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Global Existence and Asymptotic Behavior of Solutions of Nonlinear Differential Equations

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Abstract. The paper addresses two open problems related to global existence of solutions with a "linear-like" behavior at infinity. For a class of second-order nonlinear differential equations, we establish global existence of solutions under milder assumption on the rate of decay of the coefficient. Furthermore, as opposed to results reported in the literature, we prove for another class of second-order nonlinear differential equations that the region of the initial data for the solutions with desired asymptotic behavior is unbounded and proper.

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1. Introduction

The second-order nonlinear ordinary differential equation

$$u'' + f(t, u, u') = 0, \qquad t \geq t_0, \tag{1}$$

attracts constant interest of researchers because of its importance for mathematical modeling of different physical, chemical or biological systems. Numerous papers published recently are concerned with local and global existence of solutions of Eq. (1) and its particular cases, uniqueness, continuation, asymptotic behavior of solutions (including boundedness, prescribed asymptotic behavior, oscillation and nonoscillation), stability properties, etc.

Since Eq. (1) can be viewed as a nonlinear perturbation of a very simple differential equation

$$u'' = 0$$

that has solutions of the form $u(t) = at + b$, numerous authors were interested in establishing sufficient conditions for the so-called "linear-like" behavior of

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solutions at infinity referred to as "Property (L)" in [12]. This property reads as follows: for every continuable solution of (1) (or its particular cases) there exists a real constant a such that

$$\lim_{t\to+\infty} u'(t) = \lim_{t\to+\infty} \frac{u(t)}{t} = a. \tag{2}$$

The standard assumption on the nonlinear function $f(t,u,u')$ is

$$|f(t,u,u')| \le h_1(t)g_1(|u'|) + h_2(t)g_2\left(\frac{|u|}{t}\right) + h_3(t), \tag{3}$$

where the functions $h_i(t)$, $i = 1,2,3$, are continuous, nonnegative and integrable on $[t_0,+\infty)$, while $g_1(s)$ and $g_2(s)$ are continuous, nonnegative, nondecreasing functions which satisfy certain integral conditions.

Among numerous papers concerned with existence of solutions of various classes of linear and nonlinear differential equations possessing property (L), we would like to refer to recent papers by Cohen [1], Constantin [2], Kusano and Trench [7, 8], Meng [9], Mustafa and Rogovchenko [10], S. Rogovchenko and Yu. Rogovchenko [12], Rogovchenko [13], Rogovchenko and Villari [14], Tong [15] and Trench [16]. The reader may consult the papers by the present authors [10] and by S. Rogovchenko and Yu. Rogovchenko [12] where further details and additional references can be retrieved.

We also note that the study of linear-like solutions of second-order ordinary differential equations has attracted special interest of researchers dealing with radial solutions of certain classes of elliptic equations. We address the reader to recent papers by Fukagai [4], Kusano, Naito and Usami [6], Kusano and Trench [7, 8], Usami [17], Zhang [19], and the references therein.

As it has been pointed out by Kusano and Trench [8, p. 381], most results on the existence of solutions with prescribed asymptotic behavior for nonlinear equations are "local" near infinity, in the sense that solutions with the required behavior are shown to exist only for t large enough. Only a few papers (see, for instance, [7], [8], [10], [11]) provide global conditions which imply existence of solutions on $[t_0,\infty)$ for some real t_0.

Global existence of solutions of Eq. (1) with the linear-like behavior at infinity has been studied very recently by the authors [10] under the following assumption on the nonlinearity:

$$|f(t,u,v)| \le h(t)\left[p_1\left(\frac{|u|}{t}\right) + p_2(|v|)\right], \tag{4}$$

where $h, p_1, p_2 : (0,+\infty) \to (0,+\infty)$ are continuous, p_1 and p_2 are nondecreasing, h satisfies

$$\int_{t_0}^{\infty} h(s)ds < +\infty, \tag{5}$$

and

$$G(+\infty) = +\infty, \tag{6}$$

where

$$G(x) = \int_{t_0}^{x} \frac{ds}{p_1(s) + p_2(s)}.$$

It has been proved, among other results, that

(A) for every pair of real numbers u_0, u_1, there exists at least one solution $u(t)$ of Eq. (1) satisfying initial conditions

$$u(t_0) = u_0, \qquad u'(t_0) = u_1, \tag{7}$$

such that $u(t) = at + o(t)$ as $t \to +\infty$ [10, Theorem 3];

(B) for every real a, there exists a solution $u(t)$ of Eq. (1) defined on $[t_0, +\infty)$ with the asymptotic representation $u(t) = at + o(t)$ as $t \to +\infty$ [10, Theorem 4].

