Funkcialaj Ekvacioj, 47 (2004) 205–224

On the Solvability of Complete Abstract Differential Equations of Elliptic Type

By

Angelo Favini*, Rabah Labbas, Hiroki Tanabe and Atsushi Yagi

(Università degli Studi di Bologna, Italia, Université du Havre, France, Otemon Gakuin University, Japan and Osaka University, Japan)

Abstract. In this work we give some new results on complete abstract second order differential equations of elliptic type in a Banach space. The existence and the uniqueness of the strict solution are proved under some natural assumptions generalising previous theorems on the subject.

Key Words and Phrases. Second order abstract differential equation, Boundary condition, Analytic semigroup, Strongly continuous group.

2000 *Mathematics Subject Classification Numbers.* 35J25, 35J40, 47D06.

1. Introduction and hypotheses

Let us consider the second order abstract differential equation

$$(1) \qquad u''(t) + 2Bu'(t) + Au(t) = f(t), \qquad t \in (0,1),$$

together with the boundary conditions

$$(2) \qquad \begin{cases} u(0) = u_0, \\ u(1) = u_1. \end{cases}$$

Here f is a continuous X-valued function on $[0,1]$, X being a complex Banach space, u_0, u_1 are given elements of $D(A)$, the domain of A, A and B are two closed linear operators in X.

We seek for a strict solution $u(.)$ to (1), (2), i.e., a function

$$u \in C^2([0,1];X) \cap C^1([0,1];D(B)) \cap C([0,1];D(A)),$$

satisfying (1) and (2).

Our main goal is to give an alternative approach with respect to recent results due to El Haial and Labbas [1]. To this end we will assume that

$$(3) \qquad \begin{cases} B^2 - A \text{ is a densely defined closed linear operator on } X \text{ such that} \\ \forall \lambda \geqslant 0,\ \exists (\lambda I + B^2 - A)^{-1} \in L(X) \text{ with} \\ \|(\lambda I + B^2 - A)^{-1}\|_{L(X)} \leqslant C/(1+\lambda), \end{cases}$$

* The research was partially supported by italian MIUR and by University of Bologna, funds for selected research topics. It fits the program of GNAMPA.

(it is well known that hypothesis (3) implies that $-(B^2 - A)^{1/2}$ is the infinitesimal generator of an analytic semigroup $\{V(t)\}$, $t \geqslant 0$, in X),

$$(4) \qquad \begin{cases} (B^2 - A) \text{ and } B \text{ commute in the resolvent sense, i.e.} \\ (\lambda I + B^2 - A)^{-1}(B - \mu I)^{-1} = (B - \mu I)^{-1}(\lambda I + B^2 - A)^{-1} \\ \forall \mu \in \rho(B),\ \forall \lambda \geqslant 0, \end{cases}$$

$$(5) \qquad B \text{ generates a strongly continuous group } \{e^{tB}\} \text{ on } X,$$

and

$$(6) \qquad D(A) \subseteq D(B^2),$$

$$(7) \qquad D((B^2 - A)^{1/2}) \subseteq D(B).$$

Observe that (4) together with (6) and (7) yields

$$(8) \qquad B^2(B^2 - A)^{-1} = B(B^2 - A)^{-1/2}B(B^2 - A)^{-1/2} \in L(X);$$

moreover we have

$$(9) \qquad A(B^2 - A)^{-1} \in L(X).$$

Notice that if A, B are two self-adjoint operators in a Hilbert space X such that $B^2 - A$ is selfadjoint as well and (3), (6) hold, then

$$D((B^2 - A)^{1/2}) \subseteq D(|B|) \subseteq D(B)$$

so that (7) is verified, too.

Differently of [1], whose assumptions are similar to ours, we give a direct appraoch to problem (1), (2), extending to the case $B \neq 0$ the pioneering work by S. G. Krein [4], pp. 249–270.

In fact, a representation formula of the solution is found taking into account the basic properties of analytic semigroups.

The plan of the paper is as follows. Section 2 contains the abstract results. In section 3 we give some examples of application to partial differential equations.

2. Main results

We shall establish the main following result

Theorem 1. *Under assumptions* (3)∼(7), *if both* $B + (B^2 - A)^{1/2}$ *and* A *have a bounded inverse, then for all* $f \in C^\theta([0,1]; X)$, $0 < \theta < 1$ *and any* $u_0, u_1 \in D(A)$, *problem* (1)–(2) *has a unique strict solution on* $[0,1]$.

Here $C^\theta([0,1]; X)$ denotes the space of all X-valued Hölder continuous functions on $[0,1]$ with exponent θ.

For the proof of this Theorem, we need some Lemmas.

Lemma 2. *Assume that* A *and* $B+(B^2-A)^{1/2}$ *have bounded inverses. Then also* $B-(B^2-A)^{1/2}$ *is boundedly invertible, with*

$$\begin{cases} (B-(B^2-A)^{1/2})^{-1} = (B+(B^2-A)^{1/2})A^{-1}, \\ (B+(B^2-A)^{1/2})^{-1} = (B-(B^2-A)^{1/2})A^{-1}. \end{cases} \tag{10}$$

Proof. In view of assumption (4) we have

$$B(B^2-A)^{-1/2}y = (B^2-A)^{-1/2}By \qquad \forall y \in D(B). \tag{11}$$

Since $(B^2-A)^{1/2}A^{-1}x = (B^2-A)^{-1/2}(B^2-A)A^{-1}x \in D((B^2-A)^{1/2}) \subseteq D(B)$ for all $x \in X$ (see (6), (7)), it follows from (11) that

$$\begin{aligned} BA^{-1}x &= B(B^2-A)^{-1/2}(B^2-A)^{1/2}A^{-1}x \\ &= (B^2-A)^{-1/2}B(B^2-A)^{1/2}A^{-1}x \qquad \forall x \in X. \end{aligned}$$

This implies that $D(A) \subseteq D((B^2-A)^{1/2}B) \cap D(B(B^2-A)^{1/2})$ and

$$(B^2-A)^{1/2}BA^{-1}x = B(B^2-A)^{1/2}A^{-1}x \qquad \forall x \in X. \tag{12}$$

