+2\alpha}_{\infty,1}} \tag{4.5}$$

for all $0<t<t'$ *and all* $\alpha>0$.

Proof. (a) We first recall that for $f\in L^\infty(\boldsymbol{R}^n)$

$$f\in BUC \qquad \text{if and only if } \|e^{t\Delta}f - f\|_\infty \to 0 \text{ as } t\downarrow 0, \tag{4.6}$$

which is proved by [18]. Then BUC-norm tends to zero as $t\downarrow 0$ obviously.

On the other hand, since $2^{sj}\|\varphi_j*(e^{t\Delta}f-f)\|_\infty \le 2\cdot 2^{sj}\|\phi_j*f\|_\infty \in l^1$, the Lebesgue theorem on l^1 yields

$$\begin{aligned} &\lim_{t\downarrow 0}\sum_{j=-\infty}^{\infty} 2^{sj}\|\varphi_j*(e^{t\Delta}f-f)\|_\infty \\ &\quad = \sum_{j=-\infty}^{\infty} 2^{sj}\lim_{t\downarrow 0}\|\tilde{\varphi}_j*\varphi_j*(e^{t\Delta}-I)f\|_\infty \\ &\quad \le \sum_{j=-\infty}^{\infty} 2^{sj}\lim_{t\downarrow 0}\|(e^{t\Delta}-I)\tilde{\varphi}_j\|_{L^1}\|\varphi_j*f\|_\infty = 0. \end{aligned} \tag{4.7}$$

From (4.6) and (4.7) we obtain (4.4).

(b) Giga-Inui-Matsui [18] showed that

$$\|(e^{h\Delta}-I)f\|_\infty \le Ch^\alpha\|(-\Delta)^\alpha f\|_\infty \tag{4.8}$$

for all $\alpha > 0$ and all $f \in D((-\Delta)^\alpha)$. Here, the definition of $D((-\Delta)^\alpha)$ is mentioned by e.g. [18]. Then, obviously,

$$\|e^{t'\Delta}f - e^{t\Delta}f\|_{\dot{B}^s_{\infty,1}} = \sum_{j=-\infty}^{\infty} 2^{sj}\|(e^{(t'-t)\Delta} - I)e^{t\Delta}\varphi_j * f\|_\infty$$

$$\leq \sum_{j=-\infty}^{\infty} 2^{sj}C(\alpha)(t'-t)^\alpha\|(-\Delta)^\alpha\varphi_j * e^{t\Delta}f\|_\infty,$$

which implies (4.5). □

In two dimension, taking rotation to the first equation of (B′), we get the rotation equation:

(Rot) $$\omega_t - \Delta\omega + u\cdot\nabla\omega = \mathrm{rot}\, g\theta + \mathrm{rot}\, f,$$

where $\omega = \mathrm{rot}\, u$ is a scalar function. If $\theta \equiv 0$ and $f \equiv 0$, the maximum principle yields

$$\|\omega(t)\|_\infty \leq \|\omega(0)\|_\infty.$$

In [20], this estimate plays an important role in proving the global solution to the 2-dimensional Navier-Stokes equations. However, we can not apply the maximum principle for the rotation equation directly, nevertheless, $u, \partial\theta, \partial f$ and ω are also bounded. To overcome this difficulty, we need the condition $\theta \in W^{1,q}$ for $2 \leq q < \infty$. Kato-Ponce [26] also proved the L^q-version for $1 < q < \infty$ in the case of $\theta \equiv 0$.

Lemma 4.3. *Let $2 \leq q < \infty$, $i = 1, 2, \dots, n$, and let $v_0 \in L^\infty(\boldsymbol{R}^n)$ be a scalar function. Assume that $v \in L^\infty(0, T; L^\infty(\boldsymbol{R}^n))$ is a solution of*

(4.9) $$v_t - \Delta v + a(x,t)\cdot\nabla v = \partial_i\theta(x,t) + f(x,t) \qquad \text{with } v(0) = v_0.$$

Here $a \in L^\infty(0, T; W^{1,\infty})$ is a n-dimensional vector-valued function with div $a = 0$, *$\theta \in L^\infty(0, T; W^{1,\infty} \cap W^{1,q})$ and $f \in L^\infty(0, T; L^\infty)$. Then there exist v_1 and v_2 such that*

(4.10) $$v(t) = v_1(t) + v_2(t) \qquad \text{for } 0 < t < T,$$

(4.11) $$\|v_1(t)\|_\infty \leq \|v_0\|_\infty + \int_0^t \|f(\tau)\|_\infty d\tau \qquad \text{for } 0 < t < T,$$

(4.12) $$\|v_2(t)\|_q \leq C(q)\left(\int_0^t \|\theta(\tau)\|_q^2 d\tau\right)^{1/2} \qquad \text{for } 0 < t < T.$$

Proof. Thanks to the uniqueness of mild solution in $L^\infty(0, T; L^\infty)$ to (4.9), the solution v to (4.9) can be divided into two functions as follows:

$$v = v_1 + v_2, \tag{4.13}$$

$$\begin{cases} \partial_t v_1 - \Delta v_1 + a \cdot \nabla v_1 = f, \\ v_1|_{t=0} = v_0, \end{cases} \tag{4.14}$$

$$\begin{cases} \partial_t v_2 - \Delta v_2 + a \cdot \nabla v_2 = \partial_i \theta, \\ v_2|_{t=0} = 0. \end{cases} \tag{4.15}$$