This paper continues studies on global existence of solutions with linear-like behavior at infinity and is concerned with two open problems discussed in Sections 2 and 3. The first problem is related with the integrability condition (5) and its analogues in [1], [2], [9], [10], [12], [13], [15] and [16]. Is it necessary for this type of asymptotic behavior, or it is only technical? As a motivating example, consider a simple linear differential equation

$$u'' + \frac{1}{t}\left(u' - \frac{u}{t}\right) = 0, \qquad t \geq 1. \tag{8}$$

Clearly, assumption (5) fails to hold, although a straightforward computation shows that Eq. (8) has a two-parametric family of solutions

$$u(t) = C_1 t + C_2 t^{-1}$$

defined on $[1, \infty)$ and possessing the property (L).

The second problem is concerned with geometric properties of the region of linear-like behavior (we adopt this concept, similar to that used in stability theory, to denote the regions in the (u, u')-plane where initial data (u_0, u_1) for solutions that possess property (L) are located). Should these regions be bounded as in [3], [12], and [18], or there are other possibilities?

The class of equations with nonlinearities that satisfy condition (4) is very large. As it has been shown by the authors in [10, Section 5.2], condition (5) is necessary only when the information one has about Eq. (1) is the inequality

(4). However, if we slightly modify conditions imposed on the functions in the inequality (4), restriction (5) becomes no longer necessary and can be replaced with a milder one. Namely, in this paper we can have $p_i(0) = 0$, possibility that was excluded by requiring that $p_i(w) > 0$ for all $w \geq 0$ in our paper [10], and this modification of the basic assumption on the nonlinearity f enables us to prove global existence of solutions with linear-like behavior at infinity without imposing condition (5).

Furthermore, in this paper we do not require condition (6). Therefore, in general, one can establish existence of solutions of Eq. (1) that possess property (L) only *for a part* of solutions with initial data satisfying an additional condition. As opposed to the papers by Dannan [3], S. Rogovchenko and Yu. Rogovchenko [12], and Waltman [18], where the regions of linear-like behavior have different nature but are always *bounded*, the novelty of Theorem 6 also lies in the fact that for the considered class of equations the region of linear-like behavior is *unbounded* and *proper* (that is, it is neither void nor $\boldsymbol{R}^2$).

The model equation motivating our study is

$$u'' + q(t)\left(u' - \frac{u}{t}\right)^n = 0, \qquad t \geq t_0, \tag{9}$$

where $n \geq 1$ is an integer, and the function $q(t)$ is continuous. Using the elementary inequality

$$(x+y)^{2n} \leq (2\max\{|x|,|y|\})^{2n} \leq 2^{2n}(x^{2n} + y^{2n})$$

that holds for all real numbers x and y, it is not difficult to check that the nonlinearity in Eq. (9) satisfies condition (4), where $h(t) = 2^{2n}|q(t)|$, $p_i(w) = w^n$, $i = 1, 2$.

We point out that not all solutions of Eq. (9) exist in the future (that is, are not defined for all $t \geq t_0$). For instance, differential equation

$$u'' + \frac{1}{t^2}\left(u' - \frac{u}{t}\right)^2 = 0, \qquad t \geq 1, \tag{10}$$

admits a solution

$$u(t) = 2t \ln \frac{2-t}{2+t}, \qquad t \in [1, 2) \tag{11}$$

which blows up in finite time, as well as a one-parametric family of solutions

$$u(t) = Ct, \qquad C \in \boldsymbol{R},$$

that possess property (L) and exist on the entire interval $[1, +\infty)$. This example prompts that the choice of initial data affects significantly global existence and asymptotic properties of solutions of Eq. (9).

2. Global solutions with linear-like behavior

In this section, we are concerned with the nonlinear differential equation

$$u'' + a(t)g\left(u' - \frac{u}{t}\right) = 0, \qquad t \geq t_0, \tag{12}$$

where $g(x) \stackrel{\text{def}}{=} f(x)x^{2n}$, the functions $a : [t_0, +\infty) \to [0, +\infty)$ and $f : \boldsymbol{R} \to \boldsymbol{R}$ are continuous, $xf(x) > 0$ for all $x \neq 0$, $t_0 \geq 1$, and $n \geq 1$ is an integer.

First we note that

$$\begin{aligned} \left| g\left(u' - \frac{u}{t}\right) \right| &\leq \sup_{|v| \leq |u'| + |u|/t} |g(v)| \\ &\leq \sup_{|v| \leq 2|u'|} |g(v)| + \sup_{|v| \leq 2|u|/t} |g(v)| \\ &= p_1(|u'|) + p_2\left(\frac{|u|}{t}\right), \end{aligned}$$

where the functions $p_i(w)$ are continuous, nondecreasing, nonnegative and $p_i(w) > 0$ for all $w > 0$. Therefore, the nonlinearity in Eq. (12) satisfies condition (4).