Let

$$\begin{aligned} z &\in D((B-(B^2-A)^{1/2})(B+(B^2-A)^{1/2})) \\ &= D((B^2-A)^{1/2}(B+(B^2-A)^{1/2})), \end{aligned}$$

and

$$\begin{aligned} x &= (B^2-A)^{1/2}(B+(B^2-A)^{1/2})z, \\ y &= (B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}x. \end{aligned} \tag{13}$$

Then $y \in D((B^2-A)^{1/2})$ and

$$(B^2-A)^{1/2}y = (B+(B^2-A)^{1/2})^{-1}x \in D((B^2-A)^{1/2}).$$

This yields $y \in D(A)$. Hence, with the aid of (12) one has $By \in D((B^2-A)^{1/2})$ and

$$\begin{aligned} x &= (B+(B^2-A)^{1/2})(B^2-A)^{1/2}y = B(B^2-A)^{1/2}y + (B^2-A)y \\ &= (B^2-A)^{1/2}By + (B^2-A)y = (B^2-A)^{1/2}(B+(B^2-A)^{1/2})y. \end{aligned} \tag{14}$$

It follows from (13) and (14) that $z = y \in D(A)$. Hence

$$D((B-(B^2-A)^{1/2})(B+(B^2-A)^{1/2})) \subseteq D(A).$$

Since the opposite inclusion is obvious in view of (6), we conclude that

$$D((B-(B^2-A)^{1/2})(B+(B^2-A)^{1/2}))=D(A)$$

and

$$(B-(B^2-A)^{1/2})(B+(B^2-A)^{1/2})A^{-1} \tag{15}$$
$$= B^2A^{-1}-(B^2-A)^{1/2}BA^{-1}+B(B^2-A)^{1/2}A^{-1}-(B^2-A)A^{-1}$$
$$= I,$$

which proves the surjectivity of $B-(B^2-A)^{1/2}$.

It remains to show that $B-(B^2-A)^{1/2}$ is injective. Let $(B-(B^2-A)^{1/2})y=0$. Then by assumption there exists $x\in X$ such that $y=(B+(B^2-A)^{1/2})^{-1}x$ and hence

$$(B-(B^2-A)^{1/2})(B+(B^2-A)^{1/2})^{-1}x=0. \tag{16}$$

Computing $(B+(B^2-A)^{1/2})^2$ as in (15), we see from (6) and (12) that

$$D((B+(B^2-A)^{1/2})^2)=D(A),$$
$$(B+(B^2-A)^{1/2})^2z=2(B^2+B(B^2-A)^{1/2})z-Az \qquad \forall z\in D(A).$$

This implies by assumption that

$$(B+(B^2-A)^{1/2})^{-2}x\in D(A) \qquad \forall x\in X.$$

Thus (16) can be written as

$$0=(B-(B^2-A)^{1/2})(B+(B^2-A)^{1/2})A^{-1}A(B+(B^2-A)^{1/2})^{-2}x$$
$$=A(B+(B^2-A)^{1/2})^{-2}x,$$

where we have used (15). Now our assumption yields that $y=(B+(B^2-A)^{1/2})^{-1}x=0$, as desired.

We also prove

Lemma 3. *If A and $B+(B^2-A)^{1/2}$ have bounded inverses, then for all $x\in X$,*

$$\int_0^t e^{-sB}V(s)x\,ds=J_-(t,x)=(B+(B^2-A)^{1/2})^{-1}(I-e^{-tB}V(t))x, \tag{17}$$

$$\int_0^t e^{sB}V(s)x\,ds=J_+(t,x)=(-B+(B^2-A)^{1/2})^{-1}(I-e^{tB}V(t))x, \tag{18}$$

$$J_+(t,x)=(B+(B^2-A)^{1/2})A^{-1}(e^{tB}V(t)-I)x. \tag{19}$$

Proof. Let $x \in D((B^2 - A)^{1/2})$. Then assumption (4) implies that

$$
\begin{aligned}
(20) \qquad & (B^2 - A)^{1/2} \int_0^t e^{-sB} V(s)x \, ds \\
&= \int_0^t e^{-sB} V(s)(B^2 - A)^{1/2} x \, ds \\
&= -\int_0^t e^{-sB} \frac{\partial V}{\partial s}(s)x \, ds = x - e^{-tB} V(t)x \\
&\quad - \int_0^t e^{-sB} B(B^2 - A)^{-1/2} V(s)(B^2 - A)^{1/2} x \, ds \\
&= x - e^{-tB} V(t)x - B \int_0^t e^{-sB} V(s)x \, ds.
\end{aligned}
$$

Therefore, $J_-(t, x) \in D((B^2 - A)^{1/2})$ and

$$(21) \qquad (B + (B^2 - A)^{1/2}) J_-(t, x) = (I - e^{-tB} V(t))x.$$

Suppose now $x \in X$. Take a sequence $(x_n)_{n \in \mathbf{N}} \in D((B^2 - A)^{1/2})$ such that $x_n \to x$. Then

$$J_-(t, x_n) \to J(t, x) \qquad \text{as } n \to \infty,$$

and

$$(I - e^{-tB} V(t))x_n \to (I - e^{-tB} V(t))x \qquad \text{as } n \to \infty.$$

On the other hand,

$$(22) \qquad (I - e^{-tB} V(t))x_n = (B + (B^2 - A)^{1/2}) J_-(t, x_n)$$

by (21). Since $(B + (B^2 - A)^{1/2})$ is closed, it follows that $J(t, x) \in D((B^2 - A)^{1/2})$ and

$$(B + (B^2 - A)^{1/2}) J_-(t, x) = (I - e^{-tB} V(t))x,$$

as desired.

Notice that

$$(23) \quad (B^2 - A)^{1/2} J_-(t, x) = (B^2 - A)^{1/2} (B + (B^2 - A)^{1/2})^{-1} (I - e^{-tB} V(t))x.$$

The second part is an obvious consequence of the first one and Lemma 2.