We first estimate the solution v_1 to (4.14). To this end, for a moment, we assume $v_0 \in L^q \cap L^\infty$ for some $1 < q < \infty$. By [26, Lemma 4.1] we have

$$\|v_1\|_q \le \|v_0\|_q + \int_0^t \|f(\tau)\|_q d\tau. \tag{4.16}$$

Then letting $q \to \infty$, we have

$$\|v_1\|_\infty \le \|v_0\|_\infty + \int_0^t \|f(\tau)\|_\infty d\tau \tag{4.17}$$

for all $v_0 \in L^q \cap L^\infty$. To prove (4.17) for general $v_0 \in L^\infty$ without any integrability assumption, we use a cut-off function

$$\chi_N(x) = \begin{cases} 1 & \text{for } |x| \le N, \\ 0 & \text{for } |x| > N. \end{cases}$$

Let V_N be the unique solution to

$$\begin{cases} \partial_t v - \Delta v + a \cdot \nabla v = \chi_N f, \\ v|_{t=0} = \chi_N v_0. \end{cases}$$

Then by (4.17) we have

$$\|V_N\|_\infty \le \|\chi_N v_0\|_\infty + \int_0^t \|\chi_N f(\tau)\|_\infty d\tau \le \|v_0\|_\infty + \int_0^t \|f(\tau)\|_\infty d\tau \tag{4.18}$$

for all N positive integer. Since $\chi_N v_0 \to v_0$ weak-star in $L^\infty(\boldsymbol{R}^n)$ and $\chi_N f \to f$ weak-star in $L^\infty(0, T; L^\infty)$ as $N \to \infty$, there exist a subsequence $\{V_{N'}\}$ of $\{V_N\}$ and the limit v' such that $V_{N'} \to v'$ weak-star in $L^\infty(0, T; L^\infty)$ and v' satisfies (4.14). By the uniqueness of solutions to (4.14) we have $v_1(t) = v'(t)$ almost all $t \in (0, T)$ and hence we obtain

$$\|v_1(t)\|_\infty \le \liminf_{N' \to \infty} \|V_{N'}\|_\infty \le \|v_0\|_\infty + \int_0^t \|f(\tau)\|_\infty d\tau. \tag{4.19}$$

Next we estimate v_2. Taking inner product in L^2 between (4.15) and $|v_2|^{q-2} v_2$, we easily obtain the following estimate for $2 \le q < \infty$,

$$\frac{1}{q}\frac{\partial}{\partial t}\|v_2\|_q^q + (q-1)\|\,|v_2|^{(q-2)/2}\nabla v_2\|_2^2 \le \left|\int_{\boldsymbol{R}^n} (\partial_i\theta)|v_2|^{q-2}v_2\,dx\right|$$

$$= (q-1)\left|\int_{\boldsymbol{R}^n} \theta|v_2|^{q-2}\partial_i v_2\,dx\right|$$

$$\le (q-1)\|\theta\|_q\|\,|v_2|^{(q-2)/2}\|_{2q/(q-2)}\|\,|v_2|^{(q-2)/2}\nabla v_2\|_2$$

$$\le \frac{q-1}{2}\|\,|v_2|^{(q-2)/2}\nabla v_2\|_2^2 + \frac{q-1}{2}\|\theta\|_q^2\|v_2\|_q^{q-2}.$$

We thus obtain

$$\|v_2(t)\|_q^q \le \frac{q(q-1)}{2}\int_0^t \|\theta(s)\|_q^2\|v_2(s)\|_q^{q-2}ds. \tag{4.20}$$

Since $q \ge 2$, (4.20) and the usual calculation yield (4.12). □

5. Proofs of Theorems 1, 3 and 4

In this section we shall give the proofs of Theorems 1, 3 and 4. We postpone the proof of Theorem 2 until we outline it in Appendix, since Theorem 2 is not a central issue and its proof is carried out by standard argument.

We first prove Theorem 1 by using iteration.

Proof of Theorem 1. We construct a solution to $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$, using the routine iteration method. Let $0 < T < 1$ and $0 < t < T$. For j positive integer we define $\{u_j(t)\}$ and $\{\theta_j(t)\}$, inductively, as follows:

$$u_1(t) \equiv e^{t\Delta}u_0, \qquad \theta_1(t) \equiv e^{t\Delta}\theta_0,$$

$$u_{j+1}(t) \equiv u_1(t) - \int_0^t e^{(t-s)\Delta}P(u_j\cdot\nabla u_j)(s)ds$$

$$+ \int_0^t e^{(t-s)\Delta}P(g\theta_j)(s)ds + \int_0^t e^{(t-s)\Delta}f(s)ds,$$

$$\theta_{j+1}(t) \equiv \theta_1(t) - \int_0^t e^{(t-s)\Delta}(u_j\cdot\nabla\theta_j)(s)ds.$$

We notice that $\operatorname{div} u_j = 0$ for all j non-negative integer, since $\operatorname{div} f = 0$ in distribution sense. Then we have

$$u_j\cdot\nabla\theta_j = \nabla\cdot(u_j\theta_j) = \sum_{k=-\infty}^{\infty}\varphi_k * \nabla\cdot(u_j\theta_j) = \sum_{k=-\infty}^{\infty}\varphi_k * (u_j\cdot\nabla\theta_j) \qquad \text{in } \mathscr{S}',$$

when $u_j, \theta_j \in L^\infty$; see (3.2). Since the condition $\theta_0 \in \dot{B}^0_{\infty,1}$ implies $\theta_1(t) = \sum_{k=-\infty}^{\infty} \varphi_k * \theta_1(t)$ in $\mathscr{S}'$, we observe that

$$\theta_{j+1}(t) = \sum_{k=-\infty}^{\infty} \varphi_k * \theta_{j+1}(t) \qquad \text{in } \mathscr{S}' \tag{5.1}$$

when $u_j, \theta_j \in L^\infty(0,T;L^\infty)$. We now assume that

$$u_j \in C_w([0,T];L^\infty), \qquad t^{1/2}\nabla u_j \in L^\infty(0,T;L^\infty), \tag{5.2}$$

$$\theta_j \in C([0,T];\dot{B}^0_{\infty,1}), \qquad t^{1/2}\nabla\theta_j \in L^\infty(0,T;L^\infty) \tag{5.3}$$

for $j \geq 1$. We put $M_j = M_j(T)$ defined by

$$\begin{aligned} M_j \equiv &\sup_{0\leq t\leq T} \|u_j(t)\|_\infty + \sup_{0\leq t\leq T} t^{1/2}\|\nabla u_j(t)\|_\infty \\ &+ \sup_{0\leq t\leq T} \|\theta_j(t)\|_{\dot{B}^0_{\infty,1}} + \sup_{0\leq t\leq T} t^{1/2}\|\nabla\theta_j(t)\|_\infty. \end{aligned}$$