Theorem 1. *All solutions of Eq.* (12)
(a) *exist in the future;*
(b) *possess the property* (2).

Proof. Consider a pair of numbers $u_0, u_1 \in \boldsymbol{R}$. Clearly, there exists a solution $u(t)$ (not necessarily unique) of the differential equation

$$u'' + a(t)f\left(u' - \frac{u}{t}\right)\left(u' - \frac{u}{t}\right)^{2n} = 0$$

satisfying the initial condition (7) and defined on the maximal interval $[t_0, T)$, where $T = T_u \leq +\infty$. Using the transformation

$$u''(t) = t^{-1}[tu'(t) - u(t)]', \qquad t \in [t_0, T),$$

we write Eq. (12) as

$$\frac{(tu' - u)'}{t} + a(t)f\left(\frac{tu' - u}{t}\right)\left(\frac{tu' - u}{t}\right)^{2n} = 0, \qquad t \in [t_0, T). \tag{13}$$

Introducing the functions $v(t) = tu'(t) - u(t)$ and $z(t) = t^{-1}v(t)$, we write Eq. (13) first as

$$v' + ta(t)f\left(\frac{v}{t}\right)\left(\frac{v}{t}\right)^{2n} = 0, \qquad t \in [t_0, T),$$

and then in the form

$$tz' + z + ta(t)f(z)z^{2n} = 0, \qquad t \in [t_0, T).$$

We divide now the proof into the following steps.

(i) First, we prove that all solutions $z(t)$ of the Cauchy problem

$$z' = -t^{-1}z - a(t)f(z)z^{2n}, \qquad t \geq t_0, \tag{14}$$

$$z(t_0) = z_0, \tag{15}$$

exist in the future.

(ii) Second, we show that for any solution $z(t)$ of (14), (15) one has $z(t) = O(t^{-1})$ as $t \to +\infty$.

To prove claim (i), let $z_0(t)$ be a solution of the Cauchy problem (14), (15) defined for $t \in [t_0, T)$. We distinguish the following two cases.

Case 1: Assume that $z_0 = 0$. It follows from (14), (15) that $z(t) = 0$ is a solution of the given Cauchy problem for $t \geq t_0$. We shall prove that it is unique. Suppose, for the sake of contradiction, that there is another solution $z(t)$ which is not identically zero for $t \geq t_0$. Without loss of generality, we can assume that there exists a $t_1 \in (t_0, T)$ such that $z(t_1) > 0$. Then, by the continuity argument, there exists a $t_2 \in [t_0, t_1)$ such that

$$z(t_2) = 0 \qquad \text{and} \qquad z(t) > 0 \qquad \text{for } t \in (t_2, t_1].$$

Since $z(t_2) = 0 < z(t_1)$, there exists a $t_3 \in (t_2, t_1)$ such that

$$z(t_2) = 0 < z(t_3) < z(t_1). \tag{16}$$

Using the sign property of the function $f(x)$, we deduce that $z'(t) < 0$ for $t \in (t_2, t_1]$. The latter inequality implies $z(t_3) > z(t_1)$, which contradicts (16). Thus, $z(t) \leq 0$ for $t \in [t_0, T)$. Repeating the argument, we conclude that there exists no $t_4 \in (t_0, T)$ such that $z(t_4) < 0$. Hence, $z(t) = z_0(t) = 0$, $t \in [t_0, T)$, is the only solution of the Cauchy problem (14), (15), and $T = +\infty$.

Case 2: Let $z_0 \neq 0$. As above, all solutions of the Cauchy problem (14), (15) are defined for $t \in [t_0, T)$, where $T \leq +\infty$, and can be classified in accordance with their asymptotic behavior as follows:

Type (I): solutions vanishing eventually;

Type (II): positive decreasing solutions;

Type (III): negative increasing solutions.

Clearly, all solutions of the Cauchy problem (14), (15) are bounded on their maximal intervals of existence and therefore they exist in the future. The validity of claim (i) is established.

To prove claim (ii), consider $z_0 > 0$ and assume first that $z_0(t)$ is a solution of the Cauchy problem (14), (15) of type (II), that is, $z_0(t) > 0$ for $t \in [t_0, +\infty)$.