Remark 1. If $0 < t < 1$, then

$$J_+(1 - t, x) = \int_0^{1-t} e^{sB} V(s)x \, ds = \int_t^1 e^{(s-t)B} V(s - t)x \, ds.$$

We need a further lemma as follows.

Lemma 4. *Under assumptions* (3)~(7), *if* A *and* $B+(B^2-A)^{1/2}$ *have bounded inverses, the operators* $\pm B-(B^2-A)^{1/2}$ *generate analytic semigroups* $\{U_1(t)\}$ *and* $\{U_2(t)\}$, *respectively, where*

$$U_1(t)=e^{tB}V(t), \qquad U_2(t)=e^{-tB}V(t).$$

Proof. We recall that according to our previous assumptions, $\pm B-(B^2-A)^{1/2}$ are closed linear operators. We next show that they are generators of C_0-semigroups.

In view of the commutativity of B and $(B^2-A)^{1/2}$, $T(t)=e^{tB}V(t)$ is a C_0-semigroup. Let E be the infinitesimal generator of $T(t)$. Then for $x\in D((B^2-A)^{1/2})$

$$\frac{T(h)-I}{h}x=\frac{e^{hB}V(h)-I}{h}x=e^{hB}\frac{V(h)-I}{h}x+\frac{e^{hB}-I}{h}x$$
$$\to -(B^2-A)^{1/2}x+Bx$$

as $h\to 0$. Hence

$$E\supseteq B-(B^2-A)^{1/2}. \tag{24}$$

Let x be an arbitrary element of $D(E)$. Then in view of Lemma 2, there exists an element $y\in D(B-(B^2-A)^{1/2})$ such that $Ex=(B-(B^2-A)^{1/2})y$. Combination of this and (24) yields $Ex=Ey$. Therefore, if E is one-to-one, then $x=y$, and $E=B-(B^2-A)^{1/2}$ follows.

Proof of the injectivity of E.

Suppose that $x\in D(E)$ with $Ex=0$. Then $(d/dt)(T(t)x)=T(t)Ex=0$. Hence

$$e^{tB}V(t)x=T(t)x=x.$$

This implies

$$V(t)x=e^{-tB}x\in D((B^2-A)^{1/2})\subseteq D(B)$$

for any $t>0$. Letting $h\to 0$ in

$$\frac{V(h)-I}{h}V(t)x=\frac{V(t+h)-V(t)}{h}x=\frac{e^{-(t+h)B}-e^{-tB}}{h}x$$
$$=\frac{e^{-hB}-I}{h}e^{-tB}x$$

yields

$$-(B^2 - A)^{1/2} V(t)x = -Be^{-tB}x.$$

Consequently,

$$\begin{aligned}(B - (B^2 - A)^{1/2})V(t)x &= BV(t)x - (B^2 - A)^{1/2}V(t)x \\ &= Be^{-tB}x - (B^2 - A)^{1/2}V(t)x \\ &= 0.\end{aligned}$$

Hence $V(t)x = 0$ for any $t > 0$. Letting $t \to 0$ one concludes $x = 0$.

An analogous argument repeats for $-B - (B^2 - A)^{1/2}$.

It follows that

(25) $$V(t)e^{\pm tB} = e^{t(\pm B - (B^2 - A)^{1/2})} = e^{\pm tB}V(t) := S(t).$$

We prove that $S(.)$ is differentiable on $(0, \infty)$. In fact for all $x \in X$ and $t > 0$

$$\begin{aligned}\frac{dS}{dt}(t)x &= \frac{d}{dt}(e^{\pm tB}(B^2 - A)^{-1/2}(B^2 - A)^{1/2}V(t)x) \\ &= \pm e^{\pm tB}B(B^2 - A)^{-1/2}(B^2 - A)^{1/2}V(t)x \\ &\quad \mp e^{\pm tB}(B^2 - A)^{-1/2}(B^2 - A)V(t)x.\end{aligned}$$

On the other hand,

$$(B \pm (B^2 - A)^{1/2})S(t) = (B \pm (B^2 - A)^{1/2})(B^2 - A)^{-1/2}(B^2 - A)^{1/2}V(t)e^{\pm tB}$$

yields the estimate

$$\|(B \pm (B^2 - A)^{1/2})S(t)\|_{L(X)} \leqslant C/t, \qquad t > 0.$$

By A. Pazy [6], pp. 61–62, Theorem 5.2, our assertion follows.

Proof of Theorem 1. Let $f \in C^\theta([0,1]; X)$, where $\theta \in]0,1[$. Our first step consists in finding a particular solution $\bar{u}(.)$ to (1). Let us introduce $\bar{u}(.)$ by

(26) $$\begin{aligned}\bar{u}(t) = &-\frac{1}{2}\int_0^t e^{-(t-s)B}V(t-s)(B^2 - A)^{-1/2}f(s)ds \\ &-\frac{1}{2}\int_t^1 e^{-(t-s)B}V(s-t)(B^2 - A)^{-1/2}f(s)ds\end{aligned}$$

for $0 \leqslant t \leqslant 1$.

It is readily seen that there exists the derivative $\bar{u}'(t)$ and

$$\begin{aligned}
\bar{u}'(t) &= \frac{1}{2}\int_0^t e^{-(t-s)B}V(t-s)B(B^2-A)^{-1/2}f(s)ds \\
&\quad + \frac{1}{2}\int_t^1 e^{-(t-s)B}V(s-t)B(B^2-A)^{-1/2}f(s)ds \\
&\quad + \frac{1}{2}\int_0^t e^{-(t-s)B}V(t-s)f(s)ds \\
&\quad - \frac{1}{2}\int_t^1 e^{-(t-s)B}V(s-t)f(s)ds \\
&= \frac{1}{2}(v_1(t)+v_2(t)+v_3(t)-v_4(t)).
\end{aligned} \tag{27}$$