Lemma 4.1 implies

$$\|P\nabla\cdot e^{t\Delta}(u\otimes u)\|_\infty \leq \|P\nabla\cdot e^{t\Delta}(u\otimes u)\|_{\dot{B}^0_{\infty,1}} \leq C\|e^{t\Delta}(u\otimes u)\|_{\dot{B}^1_{\infty,1}} \leq Ct^{-1/2}\|u\|_\infty^2,$$

where $u\otimes u = (u^i u^j)_{i,j=1,\dots,n}$ and $\nabla\cdot(u\otimes u) = \sum_{i=1}^n \partial_i(u^i u)$. Then we have

$$\sup_{0<t<T} \|u_{j+1}(t)\|_\infty \leq C_1\|u_0\|_\infty + C_2T^{1/2}M_j^2 + C_3T|g|M_j + T\|f\|_{L^\infty(0,1:L^\infty)}. \tag{5.4}$$

Similarly, by Lemma 4.1 we have

$$\begin{aligned} \sup_{0<t<T} t^{1/2}\|\nabla u_{j+1}(t)\|_\infty \leq\ &C_1\|u_0\|_\infty + C_2T^{1/2}M_j^2 + C_3T|g|M_j \\ &+ C_4T\|f\|_{L^\infty(0,1:L^\infty)}, \end{aligned} \tag{5.5}$$

$$\sup_{0<t<T} \|\theta_{j+1}(t)\|_{\dot{B}^0_{\infty,1}} \leq C_1\|\theta_0\|_{\dot{B}^0_{\infty,1}} + C_2T^{1/2}M_j^2, \tag{5.6}$$

$$\sup_{0<t<T} t^{1/2}\|\nabla\theta_{j+1}(t)\|_\infty \leq C_1\|\theta_0\|_{\dot{B}^0_{\infty,1}} + C_2T^{1/2}M_j^2. \tag{5.7}$$

Then from (5.4)–(5.7) we obtain

$$\begin{aligned} M_{j+1} \leq\ &4C_1(\|u_0\|_\infty + \|\theta_0\|_{\dot{B}^0_{\infty,1}}) + 4C_2T^{1/2}M_j^2 \\ &+ 4C_3T|g|M_j + 4C_4\|f\|_{L^\infty(0,1;L^\infty)}. \end{aligned} \tag{5.8}$$

By induction and (5.1) we thus see that (5.2) and (5.3) hold for all j. Moreover, we have

$$M_{j+1} < \frac{1 - 4C_3 T|g|}{8C_2 T^{1/2}} \equiv M,$$

provided that

$$(5.9) \quad T \le \frac{1}{\{256C_2[C_1(\|u_0\|_\infty + \|\theta_0\|_{\dot{B}^0_{\infty,1}}) + C_4\|f\|_{L^\infty(0,1:L^\infty)}]\}^2 + 8C_3|g| + 1}.$$

Similar to M_j, letting $F_j = F_j(T)$ defined by

$$F_j \equiv \sup_{0\le t\le T} \|u_{j+1}(t) - u_j(t)\|_\infty + \sup_{0\le t\le T} t^{1/2}\|\nabla u_{j+1}(t) - \nabla u_j(t)\|_\infty$$
$$+ \sup_{0\le t\le T} \|\theta_{j+1}(t) - \theta_j(t)\|_{\dot{B}^0_{\infty,1}} + \sup_{0\le t\le T} t^{1/2}\|\nabla\theta_{j+1}(t) - \nabla\theta_j(t)\|_\infty$$

for all j non-negative integer. In the same way as in proving (5.8), we obtain

$$F_{j+1} \le \frac{5}{8}F_j \qquad \text{and hence } \sum_{j=0}^\infty F_j < \infty$$

under the condition (5.9). Therefore, there exist the limit functions $u \in L^\infty(0,T;L^\infty)$ and $\theta \in L^\infty(0,T;\dot{B}^0_{\infty,1})$ such that

$$t^{1/2}\nabla u \in L^\infty(0,T;L^\infty), \qquad t^{1/2}\nabla\theta \in L^\infty(0,T;L^\infty),$$

$$\sup_{0\le t\le T} \|u(t) - u_j(t)\|_\infty + \sup_{0\le t\le T} t^{1/2}\|\nabla u(t) - \nabla u_j(t)\|_\infty + \sup_{0\le t\le T} \|\theta(t) - \theta_j(t)\|_{\dot{B}^0_{\infty,1}}$$
$$+ \sup_{0\le t\le T} t^{1/2}\|\nabla\theta(t) - \nabla\theta_j(t)\|_\infty \to 0 \qquad \text{as } j \to \infty$$

under the condition (5.9). We easily show that (u,θ) satisfies $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$, that is, (u,θ) is a mild solution to $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$.