Then $z_0(t)$ is the unique solution of the Cauchy problem

$$z' = -t^{-1}z - a(t)f(z_0(t))z^{2n}, \qquad t \in [t_0, +\infty), \tag{17}$$

$$z(t_0) = z_0 > 0, \tag{18}$$

where

$$z_0 = u_1 - t_0^{-1}u_0. \tag{19}$$

Integration of Bernoulli equation (17) requires a new variable

$$w(t) = [z(t)]^{1-2n}.$$

Then $w(t_0) = z_0^{1-2n} \overset{\text{def}}{=} w_0 > 0$ and

$$\begin{aligned} w'(t) &= (1-2n)z'(t)[z(t)]^{-2n} \\ &= (1-2n)[-t^{-1}z(t) - a(t)f(z_0(t))[z(t)]^{2n}][z(t)]^{-2n} \\ &= (2n-1)[t^{-1}w(t) + a(t)f(z_0(t))], \qquad t \in [t_0, +\infty). \end{aligned}$$

Solving the Cauchy problem

$$\begin{aligned} w' &= (2n-1)[t^{-1}w(t) + a(t)f(z_0(t))], \qquad t \in [t_0, +\infty), \\ w(t_0) &= w_0 > 0, \end{aligned}$$

by variation of constants method, we obtain

$$w(t) = t^{2n-1}\left[\frac{w_0}{t_0^{2n-1}} + (2n-1)\int_{t_0}^{t} \frac{a(s)f(z_0(s))}{s^{2n-1}}\,ds\right], \qquad t \in [t_0, +\infty).$$

Coming back to Bernoulli equation (17), we have

$$\begin{aligned} z(t) &= [w(t)]^{-1/(2n-1)} \\ &= \left[t\left(\frac{w_0}{t_0^{2n-1}} + (2n-1)\int_{t_0}^{t} \frac{a(s)f(z_0(s))}{s^{2n-1}}\,ds\right)^{1/(2n-1)}\right]^{-1}, \qquad t \in [t_0, +\infty). \end{aligned}$$

Hence, $z_0(t)$ satisfies for $t \in [t_0, +\infty)$ the integral equation

$$z_0(t) = \left[t\left(\frac{w_0}{t_0^{2n-1}} + (2n-1)\int_{t_0}^{t} \frac{a(s)f(z_0(s))}{s^{2n-1}}\,ds\right)^{1/(2n-1)}\right]^{-1}. \tag{20}$$

Since $z_0(t) > 0$ for all $t \geq t_0$, we deduce that $f(z_0(s)) > 0$ for all $s \geq t_0$ and

$$0 < z_0(t) \leq \frac{t_0}{w_0^{1/(2n-1)}}\frac{1}{t} = \frac{z_0 t_0}{t} \qquad \text{for } t \geq t_0.$$

Therefore,

$$z_0(t) = O(t^{-1}) \qquad \text{as } t \to +\infty. \tag{21}$$

Claim (ii) for solutions of type (II) is established.

Consider now solutions of type (III) taking $z_0 < 0$ and assuming that $z_0(t) < 0$, $t \in [t_0, +\infty)$, is a solution of the Cauchy problem (14), (15). Following the same lines, we conclude that $z_0(t)$ satisfies Eq. (20). Furthermore, since $f(z_0(s)) < 0$ for all $s \geq t_0$, we deduce that

$$0 > z_0(t) \geq \frac{z_0 t_0}{t} \qquad \text{for } t \geq t_0,$$

and thus solution $z_0(t)$ satisfies (21). Hence, claim (ii) is established for all solutions of types (II) and (III), and is evident for solutions of type (I).

Let $z_0(t)$ be a solution of (14), (15), where z_0 is defined by (19). Consider another Cauchy problem with the nonhomogeneous term $z_0(t)$:

$$u' - \frac{u}{t} = z_0(t), \qquad t \in [t_0, T), \tag{22}$$

$$u(t_0) = u_0. \tag{23}$$

Clearly, the unique solution $u(t)$ of (22), (23) is a solution of Eq. (12) on $[t_0, T)$. Applying variation of constants method, we obtain for $t \in [t_0, T)$

$$u(t) = t\left[\frac{u_0}{t_0} + \int_{t_0}^{t} \frac{z_0(s)}{s}\, ds\right], \tag{24}$$

$$u'(t) = \frac{u_0}{t_0} + \int_{t_0}^{t} \frac{z_0(s)}{s}\, ds + z_0(t). \tag{25}$$

Eqs. (24) and (25) help to establish easily the following two facts.

(a) Assuming, for the sake of contradiction, that $T < +\infty$ and using the estimate

$$\sup_{t \in [t_0, T)} [|u'(t)| + |u(t)|] \leq \sup_{t \in [t_0, T)} |z_0(t)| + (1 + T)\left[\frac{|u_0|}{t_0} + \int_{t_0}^{T} \frac{|z_0(s)|}{s}\, ds\right] < +\infty,$$

we conclude that $T = +\infty$.