We will prove that $v_1(t) \in D(B)$ and $Bv_1(.) \in C([0,1];X)$. Now, from commutativity of the involved operators and (23),

$$\begin{aligned}
Bv_1(t) &= (I+A(B^2-A)^{-1})(B^2-A)^{1/2}\int_0^t e^{-(t-s)B}V(t-s)f(s)ds \\
&= (I+A(B^2-A)^{-1})(B^2-A)^{1/2}\int_0^t e^{-(t-s)B}V(t-s) \\
&\quad (f(s)-f(t))ds + (I+A(B^2-A)^{-1})\{(B^2-A)^{1/2} \\
&\quad (B+(B^2-A)^{1/2})^{-1}(I-e^{-tB}V(t))f(t)\} \\
&= (I+A(B^2-A)^{-1})\int_0^t e^{-(t-s)B}\frac{\partial V}{\partial s}(t-s)(f(s)-f(t))ds \\
&\quad + (I+A(B^2-A)^{-1})\{(B^2-A)^{1/2} \\
&\quad (B+(B^2-A)^{1/2})^{-1}(I-e^{-tB}V(t))f(t)\}.
\end{aligned} \tag{28}$$

Analogously,

$$\begin{aligned}
Bv_2(t) &= (I+A(B^2-A)^{-1})\int_t^1 e^{-(t-s)B}\frac{\partial V}{\partial s}(s-t)(f(t)-f(s))ds \\
&\quad + (I+A(B^2-A)^{-1})\{(B^2-A)^{1/2} \\
&\quad (B-(B^2-A)^{1/2})^{-1}(e^{(1-t)B}V(1-t)-I)f(t)\},
\end{aligned}$$

$$Bv_3(t) = B(B^2 - A)^{-1/2}(B^2 - A)^{1/2}\int_0^t V(t-s)e^{-(t-s)B}f(s)ds$$

$$= B(B^2 - A)^{-1/2}\int_0^t e^{-(t-s)B}\frac{\partial V}{\partial s}(s-t)(f(s) - f(t))ds$$

$$+ B(B^2 - A)^{-1/2}\{(B^2 - A)^{1/2}$$

$$(B + (B^2 - A)^{1/2})^{-1}(I - e^{-tB}V(t))f(t)\},$$

$$Bv_4(t) = B(B^2 - A)^{-1/2}\int_t^1 e^{-(t-s)B}\frac{\partial V}{\partial s}(s-t)(f(t) - f(s))ds$$

$$+ B(B^2 - A)^{-1/2}\{(B^2 - A)^{1/2}$$

$$(B - (B^2 - A)^{1/2})^{-1}(e^{(1-t)B}V(1-t) - I)f(t)\}.$$

Let us introduce the notation

$$G(t) = \int_0^t e^{-(t-s)B}\frac{\partial V}{\partial s}(t-s)(f(s) - f(t))ds, \tag{29}$$

$$H(t) = \int_t^1 e^{-(t-s)B}\frac{\partial V}{\partial s}(s-t)(f(t) - f(s))ds. \tag{30}$$

Then

$$\begin{aligned} 2B\bar{u}'(t) &= (B^2(B^2 - A)^{-1} + B(B^2 - A)^{-1/2})G(t) \\ &- (-B^2(B^2 - A)^{-1} + B(B^2 - A)^{-1/2})H(t) \\ &+ B(B + (B^2 - A)^{1/2})^{-1}(B(B^2 - A)^{-1/2} + I)(I - e^{-tB}V(t))f(t) \\ &+ B(-B + (B^2 - A)^{1/2})^{-1}(B(B^2 - A)^{-1/2} - I) \\ &\{(I - e^{(1-t)B}V(1-t))f(t)\}. \end{aligned} \tag{31}$$

In order to show that $\bar{u}(t) \in D(A)$ and $A\bar{u}(.)$ is continuous, we write

$$A\bar{u}(t) = -\frac{1}{2}A(B^2 - A)^{-1}(B^2 - A)^{1/2}\int_0^t e^{-(t-s)B}V(t-s)f(s)ds \tag{32}$$

$$
\begin{aligned}
& -\frac{1}{2}A(B^2-A)^{-1}(B^2-A)^{1/2}\int_t^1 e^{-(t-s)B}V(s-t)f(s)ds \\
&= -\frac{1}{2}A(B^2-A)^{-1}(G(t)+H(t)) \\
& -\frac{1}{2}A(B^2-A)^{-1/2}(B+(B^2-A)^{1/2})^{-1}(I-e^{-tB}V(t))f(t) \\
& -\frac{1}{2}A(B^2-A)^{-1/2}(-B+(B^2-A)^{1/2})^{-1}(I-e^{(1-t)B}V(1-t))f(t).
\end{aligned}
$$

To prove that $\bar{u}(.)$ is two-times continuously differentiable, we use for Lemma 4.

We in fact have the equivalent expression to $\bar{u}'(t)$ (in formula (27))

$$
\begin{aligned}
(33)\quad \bar{u}'(t) &= \frac{1}{2}(B+(B^2-A)^{1/2})(B^2-A)^{-1/2}\int_0^t e^{-(t-s)(B+(B^2-A)^{1/2})}f(s)ds \\
& -\frac{1}{2}(-B+(B^2-A)^{1/2})(B^2-A)^{-1/2}\int_t^1 e^{(t-s)(-B+(B^2-A)^{1/2})}f(s)ds.
\end{aligned}
$$

Since f is Hölder-continuous, $\bar{u}'(.)$ is differentiable and

$$
\begin{aligned}
\bar{u}''(t) &= f(t) - \frac{1}{2}[(B+(B^2-A)^{1/2})(B^2-A)^{-1/2}]^2 \\
& \left((B^2-A)^{1/2}\int_0^t e^{-(t-s)(B+(B^2-A)^{1/2})}f(s)ds\right) \\
& -\frac{1}{2}[(-B+(B^2-A)^{1/2})(B^2-A)^{-1/2}]^2 \\
& \left((B^2-A)^{1/2}\int_t^1 e^{(t-s)(-B+(B^2-A)^{1/2})}f(s)ds\right).
\end{aligned}
$$