The proof of the uniqueness of mild solution to $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$ in the class (3.1) is standard argument, so we may omit it; see e.g. [17]. □

Proof of Theorem 3. Since (u,θ,p) satisfies (B′) in distribution sense, we have

$$(5.10) \quad \int_0^T \left\{ \left(u, -\frac{\partial}{\partial t}v - \Delta v\right) - (u\cdot\nabla v, u) - (g\theta, v) - (p, \operatorname{div} v) - (f, v) \right\} ds = 0$$

$$(5.11) \quad \int_0^T \left\{ \left(\theta, -\frac{\partial}{\partial t}v - \Delta v\right) - (u\cdot\nabla v, \theta) \right\} ds = 0$$

for all $v \in C_0^\infty((0,T)\times \boldsymbol{R}^n)$ vector-valued. Substituting $v = \nabla\Phi$ into (5.10), we obtain

$$(5.12) \qquad \int_0^T \{-(u^m \partial_m \partial_i \Phi, u^i) - (g\theta, \nabla\Phi) - (p, \Delta\Phi)\} ds = 0$$

for all $\Phi \in C_0^\infty((0,T) \times \boldsymbol{R}^n)$ scalar function, since $\operatorname{div} u = \operatorname{div} f = 0$ in distribution sense. We now set $\Psi \in C_0^\infty((0,T) \times \boldsymbol{R}^n)$ vector-valued and

$$\Phi_N = (-\Delta)^{-1} \operatorname{div} S_N \Psi$$

for $N \geq 1$, where $S_N f = \sum_{k=-N}^N \varphi_k * f$. Then we substitute Φ_N into (5.12) to obtain

$$(5.13) \quad \int_0^T \{-(u^m R_i R_l S_N \partial_m \Psi^l, u^i) - (g^i\theta, R_i R_m S_N \Psi^m) + (p, \operatorname{div} S_N \Psi)\} ds = 0.$$

To observe the limit of the first term of (5.13), we easily see that

$$(5.14) \qquad \begin{aligned} \|\partial_m \Psi\|_{\dot{B}^0_{1,1}} &= \sum_{j<0} \|\varphi_j * \partial_m \Psi\|_1 + \sum_{j \geq 0} \|\varphi_j * \partial_m \Psi\|_1 \\ &\lesssim \sum_{j<0} 2^j \|(\partial_m \varphi)_j * \Psi\|_1 + \sum_{j \geq 0} 2^{-2j} \|((-\Delta)^{-1}\varphi)_j * \Delta \partial_m \Psi\|_1 \\ &\lesssim \|\Psi\|_{W^{3,1}}. \end{aligned}$$

Then we have $\partial_m \Psi \in L^1(0,T; \dot{B}^0_{1,1})$ and hence

$$(5.15) \qquad S_N R_i R_l \partial_m \Psi^l \to R_i R_l \partial_m \Psi^l \qquad \text{in } L^1(0,T; \dot{B}^0_{1,1}) \text{ as } N \to \infty.$$

To estimate the second term of (5.13), we recall the definition of Riesz transform (2.5), which implies that

$$(5.16) \quad (g^i\theta(t), R_i R_m S_N \Psi^m(t)) \to (R_i R_m(g^i\theta(t)), \Psi^m(t)) \qquad \text{as } N \to \infty.$$

Since

$$\begin{aligned} |(g^i\theta(t), R_i R_m S_N \Psi^m(t))| &= |(R_i R_m S_N (g^i\theta(t)), \Psi^m(t))| \\ &\leq |g| \, \|\theta(t)\|_{\dot{B}^0_{\infty,1}} \|\Psi(t)\|_1 \in L^1(0,T), \end{aligned}$$

from (5.16) and the Lebesgue dominated convergence theorem we obtain

$$(5.17) \quad \int_0^T (g^i\theta, R_i R_m S_N \Psi^m) ds \to \int_0^T (R_i R_m(g^i\theta), \Psi^m) ds \qquad \text{as } N \to \infty.$$

To estimate the third term of (5.13), by the hypothesis on p we have $p(t) \in \dot{B}^{-\kappa}_{\infty,\infty} + \dot{\mathscr{A}}^1_{\infty,1}$ almost all $t \in (0,T)$, which implies

$$(5.18) \qquad \nabla p(t) = \lim_{N \to \infty} S_N \nabla p(t) \qquad \text{in } \mathscr{S}' \text{ as } N \to \infty$$

for almost all $t \in (0,T)$. See the definitions (2.2) and (2.4) of $\dot{B}^{-\kappa}_{\infty,\infty}$ and $\dot{\mathscr{A}}^1_{\infty,1}$. In the same way as in proving (5.14) we easily see

$$\|\mathrm{div}\ \Psi\|_{\dot{B}^{\kappa}_{1,1}} + \|\mathrm{div}\ \Psi\|_{\dot{B}^{-1}_{1,\infty}} \lesssim \|\Psi\|_{W^{\kappa+2,1}}.$$

Letting $p = p^1 + p^2$, then we have

$$(5.19)\quad \begin{aligned} |(p, S_N\ \mathrm{div}\ \Psi)| &\le \sum_{j=-N}^{N} \{|(p^1, \varphi_j * \mathrm{div}\ \Psi)| + |(p^2, \varphi_j * \mathrm{div}\ \Psi)|\} \\ &\le \sum_{j=-N}^{N} \{|(\tilde{\varphi}_j * p^1, \varphi_j * \mathrm{div}\ \Psi)| + |(\tilde{\varphi}_j * p^2, \varphi_j * \mathrm{div}\ \Psi)|\} \\ &\le \|p^1\|_{\dot{B}^{-\kappa}_{\infty,\infty}} \|\mathrm{div}\ \Psi\|_{\dot{B}^{\kappa}_{1,1}} + \|p^2\|_{\dot{\mathscr{A}}^1_{\infty,1}} \|\mathrm{div}\ \Psi\|_{\dot{B}^{-1}_{1,\infty}}, \end{aligned}$$

which implies $|(p(t), S_N\ \mathrm{div}\ \Psi(t))| \le C\|p(t)\|_{\dot{B}^{-\kappa}_{\infty,\infty}+\dot{\mathscr{A}}^1_{\infty,1}} \|\Psi(t)\|_{W^{\kappa+2,1}} \in L^1(0,T)$. Here the constant C is independent of N. From this and (5.18) we obtain

$$(5.20)\qquad \int_0^T (p, \mathrm{div}\ S_N\Psi)ds \to \int_0^T (p, \mathrm{div}\ \Psi)ds \qquad \text{as } N \to \infty.$$