(b) Using the fact that $z_0(t) = O(t^{-1})$ as $t \to +\infty$, we deduce that the improper integral

$$\int_{t_0}^{\infty} \frac{z_0(s)}{s}\, ds$$

converges, and thus

$$\lim_{t\to+\infty} u'(t) = \frac{u_0}{t_0} + \int_{t_0}^{\infty} \frac{z_0(s)}{s}\, ds \in \boldsymbol{R}.$$

Therefore, all solutions of Eq. (12) exist in the future and possess property (2), which completes the proof. ■

Remark 2. It follows from (20) that the only solution of Eq. (14) of type (I) is the trivial one. In fact, suppose to the contrary that $z_0(t)$ is a solution of type (I) such that for some t_1 one has $z_0(t) > 0$, $t \in [t_0, t_1)$ and $z_0(t) = 0$ for all $t \geq t_1$. Then $z_0(t)$ satisfies on $[t_0, t_1)$ Eq. (20) and

$$\begin{aligned} z_0(t_1) = 0 &= \lim_{t\to t_1-} z_0(t) \\ &= \left[t_1 \left(\frac{w_0}{t_0^{2n-1}} + (2n-1) \int_{t_0}^{t_1} \frac{a(s) f(z_0(s))}{s^{2n-1}}\, ds \right)^{1/(2n-1)} \right]^{-1} > 0, \end{aligned}$$

a contradiction.

Example 3. Consider the nonlinear differential equation

$$u'' + \alpha(t^{-2} + C^{-2} t^{2\alpha}) \frac{(tu' - u)^3}{t^2 + (tu' - u)^2} = 0, \qquad t \geq 1, \tag{26}$$

where $C \neq 0$ and $\alpha > 0$ are real constants. Here $n = 1$, $f(z) = z(1 + z^2)^{-1}$, $a(t) = \alpha(t^{-1} + C^{-2} t^{2\alpha+1})$, and $t_0 = 1$. It is not difficult to verify all conditions of Theorem 1. In fact, Eq. (26) has the exact solution $u(t) = C(1+\alpha)^{-1}(t - t^{-\alpha})$ which exists for $t \geq 1$ and possesses the property (L).

Theorem 1 demonstrates that the linear-like asymptotic behavior of solutions to Eq. (1) does not inevitably require strong integral restrictions on the functions $h_i(t)$. In fact, assumptions on coefficients imposed by Cohen [1], Constantin [2], Meng [9], Mustafa and Rogovchenko [10], S. Rogovchenko and Yu. Rogovchenko [12], Rogovchenko [13], Tong [15], Trench [16] and Waltman [18] are merely due to demonstration technique.

3. Unbounded domains of initial data

In this section we are concerned with Eq. (1) under the assumption that the real-valued continuous function $f(t, u, u')$ satisfies

$$|f(t, u, u')| \leq a(t) g\left(\left| u' - \frac{u}{t} \right| \right). \tag{27}$$

In what follows, we suppose that $t \geq t_0 \geq 1$, the function $a(t)$ is continuous, nonnegative and satisfies the growth condition

$$\int_{t_0}^{\infty} \frac{a(s)}{s^{\alpha}}\, ds < +\infty \qquad \text{for some } \alpha \in (0,1).$$

Furthermore, let $g(s)$ be continuous, nondecreasing, positive for $s > 0$, and such that for every $x \geq t_0$

$$xg(s) \leq g(x^{1-\alpha}s), \qquad s \geq 0. \tag{28}$$

The class of functions satisfying these assumptions is non-empty. For instance, one may think of $g(s) = s^{1/(1-\alpha)}$, $s \geq 0$.

Fix a $c > 0$. For $u \geq c$, we introduce the function $G(u)$ by

$$G(u) \stackrel{\text{def}}{=} \int_c^u \frac{ds}{g(s)}$$

and require that

$$\int_{t_0}^{\infty} \frac{a(s)}{s^{\alpha}}\, ds < G(+\infty).$$

Note that by (28) the value $G(+\infty)$ is finite. Indeed, let $x = x^{1/(1-\alpha)}$ and $s = 1$ in (28). Then

$$\int_{t_0}^{\infty} \frac{dx}{g(x)} \leq \int_{t_0}^{\infty} \frac{dx}{g(1)x^{1/(1-\alpha)}} = \frac{1-\alpha}{\alpha}[g(1)t_0^{\alpha/(1-\alpha)}]^{-1} < +\infty.$$

In order to establish the main result in this section, we need two auxiliary lemmas.

Lemma 4. *Let $a, b > 0$ be real constants, $\mathbf{u}_0 \in \boldsymbol{R}^n$ be a given vector,*

$$U \stackrel{\text{def}}{=} \{\mathbf{u} \in \boldsymbol{R}^n : \|\mathbf{u} - \mathbf{u}_0\| \leq b\},$$

$$V \stackrel{\text{def}}{=} \left\{\mathbf{v} \in \boldsymbol{R}^n : \|\mathbf{v}\| \leq \frac{a}{t_0}[b + \|\mathbf{u}_0\|]\right\},$$

and let the vector function $\mathbf{g} : D = [t_0, t_0 + a] \times U \times V \to \boldsymbol{R}^n$ be continuous.