We now observe that, from previous calculation (compare with $Bv_3(t)$ and $Bv_4(t)$),

$$
\begin{aligned}
&(B^2-A)^{1/2}\int_0^t e^{-(t-s)(B+(B^2-A)^{1/2})}f(s)ds = (B^2-A)^{1/2}v_3(t) \\
&\quad = G(t) + (B^2-A)^{1/2}(B+(B^2-A)^{1/2})^{-1}(I-e^{-tB}V(t))f(t), \\
&(B^2-A)^{1/2}\int_t^1 e^{(t-s)(-B+(B^2-A)^{1/2})}f(s)ds = (B^2-A)^{1/2}v_4(t) \\
&\quad = H(t) - (B^2-A)^{1/2}(-B+(B^2-A)^{1/2})^{-1}(e^{(1-t)B}V(1-t)-I)f(t),
\end{aligned}
$$

so that

$$\bar{u}''(t) = f(t) - \frac{1}{2}[(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}]^2 G(t) \tag{34}$$

$$- \frac{1}{2}(B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}(I - e^{-tB}V(t))f(t)$$

$$- \frac{1}{2}[(-B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}]^2 H(t)$$

$$+ \frac{1}{2}(-B + (B^2 - A)^{1/2})(B^2 - A)^{-1/2}(e^{(1-t)B}V(1 - t) - I)f(t).$$

Therefore

$$\bar{u}''(t) + 2B\bar{u}'(t) + A\bar{u}(t) \tag{35}$$

$$= f(t) - \frac{1}{2}[I + B(B^2 - A)^{-1/2}]^2 G(t) - \frac{1}{2}[I - B(B^2 - A)^{-1/2}]^2 H(t)$$

$$- \frac{1}{2}(I + B(B^2 - A)^{-1/2})(I - e^{-tB}V(t))f(t)$$

$$+ \frac{1}{2}(I - B(B^2 - A)^{-1/2})(e^{(1-t)B}V(1 - t) - I)f(t)$$

$$+ (B^2(B^2 - A)^{-1} + B(B^2 - A)^{-1/2})G(t)$$

$$- (-B^2(B^2 - A)^{-1} + B(B^2 - A)^{-1/2})H(t)$$

$$+ B(B^2 - A)^{-1/2}(I - e^{-tB}V(t))f(t)$$

$$- B(B^2 - A)^{-1/2}(I - e^{(1-t)B}V(1 - t))f(t)$$

$$- \frac{1}{2}A(B^2 - A)^{-1}(G(t) + H(t))$$

$$- \frac{1}{2}A(B^2 - A)^{-1/2}(B + (B^2 - A)^{1/2})^{-1}(I - e^{-tB}V(t))f(t)$$

$$- \frac{1}{2}A(B^2 - A)^{-1/2}(-B + (B^2 - A)^{1/2})^{-1}(I - e^{(1-t)B}V(1 - t))f(t).$$

We have

$$- \frac{1}{2}(I + B(B^2 - A)^{-1/2}) + B(B^2 - A)^{-1/2} - \frac{1}{2}A(B^2 - A)^{-1/2}(B + (B^2 - A)^{1/2})^{-1}$$

$$= - \frac{1}{2}I + \frac{1}{2}B(B^2 - A)^{-1/2} - \frac{1}{2}A(B^2 - A)^{-1}(I + B(B^2 - A)^{-1/2})^{-1}$$

$$= \left(-\frac{1}{2}(I - B(B^2 - A)^{-1/2})(I + B(B^2 - A)^{-1/2}) - \frac{1}{2}A(B^2 - A)^{-1} \right)$$

$$((I + B(B^2 - A)^{-1/2})^{-1})$$

$$= -\frac{1}{2}[I - B^2(B^2 - A)^{-1} + A(B^2 - A)^{-1}](I + B(B^2 - A)^{-1/2})^{-1}$$

$$= 0,$$

(we have used (11)). Also

$$-\frac{1}{2}(I - B(B^2 - A)^{-1/2}) - B(B^2 - A)^{-1/2} - \frac{1}{2}A(B^2 - A)^{-1/2}(-B + (B^2 - A)^{1/2})^{-1}$$

$$= -\frac{1}{2}I - \frac{1}{2}B(B^2 - A)^{-1/2} - \frac{1}{2}A(B^2 - A)^{-1}(I - B(B^2 - A)^{-1/2})^{-1}$$

$$= \left(-\frac{1}{2}(I + B(B^2 - A)^{-1/2})(I - B(B^2 - A)^{-1/2}) - \frac{1}{2}A(B^2 - A)^{-1} \right)$$

$$((I - B(B^2 - A)^{-1/2})^{-1})$$

$$= \left(-\frac{1}{2}(I - B^2(B^2 - A)^{-1}) - \frac{1}{2}A(B^2 - A)^{-1} \right)(I - B(B^2 - A)^{-1/2})^{-1}$$

$$= 0,$$

$$-\frac{1}{2}(B(B^2 - A)^{-1/2} + I)^2 + B^2(B^2 - A)^{-1} + B(B^2 - A)^{-1/2} - \frac{1}{2}A(B^2 - A)^{-1}$$

$$= -\frac{1}{2}(B^2(B^2 - A)^{-1} + I + 2B(B^2 - A)^{-1/2}) + B^2(B^2 - A)^{-1}$$

$$+ B(B^2 - A)^{-1/2} - \frac{1}{2}A(B^2 - A)^{-1}$$

$$= \frac{1}{2}B^2(B^2 - A)^{-1} - \frac{1}{2}I - \frac{1}{2}A(B^2 - A)^{-1}$$

$$= 0,$$

and

$$-\frac{1}{2}(I - B(B^2 - A)^{-1/2})^2 - (-B^2(B^2 - A)^{-1} + B(B^2 - A)^{-1/2}) - \frac{1}{2}A(B^2 - A)^{-1}$$

$$= -\frac{1}{2}(I + B^2(B^2 - A)^{-1} - 2B(B^2 - A)^{-1/2}) + B^2(B^2 - A)^{-1}$$

$$- B(B^2 - A)^{-1/2} - \frac{1}{2}A(B^2 - A)^{-1}$$

$$= -\frac{1}{2}I + \frac{1}{2}B^2(B^2 - A)^{-1} - \frac{1}{2}A(B^2 - A)^{-1}$$

$$= 0.$$

Therefore it is verified that $\bar{u}(.)$ is a strict solution to (1) satisfying the boundary conditions

$$\begin{cases} \bar{u}(0) = -\frac{1}{2}\int_0^1 e^{sB}V(s)(B^2 - A)^{-1/2}f(s)ds \\ = -\frac{1}{2}\int_0^1 U_1(s)(B^2 - A)^{-1/2}f(s)ds, \end{cases}$$

$$\begin{cases} \bar{u}(1) = -\frac{1}{2}\int_0^1 e^{-(1-s)B}V(1-s)(B^2 - A)^{-1/2}f(s)ds \\ = -\frac{1}{2}\int_0^1 U_2(1-s)(B^2 - A)^{-1/2}f(s)ds, \end{cases}$$

where $U_1(s)$ and $U_2(s)$ are the analytic semigroups generated by $B - (B^2 - A)^{1/2}$ and $-(B + (B^2 - A)^{1/2})$ respectively (see Lemma 4).