Gathering (5.15), (5.17) and (5.20), we take the limit $N \to \infty$ for (5.13) to obtain

$$(5.21)\qquad \int_0^T \{-(u^m R_i R_l \partial_m \Psi^l, u^i) - (g^i\theta, R_i R_m \Psi^m) + (p, \mathrm{div}\ \Psi)\}ds = 0$$

for all $\Psi \in C_0^\infty((0,T) \times \boldsymbol{R}^n)$. The combination of (5.21) with (5.10) implies

$$(5.22)\quad \int_0^T \left\{\left(u, -\frac{\partial}{\partial t}v - \Delta v\right) - (u \cdot \nabla Pv, u) - (P(g\theta), v) - (f, v)\right\}ds = 0$$

for all $v \in C_0^\infty((0,T) \times \boldsymbol{R}^n)$. In the similar way as in [18, Proof of Theorem 3], from (5.22) and (5.11) we observe that (u,θ) is a mild solution $(\mathrm{I.E.B})_1$ and $(\mathrm{I.E.B})_2$ in the class (3.1). Therefore, it follows from uniqueness of mild solution in the class (3.1) that (u,θ) is uniquely determined. This uniqueness of (u,θ) and (5.21) imply that ∇p is also unique, which completes the proof of Theorem 3. □

Before proving Theorem 4, we shall state the local existence theorem again.

Lemma 5.1. *Assume that the initial data* $(u_0, \theta_0) \in L^\infty(\boldsymbol{R}^2) \times L^q(\boldsymbol{R}^2)$ *with*

$\operatorname{div} u_0 = 0$ *for some* $q \in (1, \infty)$, $f \in L^{\infty}_{loc}([0, \infty); L^{\infty}(\boldsymbol{R}^2))$ *with* $\operatorname{div} f = 0$ *and that* g *is a constant vector. Then there exist* $T > 0$ *and a unique mild solution*

$$(5.23) \qquad (u, \theta) \in C_w([0, T); L^{\infty}(\boldsymbol{R}^2)) \times C([0, T); L^q(\boldsymbol{R}^2))$$

to $(\mathrm{I.E.B})_1$ *and* $(\mathrm{I.E.B})_2$. *Here, we note that*

$$(5.24) \qquad T > \frac{C}{(\|u_0\|_{\infty} + \|\theta_0\|_q + \|f\|_{L^{\infty}(0,1;L^{\infty})})^2 + |g|^{q/(q-1)} + 1}.$$

Moreover,

$$(5.25) \qquad u \in C((0, T); W^{2,\infty}) \qquad \text{and} \qquad \theta \in C((0, T); W^{2,\infty}).$$

Since the proof of Lemma 5.1 is parallel to Theorems 1 and 2, we may omit it.

Proof of Theorem 4. We show that the time-local solution given in Lemma 5.1 can be extended to a global one. Thanks to (5.25), we may assume $u_0 \in W^{2,\infty}$, $\theta_0 \in L^q \cap W^{2,\infty}$ and assume $(u, \theta) \in C([0, T); W^{2,\infty}) \times C([0, T); W^{2,\infty} \cap W^{2,q})$. We shall prove that

$$(5.26) \qquad \sup_{0<t<T} (\|u(t)\|_{\infty} + \|\theta(t)\|_q) < \infty \qquad \text{if } T < \infty.$$

If (5.26) holds, from (5.24) we easily see that (u, θ) is a global solution by using the standard argument; see [20].

Since θ is a solution to $\theta_t - \Delta\theta + u \cdot \nabla\theta = 0$ and $\operatorname{div} u = 0$, from (4.16) and (4.19) we can obtain that for all $q' \in [q, \infty]$

$$(5.27) \qquad \|\theta(t)\|_{q'} \le \|\theta_0\|_{q'} \qquad \text{for all } t > 0.$$

Concerning the estimate of $\|u\|_{\infty}$, we use Lemma 4.3, $(\mathrm{I.E.B})_1$ and the similar method as in [28, 37]. We now recall that the Littlewood-Paley decomposition:

$$1 = \hat{\psi}_{-N}(\xi) + \sum_{j \ge -N} \hat{\varphi}_j(\xi) \qquad (\xi \in \boldsymbol{R}^2, N = 0, 1, 2, \dots,).$$

We easily see that

$$(5.28) \qquad \|u(t)\|_{\infty} \le \|u(t)\|_{B^0_{\infty,1}} \le \|\psi_{-N} * u(t)\|_{\infty} + \sum_{j=-N}^{\infty} \|\varphi_j * u(t)\|_{\infty}$$
$$\equiv I_1 + I_2$$