Then there exists a solution $\mathbf{x}(t)$ of the Cauchy problem

$$\mathbf{u}'(t) = \mathbf{g}\left(t, \mathbf{u}(t), \int_{t_0}^{t} \frac{\mathbf{u}(s)}{s}\, ds\right), \qquad \mathbf{u}(t_0) = \mathbf{u}_0 \tag{29}$$

defined on $[t_0, t_0 + \alpha]$, where $\alpha = \min\{a, bM^{-1}\}$, and

$$M = \sup_{(t,\mathbf{u},\mathbf{v}) \in D} \|\mathbf{g}(t, \mathbf{u}, \mathbf{v})\|.$$

The proof of this result is very similar to that of Peano's theorem (see Kartsatos [5, Theorem 3.1, pp. 51–52]).

Lemma 5. *Assume that the vector function* $\mathbf{g} : [t_0, t_0 + a] \times \boldsymbol{R}^n \times \boldsymbol{R}^n \to \boldsymbol{R}^n$ *is continuous and let* $\mathbf{u}_0 \in \boldsymbol{R}^n$ *be a given vector. Let* $\mathbf{x}(t)$ *be a solution of the Cauchy problem* (29) *satisfying* $\|\mathbf{x}(t)\| < \lambda$ *for as long as it exists to the right of* t_0, *where* $\lambda > 0$ *is a real constant.*

Then the solution $\mathbf{x}(t)$ *is continuable up to the point* $t = t_0 + a$.

The proof follows the same lines as in Kartsatos [5, Theorem 3.8, p. 57], and we use the definition of continuation of solutions to (29) given by Kartsatos in [5, Definition 3.5, p. 55].

Theorem 6. *Assume that*

$$\int_{t_0}^{\infty} \frac{a(s)}{s^{\alpha}}\, ds < \int_{c+|z_0|t_0^{1-\alpha}}^{\infty} \frac{du}{g(u)}. \tag{30}$$

Then every solution $u(t)$ *of Eq.* (1) *satisfying initial conditions* (7), *where* z_0 *is defined by* (19), *exists in the future. Furthermore,* $u(t) = a_u t + o(t)$ *and* $u'(t) = a_u + o(1)$ *as* $t \to +\infty$.

Proof. We use the relationship between $u(t)$ and $z(t)$ established in Theorem 1. Then Eq. (1) can be written as

$$z' = -t^{-1} z - f(t, u, u'), \qquad t \in [t_0, T),$$

where $u(t)$ is a solution of the Cauchy problem (1), (7) with the initial data satisfying (30) and $z(t) = u'(t) - t^{-1}u(t)$, $t \in [t_0, T)$. Our aim is to prove the following two assertions.

(i) Every solution of the Cauchy problem

$$z' = -t^{-1} z + a(t) g(|z|), \qquad z(t_0) = z_0, \tag{31}$$

where z_0 satisfies (30), exists in the future.

(ii) For every solution $z(t)$ in (i), there exists a real number $\lambda \in (0, 1)$ such that $z(t) = O(t^{-\lambda})$ as $t \to +\infty$.

To prove claims (i) and (ii), consider solution $z_0(t)$ of the Cauchy problem (31) defined for $t \in [t_0, T)$. Clearly, $z_0(t)$ is the unique solution of

$$z' = -t^{-1} z + a(t) g(|z_0(t)|), \qquad z(t_0) = z_0,$$

on $[t_0, T)$. Using variation of constants method, we establish that $z_0(t)$ satisfies the integral equation

$$z_0(t) = \frac{z_0 t_0}{t} + \frac{1}{t} \int_{t_0}^{t} s a(s) g(|z_0(s)|) ds, \qquad t \in [t_0, T). \tag{32}$$

Making use of (32) and assumption (28), we obtain for all $t \in [t_0, T)$

$$(33) \qquad \begin{aligned} t^{1-\alpha}|z_0(t)| &\le \frac{|z_0|t_0}{t^\alpha} + \frac{1}{t^\alpha}\int_{t_0}^{t} a(s)[sg(|z_0(s)|)]ds \\ &\le |z_0|t_0^{1-\alpha} + \frac{1}{t^\alpha}\int_{t_0}^{t} a(s)g(s^{1-\alpha}|z_0(s)|)ds \\ &\le |z_0|t_0^{1-\alpha} + \int_{t_0}^{t} \frac{1}{s^\alpha} a(s)g(s^{1-\alpha}|z_0(s)|)ds. \end{aligned}$$