We show that $\bar{u}(0), \bar{u}(1) \in D(A)$. Indeed,

$$(B^2 - A)^{1/2}\bar{u}(0) = -\frac{1}{2}\int_0^1 U_1(s)f(s)ds,$$

and

$$(B^2 - A)\bar{u}(0)$$

$$= -\frac{1}{2}(B^2 - A)^{1/2}\int_0^1 U_1(s)(f(s) - f(0))ds - \frac{1}{2}(B^2 - A)^{1/2}\int_0^1 U_1(s)f(0)ds$$

$$= \frac{1}{2}H(0) - \frac{1}{2}(B^2 - A)^{1/2}(B - (B^2 - A)^{1/2})^{-1}(U_1(1) - I)f(0).$$

Since we also have

$$B^2\bar{u}(0) = B^2(B^2 - A)^{-1}(B^2 - A)\bar{u}(0),$$

we conclude $\bar{u}(0) \in D(A)$.

On the other hand

$$(B^2 - A)^{1/2}\bar{u}(1) = -\frac{1}{2}\int_0^1 U_2(1-s)f(s)ds$$

yields

$$\begin{aligned}(B^2 - A)\bar{u}(1) \\ &= -\frac{1}{2}(B^2 - A)^{1/2}\int_0^1 U_2(1-s)(f(s) - f(1))ds \\ &\quad -\frac{1}{2}(B^2 - A)^{1/2}\int_0^1 U_2(1-s)dsf(1) \\ &= \frac{1}{2}G(1) - \frac{1}{2}(B^2 - A)^{1/2}(B + (B^2 - A)^{1/2})^{-1}(I - e^{-B}V(1))f(1),\end{aligned}$$

and

$$B^2\bar{u}(1) = B^2(B^2 - A)^{-1}(B^2 - A)\bar{u}(1),$$

so that the assertion follows.

To conclude our proof, we need a further result on the homogenous equation (1).

Lemma 5. *Under assumptions* (3)$\sim$(7), *the homogeneous equation*

$$u''(t) + 2Bu'(t) + Au(t) = 0, \qquad t \in [0,1], \tag{36}$$

has a unique strict solution $u(.)$ *satisfying*

$$u(0) = x_0, \qquad u(1) = x_1, \tag{37}$$

provided that $x_0, x_1 \in D(A)$.

Proof. It suffices to show that under the indicated assumptions, problem (36)–(37) has one strict solution. To accomplish this, we in fact furnish an explicit solution to it, precisely

$$u(t) = U_2(t)\xi_0 + U_1(1-t)\xi_1, \tag{38}$$

where

(39) $$Z = e^{-2(B^2-A)^{1/2}}$$

(40) $$\xi_0 = (I - Z)^{-1}(x_0 - U_1(1)x_1)$$

(41) $$\xi_1 = (I - Z)^{-1}(x_1 - U_2(1)x_0).$$

Notice that since the imaginary axis is contained in the resolvent set

$$\rho(-(B^2 - A)^{1/2}),$$

$I - Z$ has a bounded inverse (see Lunardi [5], p. 60):

(42) $$(I - Z)^{-1} = \frac{1}{2\pi i}\int_{\gamma_\#} \frac{e^{2z}}{1 - e^{2z}}(zI + (B^2 - A)^{1/2})^{-1}dz + I,$$

where $\gamma_\# = \gamma_1 - \gamma_2$ is a suitable curve in the complex plane (see Lunardi [5], p. 59). Moreover, the commutativity properties and assumptions on x_0, x_1 imply that $\xi_0, \xi_1 \in D(A)$. In fact it is enough to observe that

$$-A\xi_i = (I - B^2(B^2 - A)^{-1})(B^2 - A)\xi_i$$

and

$$(B^2 - A)^{1/2}\xi_0 = (I - Z)^{-1}((B^2 - A)^{1/2}x_0 - U_1(1)(B^2 - A)^{1/2}x_1)$$

$$(B^2 - A)\xi_0 = (I - Z)^{-1}((B^2 - A)x_0 - U_1(1)(B^2 - A)x_1).$$

An analogous argument holds for ξ_1.

Next,

$$\begin{aligned} u(0) &= \xi_0 + U_1(1)\xi_1 \\ &= (I - Z)^{-1}(x_0 - U_1(1)x_1) + (I - Z)^{-1}U_1(1)(x_1 - U_2(1)x_0) \\ &= (I - Z)^{-1}(I - U_1(1)U_2(1))x_0 \\ &= x_0, \end{aligned}$$

(via the commutativity assumption (4)),

$$\begin{aligned} u(1) &= U_2(1)\xi_0 + \xi_1 \\ &= U_2(1)(I - Z)^{-1}(x_0 - U_1(1)x_1) + (I - Z)^{-1}(x_1 - U_2(1)x_0) \\ &= (I - Z)^{-1}(I - U_2(1)U_1(1))x_1 \\ &= x_1. \end{aligned}$$