Since u satisfies $(\mathrm{I.B.E})_1$, by (5.27) we have

$$(5.29)\qquad I_1 = \left\| \psi_{-N} * e^{t\Delta}u_0 + \int_0^t P\nabla \cdot \psi_{-N} * e^{(t-\tau)\Delta}(u \otimes u)(\tau)d\tau \right.$$
$$\left. + \int_0^t \psi_{-N} * e^{(t-\tau)\Delta}P(g\theta)(\tau)d\tau + \int_0^t \psi_{-N} * e^{(t-\tau)\Delta}f(\tau)d\tau \right\|_\infty$$
$$\le \|u_0\|_\infty + C\int_0^t 2^{-N}\|u(\tau)\|_\infty^2 d\tau$$
$$+ C\int_0^t 2^{-2N/q'}|g|\,\|\theta(\tau)\|_{q'}d\tau + \int_0^T \|f(\tau)\|_\infty d\tau$$
$$\le \|u_0\|_\infty + C\int_0^t 2^{-N}\|u(\tau)\|_\infty^2 d\tau + CT|g|\,\|\theta_0\|_{q'} + \int_0^T \|f(\tau)\|_\infty d\tau.$$

Here we used in (5.29) that $\|P\nabla \cdot \psi_{-N}\|_{L^1} \le C\|\nabla \cdot \psi_{-N}\|_{\dot{B}^0_{1,1}} \le C2^{-N}$. Let $\omega(t) = \operatorname{rot} u(t)$. Since ω satisfies

$$\frac{\partial}{\partial t}\omega - \Delta\omega + u \cdot \nabla\omega = \operatorname{rot} f + \operatorname{rot}(g\theta),$$

by Lemma 4.3 we can decompose ω into two parts ($\omega = \omega^1 + \omega^2$) and obtain that

$$\sup_{0<t<T} \|\omega^1(t)\|_\infty \le \|\operatorname{rot} u_0\|_\infty + \int_0^T \|\operatorname{rot} f(s)\|_\infty ds,$$
$$\sup_{0<t<T} \|\omega^2(t)\|_{q'} \le C(q') \sup_{0<t<T}\left(\int_0^t \|g\theta(s)\|_{q'}^2 ds\right)^{1/2} \le C(q')T^{1/2}|g|\,\|\theta_0\|_{q'}.$$

To estimate I_2, we use the Biot-Savart law, that is,

$$\varphi_j * u = ((-\Delta)^{-1}\partial_2(\varphi_j * \omega), -(-\Delta)^{-1}\partial_1(\varphi_j * \omega)),$$

which yields

$$(5.30)\qquad \|\varphi_j * u(t)\|_\infty \le C2^{-j}\|\varphi_j * \omega\|_\infty \le C(2^{-j}\|\omega^1(t)\|_\infty + 2^{-j+2j/q'}\|\omega^2(t)\|_{q'})$$
$$\le C(2^{-j} + 2^{-j+2j/q'})K(T,q'),$$

where $K(T,q') \equiv \|\operatorname{rot} u_0\|_\infty + \int_0^T \|\operatorname{rot} f(s)\|_\infty ds + C(q')T^{1/2}|g|\,\|\theta_0\|_{q'}$. Let $q' > 2$. Then we have that

$$(5.31)\qquad I_2 \le C(2^N + 2^{N-2N/q'})K(T,q') \le C2^N K(T,q').$$

Hence, gathering (5.29) and (5.31) with (5.28), we obtain that

$$\|u(t)\|_{\infty} \le \|u_0\|_{\infty} + C\int_0^t 2^{-N}\|u(\tau)\|_{\infty}^2 d\tau + CT|g|\,\|\theta_0\|_{q'} + \int_0^T \|f(\tau)\|_{\infty} d\tau + C2^N K(T,q'). \tag{5.32}$$

Let $N = 0$, if $K(T,q') \ge \int_0^t \|u(\tau)\|_{\infty}^2 d\tau$. Let $N = [\log_2\{K(T,q')^{-1/2} \cdot (\int_0^t \|u(\tau)\|_{\infty}^2 d\tau)^{1/2}\}] + 1$, i.e., $2^N \sim K(T,q')^{-1/2}(\int_0^t \|u(\tau)\|_{\infty}^2 d\tau)^{1/2}$, if $K(T,q') < \int_0^t \|u(\tau)\|_{\infty}^2 d\tau$. (This setting is similar to [37].) Then we get

$$\|u(t)\|_{\infty} \le \|u_0\|_{\infty} + CK(T,q')^{1/2}\left\{\int_0^t \|u(\tau)\|_{\infty}^2 d\tau\right\}^{1/2} + CT|g|\,\|\theta_0\|_{q'} + \int_0^T \|f(\tau)\|_{\infty} d\tau + CK(T,q'),$$

and hence

$$\|u(t)\|_{\infty}^2 \le C\left(\|u_0\|_{\infty} + CT|g|\,\|\theta_0\|_{q'} + \int_0^T \|f(\tau)\|_{\infty} d\tau + K(T,q')\right)^2 + CK(T,q')\int_0^t \|u(\tau)\|_{\infty}^2 d\tau.$$

We apply the Gronwall inequality to obtain

$$\sup_{0<t\le T} \|u(t)\|_{\infty} \le C\left(\|u_0\|_{\infty} + CT|g|\,\|\theta_0\|_{q'} + \int_0^T \|f(\tau)\|_{\infty} d\tau + K(T,q')\right) \exp\{CTK(T,q')\}. \tag{5.33}$$

Therefore, the proof of Theorem 4 is now complete. □

6. Appendix

In this Appendix, we shall show that the mild solution (u,θ) given in Theorem 1 satisfies (B′) in the strong sense under the additional assumption $f \in C((0,T); W^{1,\infty})$.

Proof of Theorem 2. We assume the assumption of Theorem 2. We divide the proof into four parts.