Introducing the notation

$$y(t) = c + t^{1-\alpha}|z_0(t)|,$$

we deduce from (33) that

$$y(t) \le c + |z_0|t_0^{1-\alpha} + \int_{t_0}^{t} \frac{1}{s^\alpha} a(s)g(y(s))ds, \qquad t \in [t_0, T).$$

Thus, for all $t \in [t_0, T)$

$$\begin{aligned} G(y(t)) &\le G(c + |z_0|t_0^{1-\alpha}) + \int_{t_0}^{\infty} \frac{a(s)}{s^\alpha}\, ds \\ &< G(c + |z_0|t_0^{1-\alpha}) + \int_{c+|z_0|t_0^{1-\alpha}}^{\infty} \frac{du}{g(u)} = G(+\infty). \end{aligned}$$

Since the function G is invertible on $[0, G(+\infty))$, we have

$$(34) \qquad y(t) \le G^{-1}\left(G(c + |z_0|t_0^{1-\alpha}) + \int_{t_0}^{\infty} \frac{a(s)}{s^\alpha}\, ds\right) = K < +\infty.$$

It follows from (34) that

$$|z_0(t)| \le \frac{K}{t^{1-\alpha}} \le K, \qquad t \in [t_0, T),$$

which ensures that $T = +\infty$, and thus $z_0(t)$ exists in the future. Furthermore, inequality (34) implies that for any $\lambda \in (0, 1-\alpha]$

$$z_0(t) = O(t^{-\lambda}) \qquad \text{as } t \to +\infty.$$

In a similar manner, one can establish the following two assertions.

(iii) Every solution of the Cauchy problem

$$(35) \qquad z' = -t^{-1}z - a(t)g(|z|), \qquad z(t_0) = z_0,$$

where z_0 satisfies (30), exists in the future.

(iv) For every solution $z(t)$ in (iii), there exists a $\lambda \in (0,1)$ such that $z(t) = O(t^{-\lambda})$ as $t \to +\infty$.

Coming back to Eq. (1), we conclude that for $t \in [t_0, T)$

$$\begin{aligned} z' &= -\frac{1}{t} z - f(t, u, u') \\ &= -\frac{1}{t} z - f\left[t, t\left(\frac{u_0}{t_0} + \int_{t_0}^{t} \frac{z(s)}{s}\, ds\right), \frac{u_0}{t_0} + \int_{t_0}^{t} \frac{z(s)}{s}\, ds + z(t)\right]. \end{aligned}$$

Thus, for every solution $u(t)$ of the Cauchy problem (1), (7) defined on $[t_0, T)$, there exists a unique solution $z(t)$, also defined on $[t_0, T)$, of the Cauchy problem

$$z' = V\left[t, z(t), \int_{t_0}^{t} \frac{z(s)}{s}\, ds\right], \qquad z(t_0) = z_0 = u_1 - \frac{u_0}{t_0}, \tag{36}$$

where

$$V[t, z, w] \stackrel{\text{def}}{=} -\frac{1}{t} z - f\left[t, t\left(\frac{u_0}{t_0} + w\right), \frac{u_0}{t_0} + w + z\right],$$

and $z(t)$ can be viewed as a nonhomogeneous term in Eq. (22).

It follows from (27) that solution $z(t)$ of (36) satisfies for $t \in [t_0, T)$ differential inequalities

$$z' \le -t^{-1} z + a(t) g(|z|), \qquad z(t_0) = z_0,$$

and

$$z' \ge -t^{-1} z - a(t) g(|z|), \qquad z(t_0) = z_0.$$

If $z_{\min}(t)$ and $z_{\max}(t)$ are the minimal and maximal solutions of the Cauchy problems (35) and (31) on $[t_0, +\infty)$, one has

$$z_{\min}(t) \le z(t) \le z_{\max}(t), \qquad t \in [t_0, T). \tag{37}$$

Furthermore, there exists a $\lambda \in (0,1)$ such that

$$z_{\min}(t) = O(t^{-\lambda}) \qquad \text{and} \qquad z_{\max}(t) = O(t^{-\lambda}) \qquad \text{as } t \to +\infty. \tag{38}$$

Using (37), (38), and Lemma 5, we conclude the following.

(v) Every solution $z(t)$ of the Cauchy problem (36) exists in the future and is bounded.

(vi) For every solution $z(t)$ in (v), there exists a $\lambda \in (0,1)$ such that $z(t) = O(t^{-\lambda})$ as $t \to +\infty$.