Since $\xi_i \in D(A)$, in view of Lemma 2, it is readily seen that $u \in C^2([0,1]; X)$ with

$$\begin{aligned} u'(t) = &-e^{-tB}V(t)(B + (B^2 - A)^{1/2})\xi_0 \\ &- e^{(1-t)B}V(1 - t)(B - (B^2 - A)^{1/2})\xi_1, \end{aligned}$$

$$u''(t) = e^{-tB}V(t)(2B^2 - A + 2B(B^2 - A)^{1/2})\xi_0$$
$$+ e^{(1-t)B}V(1-t)(2B^2 - A - 2B(B^2 - A)^{1/2})\xi_1.$$

Therefore,

$$u''(t) + 2Bu'(t) = -e^{-tB}V(t)A\xi_0 - e^{(1-t)B}V(1-t)A\xi_1.$$

But

$$\begin{aligned}-Ae^{-tB}V(t)\xi_0 &= (B^2 - A - B^2)e^{-tB}V(t)\xi_0\\ &= (B^2 - A)e^{-tB}V(t)\xi_0 - B^2e^{-tB}V(t)\xi_0\\ &= e^{-tB}V(t)(B^2 - A)\xi_0 - e^{-tB}V(t)B^2\xi_0\\ &= -e^{-tB}V(t)A\xi_0,\end{aligned}$$

and

$$Ae^{(1-t)B}V(1-t)\xi_1 = e^{(1-t)B}V(1-t)A\xi_1.$$

This concludes the proof of the Lemma.

End of the proof of Theorem 1.

Let $\bar{\bar{u}}$ be a strict solution of problem (36)–(37) with

$$x_0 = u_0 - \bar{u}(0), \qquad x_1 = u_1 - \bar{u}(1).$$

We know $\bar{u}(i) \in D(A)$ and, from the assumptions, $u_i \in D(A)$, $i = 0, 1$, so that Lemma 5 applies. Then it is a simple matter to recognize that

$$u(.) = \bar{u}(.) + \bar{\bar{u}}(.),$$

is a unique strict solution to problem (1)–(2).

Remark 2. Assumption (3), according which $B^2 - A$ has a bounded inverse, is really essential to confine ourselves to the declared Hölder continuity of the function $f(.)$.

For example, let $X = H$ a Hilbert space over the complex numbers, $B = iK$, $A = -K^2$, K being a self-adjoint operator in H. Given $f \in C([0,1]; H)$, it is readily seen that the candidate solution $u(.)$ to (1), with homogeneous boundary conditions $u(0) = u(1) = 0$, is given by

$$u(t) = te^{-itK}\int_0^1 (s-1)e^{isK}f(s)ds + \int_0^t e^{i(s-t)K}(t-s)f(s)ds,$$

but without further regularity (either in space or in time) of function $f(.), u(.)$ need not to be a strict solution.

3. Examples

Example 1. Take $X = L^2(\boldsymbol{R})$ and let

$$\begin{cases} D(A) = H^4(\boldsymbol{R}), \quad Au = au^{(4)} - cu,\ c > 0 \\ D(B) = H^2(\boldsymbol{R}), \quad Bu = ibu'',\ b \neq 0, \\ \text{with } a + b^2 < 0. \end{cases}$$

The operator $B^2 - A$ satisfies

$$\int_{\boldsymbol{R}} (-b^2 - a)u^{(4)}(x)\overline{u(x)}dx + \int_{\boldsymbol{R}} c|u(x)|^2 dx = -(b^2 + a)\int_{\boldsymbol{R}} |u''(x)|^2 dx + c\|u\|_2^2$$

and thus condition (3) holds. On the other hand, $B^2 - A$ is a stricly positive operator and $-A$ too. Therefore, Heinz' Theorem (see Tanabe [7], p. 44) gives

$$\|(-A)^{1/2}u\|_2 \leqslant C\|(B^2 - A)^{1/2}u\|_2$$

for all $u \in D((B^2 - A)^{1/2}) = H^2(\boldsymbol{R})$. Since $\|(-A)^{1/2}u\|_2$ is equivalent to the norm on $H^2(\boldsymbol{R})$, definition of B implies that (3)$\sim$(7) are verified and thus all our preceding results apply to the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + 2ib\dfrac{\partial^3 u}{\partial x^2 \partial t} + a\dfrac{\partial^4 u}{\partial x^4} - cu = f(t,x), \quad (t,x) \in (0,1) \times \boldsymbol{R} \\ u(0,x) = u_0(x), \quad x \in \boldsymbol{R}, \\ u(1,x) = u_1(x), \quad x \in \boldsymbol{R}, \end{cases}$$

where $u_0, u_1 \in H^4(\boldsymbol{R})$, $f \in C^\theta([0,1]; L^2(\boldsymbol{R}))$.

Example 2. Take $X = L^p(\boldsymbol{R})$, $1 < p < \infty$ and define A, B by

$$\begin{cases} D(A) = W^{2,p}(\boldsymbol{R}), \quad Au = au'' + cu' - du, \\ D(B) = W^{1,p}(\boldsymbol{R}), \quad Bu = bu', \\ \text{with } a > b^2,\ b \neq 0,\ d > 0,\ c \in \boldsymbol{R}. \end{cases}$$

B generates a contraction C_0-group in X. Moreover for any $\lambda \geqslant 0$, with

$$\langle u, v\rangle = \int_{\boldsymbol{R}} u(x).\overline{v(x)}|v(x)|^{p-2}dx,$$

one has

$$\begin{aligned} &\mathrm{Re}\langle \lambda u - au'' + b^2u'' - cu' + du, u\rangle \\ &\quad = (\lambda + d)\|u\|_p^p + (a - b^2)\int_{\boldsymbol{R}} |u'(x)|^2|u(x)|^{p-2}dx \\ &\qquad + (a - b^2)(p-2)\int_{\boldsymbol{R}} |\mathrm{Re}(u'(x)\overline{u(x)})|^2|u(x)|^{p-4}dx \\ &\quad \geqslant (\lambda + d)\|u\|_p^p. \end{aligned}$$

Moreover, one sees that $\lambda + B^2 - A$ is onto X. Hence, condition (3) is verified.