Step 1: Since $u, t^{1/2}\nabla u, \theta$ and $t^{1/2}\nabla\theta$ belong to $L^{\infty}(0,T;L^{\infty})$, there holds

$$\sup_{\varepsilon<s<T'} \|u(s)\|_{\dot{B}^k_{\infty,1}} + \sup_{\varepsilon<s<T'} \|\theta(s)\|_{\dot{B}^k_{\infty,1}} < \infty \tag{6.1}$$

for all $0 < \varepsilon < T' < T$ and all $k \in (0,1)$. Now we shall show that (6.1) also holds for $k \in [1,3)$ by using the induction. We note that u satisfies

$$u(t) = e^{(t-\varepsilon)\Delta}u(\varepsilon) - \int_\varepsilon^t e^{(t-\tau)\Delta}P(u\cdot\nabla u)(\tau)d\tau + \int_\varepsilon^t e^{(t-\tau)\Delta}P(g\theta)(\tau)d\tau + \int_\varepsilon^t e^{(t-\tau)\Delta}f(\tau)d\tau,$$

and $f \in L^\infty_{loc}(0,T;\dot{B}^l_{\infty,1})$ for all $0 < l < 1$ since $W^{1,\infty} \subset \dot{B}^l_{\infty,1}$. Thus by Lemma 4.1 (ii) we have

$$\begin{aligned}
\|u(t)\|_{\dot{B}^{k+1/2}_{\infty,1}} &\lesssim (t-\varepsilon)^{-1/4}\|u(\varepsilon)\|_{\dot{B}^k_{\infty,1}} + \int_\varepsilon^t (t-\tau)^{-3/4}\|(u\otimes u)(\tau)\|_{\dot{B}^k_{\infty,1}}d\tau \\
&\quad + \int_\varepsilon^t (t-\tau)^{-1/4}\|g\theta(\tau)\|_{\dot{B}^k_{\infty,1}}d\tau + \int_\varepsilon^t (t-\tau)^{-(1/2)(k-l+1/2)}\|f(\tau)\|_{\dot{B}^l_{\infty,1}}d\tau \\
&\lesssim (t-\varepsilon)^{-1/4}\|u(\varepsilon)\|_{\dot{B}^k_{\infty,1}} + (t-\varepsilon)^{1/4}\sup_{\varepsilon<s<T'}\|u(s)\|_\infty \sup_{\varepsilon<s<T'}\|u(s)\|_{\dot{B}^k_{\infty,1}} \\
&\quad + |g|(t-\varepsilon)^{3/4}\sup_{\varepsilon<s<T'}\|\theta(s)\|_{\dot{B}^k_{\infty,1}} + (t-\varepsilon)^{3/4-k/2+l/2}\sup_{\varepsilon<s<T'}\|f(s)\|_{\dot{B}^l_{\infty,1}}
\end{aligned}$$

for all $\varepsilon < t \le T'$ and all $\max\{0, k-\frac{3}{2}\} < l < \min\{1, k+\frac{1}{2}\}$, where we applied the Hölder type inequality in Besov spaces such as

$$\|fg\|_{\dot{B}^s_{\infty,1}} \le C(n,s)(\|f\|_\infty\|g\|_{\dot{B}^s_{\infty,1}} + \|g\|_\infty\|f\|_{\dot{B}^s_{\infty,1}}) \qquad \text{for } s > 0. \tag{6.2}$$

This inequality was proved by e.g. [10]. Similarly, we have

$$\|\theta(t)\|_{\dot{B}^{k+1/2}_{\infty,1}} \lesssim (t-\varepsilon)^{-1/4}\|\theta(\varepsilon)\|_{\dot{B}^k_{\infty,1}} + (t-\varepsilon)^{1/4}\sup_{\varepsilon<s<T'}(\|u(s)\|_\infty\|\theta(s)\|_{\dot{B}^k_{\infty,1}} + \|\theta(s)\|_\infty\|u(s)\|_{\dot{B}^k_{\infty,1}}).$$

Hence, we conclude that (6.1) also holds for all $0 < s < T' < T$ and all $k \in (0,3)$.

Step 2: We shall show in this step that for all $k \in (0,3)$

$$u \in C((0,T);\dot{B}^k_{\infty,1}) \cap C((0,T);BUC) \tag{6.3}$$

$$\theta \in C((0,T);\dot{B}^k_{\infty,1}) \cap C([0,T);\dot{B}^0_{\infty,1}). \tag{6.4}$$

Let $k \in (0,3)$ and choose $\alpha \in (0,1/2)$ and $l \in (0,1)$ such that $0 < k-l < 2$ and $0 < 2\alpha + k - l < 2$. Since

$$u(t') - u(t) = (e^{t'\Delta} - e^{t\Delta})u(\epsilon) - \int_{\epsilon}^{t} (e^{(t'-s)\Delta} - e^{(t-s)\Delta})[P(\nabla \cdot (u \otimes u) - g\theta) - f](s)ds$$
$$- \int_{t}^{t'} e^{(t'-s)\Delta}[P(\nabla \cdot (u \otimes u) - g\theta) - f](s)ds$$

for $0 < \epsilon < t < t' < T$, we have by Lemmas 4.1 and 4.2 (a),(b)

$$\begin{aligned}
\|u(t') - u(t)\|_{\dot{B}^k_{\infty,1}} &\lesssim (t'-t)^{\alpha}\|e^{t\Delta}u(\epsilon)\|_{\dot{B}^{k+2\alpha}_{\infty,1}} \\
&\quad + \int_{\epsilon}^{t} [(t'-t)^{\alpha}(t-s)^{-\alpha-1/2}\|u \otimes u(s)\|_{\dot{B}^k_{\infty,1}} \\
&\quad + (t'-t)^{\alpha}(t-s)^{-\alpha}\|g\theta(s)\|_{\dot{B}^k_{\infty,1}} + (t'-t)^{\alpha}(t-s)^{-\alpha-(k-l)/2}\|f(s)\|_{\dot{B}^l_{\infty,1}}]ds \\
&\quad + \int_{t}^{t'} [(t'-s)^{-1/2}\|u \otimes u(s)\|_{\dot{B}^k_{\infty,1}} + \|g\theta(s)\|_{\dot{B}^k_{\infty,1}} + (t'-s)^{-(k-l)/2}\|f(s)\|_{\dot{B}^l_{\infty,1}}]ds \\
&\lesssim (t'-t)^{\alpha}t^{-\alpha-k/2}\|u(\epsilon)\|_{\infty} + (t'-t)^{\alpha}t^{-\alpha+1/2}\left(\sup_{\epsilon<s<t'}\|u(s)\|_{\dot{B}^k_{\infty,1}\cap L^{\infty}}\right)^2 \\
&\quad + (t'-t)^{\alpha}t^{-\alpha+1}|g|\left(\sup_{\epsilon<s<t'}\|\theta(s)\|_{\dot{B}^k_{\infty,1}}\right) \\
&\quad + (t'-t)^{\alpha}t^{1-\alpha-(k-l)/2}\left(\sup_{\epsilon<s<t'}\|f(s)\|_{\dot{B}^l_{\infty,1}}\right) \\
&\quad + (t'-t)^{1/2}\left(\sup_{\epsilon<s<t'}\|u(s)\|_{\dot{B}^k_{\infty,1}\cap L^{\infty}}\right)^2 + (t'-t)|g|\sup_{\epsilon<s<t'}\|\theta(s)\|_{\dot{B}^k_{\infty,1}} \\
&\quad + (t'-t)^{1-(k-l)/2}\sup_{\epsilon<s<t'}\|f(s)\|_{\dot{B}^l_{\infty,1}}
\end{aligned}$$

for all $0 < \epsilon < t < t' < T$, which implies $u \in C((0,T); \dot{B}^k_{\infty,1})$ for $0 < k < 3$. Similarly, we obtain $u \in C((0,T); BUC)$ and $\theta \in C((0,T); \dot{B}^k_{\infty,1}) \cap C([0,T); \dot{B}^0_{\infty,1})$ for all $0 < k < 3$.

Step 3: Next, we shall prove that (u,θ) satisfies (3.6) and (3.7) in the strong sense, showing (3.5). By (6.2) we easily obtain that for all $l > -1$ and $0 < t < t+h < T$

$$\|\nabla \cdot (u \otimes u)\|_{\dot{B}^l_{\infty,1}} \lesssim \|u \otimes u\|_{\dot{B}^{l+1}_{\infty,1}} \lesssim \|u\|^2_{\dot{B}^{l+1}_{\infty,1}\cap L^{\infty}} \tag{6.5}$$

and

$$\begin{aligned}
&\|\nabla \cdot (u \otimes u)(t+h) - \nabla \cdot (u \otimes u)(t)\|_{\dot{B}^l_{\infty,1}} \\
&\quad \lesssim (\|u(t+h)\|_{\dot{B}^{l+1}_{\infty,1}\cap L^{\infty}} + \|u(t)\|_{\dot{B}^{l+1}_{\infty,1}\cap L^{\infty}})\|u(t+h) - u(t)\|_{\dot{B}^{l+1}_{\infty,1}\cap L^{\infty}},
\end{aligned} \tag{6.6}$$

which implies that $u \cdot \nabla u \in C((0,T); \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1})$. Since (6.4) implies $g\theta \in C((0,T); \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1})$, we obtain (3.5) and that

$$P(u \cdot \nabla u), \qquad P(g\theta) \in C((0,T); \dot{B}^0_{\infty,1} \cap \dot{B}^1_{\infty,1}) \subset C((0,T); BUC). \tag{6.7}$$

On the other hand, (6.3) implies that

$$\frac{\partial}{\partial x_i} u, \; \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} u \in C((0,T); \dot{B}^0_{\infty,1}) \subset C((0,T); BUC)$$

for $i, j = 1, 2, \dots, n$, and hence

$$u \in C((0,T); W^{2,\infty}). \tag{6.8}$$

In particular, we note that

$$\Delta u \in C((0,T); BUC). \tag{6.9}$$

Now we recall that Δ generates the strong continuous and analytic semigroup $\{e^{t\Delta}\}_{t \geq 0}$ in BUC. For the detail, see e.g. [18]. Since

$$u(t') - u(t) = (e^{(t'-t)\Delta} - I)u(t) - \int_t^{t'} e^{(t'-s)\Delta}[P(u \cdot \nabla u - g\theta) - f](s)ds,$$

and since $\|e^{t\Delta}F - F\|_\infty \leq Ct^{1/2}\|\nabla F\|_\infty$ implies $\lim_{t' \downarrow t} \int_t^{t'} e^{(t'-s)\Delta}F(s)ds/(t'-t) = F(t)$ for $F \in C((0,T); W^{1,\infty})$, from (6.7)–(6.9) we obtain

$$u_t = \left(\lim_{t' \to t} \frac{u(t') - u(t)}{t' - t} \text{ in } L^\infty\right) = \Delta u - P(u \cdot \nabla u) + P(g\theta) + f \in C((0,T); BUC).$$

This and (6.8) imply (3.3) and (3.6). Similarly, we also prove (3.4) and (3.7).

Step 4: It remains to show that (u, θ, p) satisfies (B′) in the strong sense, when p is defined as in Theorem 2. We easily prove that

$$\mathrm{rot}[-(1-P)(u \cdot \nabla u) + (1-P)(g\theta)] = 0.$$

Then we have

$$\nabla p = -(1-P)(u \cdot \nabla u) + (1-P)(g\theta). \tag{6.10}$$

Therefore, from (3.6) and (6.10) we conclude that (u, θ, p) satisfies (B′) on $(0,T)$ in the strong sense. This proves Theorem 2. □

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