Let $p = 2$ and $c = 0$. Then the estimates above for $\lambda = 0$ show that the positive operator $B^2 - A$, has a bounded inverse and

$$\|(B^2 - A)^{1/2} u\|_p^2 \geqslant (a - b^2)\|u'\|_p^2,$$

implying (7). If $p \neq 2$ and c is arbitrary, we know that $B^2 - A$ is strictly positive whose inverse $(B^2 - A)^{-1}$ maps X into $W^{2,p}(\boldsymbol{R})$. Since $D((B^2 - A)^{1/2}) = W^{1,p}(\boldsymbol{R})$, (7) follows. Hence, we can apply our abstract results.

Example 3. Let $\alpha \in C([0,1])$ be a real valued function satisfying

$$\begin{cases} \alpha(x) > 0, & x \in (0,1), \\ \sqrt{\alpha} \in C^1([0,1]), \\ \alpha(0) = 0 = \alpha(1). \end{cases}$$

Let

$$X = C_0([0,1]) = \{u \in C([0,1]) : u(0) = u(1) = 0\}$$

endowed with the maximum norm.

Let B be the linear operator in $X = C_0([0,1])$ defined by

$$\begin{cases} D(B) = \left\{ u \in C_0([0,1]) \cap C^1(0,1) : \lim\limits_{x \to 0_+} \sqrt{\alpha(x)}u'(x) \right. \\ \qquad\qquad \left. = \lim\limits_{x \to 1_-} \sqrt{\alpha(x)}u'(x) = 0 \right\} \\ Bu = \sqrt{\alpha(.)}u', \quad u \in D(B). \end{cases}$$

Then it is known, see A. Favini and S. Romanelli [2], that $(B, D(B))$ generates a C_0-group of operators in X.

Take

$$\begin{cases} D(A) = \left\{ u \in C_0([0,1]) \cap C^2(0,1) : \lim\limits_{x \to 0_+; x \to 1_-} \sqrt{\alpha(x)}u'(x) \right. \\ \qquad\qquad \left. = \lim\limits_{x \to 0_+; x \to 1_-} \alpha(x)u''(x) = 0 \right\} \\ Au = aB^2 u - cu = a\sqrt{\alpha}(\sqrt{\alpha}u')' - cu = a\alpha u'' + \dfrac{a}{2}\alpha' u' - cu, \quad u \in D(A), \end{cases}$$

with

$$c > 0,\ a > 1.$$

Since

$$(B^2 - A)u = (1 - a)\sqrt{\alpha}(\sqrt{\alpha}u')' + cu,$$

we conclude, by a lattice isomorphism argument, that assumptions (3)∼(7) are verified and our results apply to

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + 2\sqrt{\alpha(x)}\dfrac{\partial^2 u}{\partial x \partial t} + a\sqrt{\alpha(x)}\dfrac{\partial}{\partial x}\left(\sqrt{\alpha(x)}\dfrac{\partial u}{\partial x}\right) - cu \\ = f(t,x), \quad (t,x) \in (0,1) \times (0,1), \\ u(0,x) = u_0(x), \quad 0 < x < 1, \\ u(1,x) = u_1(x), \quad 0 < x < 1, \\ \lim\limits_{x \to 0_+;\, x \to 1_-} \dfrac{\partial u}{\partial t}(t,x) = \lim\limits_{x \to 0_+;\, x \to 1_-} \sqrt{\alpha(x)}\dfrac{\partial^2 u}{\partial x \partial t}(t,x) = 0, \quad 0 < t < 1, \\ \lim\limits_{x \to 0_+;\, x \to 1_-} u(t,x) = \lim\limits_{x \to 0_+;\, x \to 1_-} \sqrt{\alpha(x)}\dfrac{\partial u}{\partial x}(t,x) = 0, \quad 0 < t < 1, \\ \lim\limits_{x \to 0_+;\, x \to 1_-} \alpha(x)\dfrac{\partial^2 u}{\partial x^2}(t,x) = 0, \quad 0 < t < 1, \end{cases}$$

provided that $f \in C^{\theta}([0,1]; C_0([0,1]))$, $0 < \theta < 1$ and $u_0, u_1 \in D(A)$.

Example 4. Let $X = L^2(\boldsymbol{R}^n)$, $n > 1$ and

$$\begin{cases} D(A) = H^4(\boldsymbol{R}^n), \quad Au = a\Delta^2 u - cu, \\ D(B) = H^2(\boldsymbol{R}^n), \quad Bu = ib\Delta u, \end{cases}$$

where

$$a < -b^2,\ b \neq 0,\ c > 0.$$

Then we can extend the arguments in Example 1; see Kato [3], for properties of operator $i\Delta$. Hence Theorem 1 applies.

Acknowledgements. We would like to express our hearty thanks to the referee, who read the manuscript very carefully and whose detailed comments helped to improve the quality of this work.

References

[1] El Haial, A. and Labbas, R., On the Ellipticity and Solvability of Abstract Second-order Differential Equation. Electronic Journal of Differential Equations, **57** (2001), 1–18.

[2] Favini, A. and Romanelli, S., Analytic Semigroups on $C([0,1])$ Generated by some Classes of Second Order Differential Operators. Semigroup Forum, **56** (1998), 367–372.

[3] Kato, T., *Perturbations Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg, New York, 1966.

[4] Krein, S. G., *Linear Differential Equations in Banach Space*, Moscou, 1967; English Translation: AMS, Providence, 1971.

[5] Lunardi, A., *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, 1995.
[6] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin, Heidelberg, Tokyo, 1983.
[7] Tanabe, H., *Equations of Evolution*, Pitman, London, San Francisco, Melbourne, 1979.

nuna adreso:
Angelo Favini
Dipartimento di Matematica
Università degli Studi di Bologna
Piazza di Porta S. Donato, 5
40126 Bologna
Italia

Rabah Labbas
Laboratoire de Mathématiques
Faculté de Sciences et Techniques
Université du Havre
B.P 540, 76058 Le Havre Cedex
France

Hiroki Tanabe
Department of Economics
Otemon Gakuin University
Ibaraki, Osaka 567-8502
Japan

Atsushi Yagi
Department of Applied Physics
Osaka University
Suita, Osaka 565-0871
Japan

(Ricevita la 7-an de februaro, 2003)
(Reviziita la 30-an de julio, 2003)