To complete the proof, one follows the same steps as in Theorem 1. ■

Remark 7. As it has been mentioned in the Introduction, nonlinear differential equation (10) has a non-extendable solution (11) defined on $[1,2)$. Simple computation of z_0 for this solution gives

$$z_0 \stackrel{\text{def}}{=} u'(1) - u(1)/1 = -8/3.$$

With the choice $\alpha = 1/2$ and $a(t) = t^{-2}$, $t \geq t_0 = 1$, condition (30) reads as

$$2/3 < (c + |z_0|)^{-1}.$$

Since $c > 0$ is arbitrary real number, the best possible estimate is $|z_0| < 3/2$. Initial data of the blowing up solution (11) do not belong to the region of linear-like behavior obtained in Theorem 6. This region is neither $\boldsymbol{R}^2$, nor the empty set since there are solutions satisfying $|z_0| < 3/2$ with linear-like behavior at infinity as, for instance, $u(t) = t/8$.

Example 8. Consider the nonlinear differential equation

$$u'' - \frac{1}{15t^{3/2}}\left(u' - \frac{u}{t}\right)^2\left(1 + \frac{3tu - 2t^2u'}{8(t^2+u^2)}\right) = 0, \qquad t \geq 1. \tag{39}$$

A straightforward computation yields

$$\begin{aligned}\left|\frac{1}{15}t^{-3/2}\left(u' - \frac{u}{t}\right)^2\left(\frac{1}{4}\cdot\frac{u' - (3u)/(2t)}{1 + (|u|/t)^2} - 1\right)\right| &\leq \frac{1}{15}t^{-3/2}\left|u' - \frac{u}{t}\right|^2 \\ &+ \frac{1}{60}t^{-3/2}\frac{1}{1+(|u|/t)^2}\left|u' - \frac{u}{t}\right|^3 + \frac{1}{120}t^{-3/2}\frac{(|u|/t)}{1+(|u|/t)^2}\left|u' - \frac{u}{t}\right|^2 \\ &\leq \frac{7}{80}t^{-3/2}\left(\left|u' - \frac{u}{t}\right|^2 + \left|u' - \frac{u}{t}\right|^3\right).\end{aligned}$$

Therefore, $a(t) = (7/80)t^{-3/2}$, $g(s) = s^2 + s^3$, $t_0 = 1$, and $\alpha = 1/2$. Then

$$\int_1^\infty \frac{7s^{-3/2}}{80s^{1/2}}\,ds = \frac{7}{80}, \tag{40}$$

$$\int_{c+|z_0|}^\infty \frac{ds}{s^2+s^3} \geq \int_{c+|z_0|}^\infty \frac{ds}{(s+3^{-1})^3} = \frac{1}{2(c+3^{-1}+|z_0|)^2}. \tag{41}$$

Solving the inequality

$$\frac{7}{80} < \frac{1}{2}\cdot\frac{1}{(c+3^{-1}+|z_0|)^2},$$

we conclude that the region of linear-like behavior contains the set

$$|z_0| < \sqrt{40/7} - 3^{-1} \approx 2.05723. \tag{42}$$

Thus, all solutions of Eq. (39) with initial data satisfying (42) exhibit desired behavior at infinity. Since we approximate the integral (41) instead of computing it exactly, the estimate (42) is not sharp and can be improved.

It follows from the inequality

$$\int_{c+|z_0|}^{\infty} \frac{ds}{s^2 + s^3} \le (c + |z_0|)^{-1} \le \frac{7}{80}$$

that one can find solutions of Eq. (39) with different from linear asymptotic behavior as $t \to +\infty$, if any, in the set containing initial data satisfying $|z_0| > 80/7$, although the description of this set is not precise since we only estimated (41). In fact, Eq. (39) has the exact solution $u(t) = 45t^{3/2}$ whose derivative grows without limit at infinity, and a simple computation shows that for this solution $z_0 = 45/2$. This example confirms coexistence of positive half-trajectories with qualitatively different asymptotic behavior discussed by the present authors in [10] for the general equation (1). Even for particular classes of equations studied in this paper, one can have, for instance, both linear-like solutions with bounded derivatives and solutions having unbounded derivatives, which reflects complicated nature of the problem.

4. Discussion

We would like to conclude the paper pointing out open problems and indicating possible applications and extensions of the results.

Remark 9. For Eq. (12), the coefficient $h(t)$ in the inequality (4) is $a(t)$. The only conditions imposed on it are continuity and nonnegativity, there are no integrability assumptions or restrictions on its asymptotic behavior. We believe that these hypotheses are minimal for this class of equations and cannot be relaxed. It is not possible to achieve a similar goal for the general nonlinear differential equation (1). Our efforts aimed on relaxing integrability conditions on the coefficients imposed by most authors, see Cohen [1], Constantin [2], Kusano and Trench [7, 8], Meng [9], Mustafa and Rogovchenko [10], S. Rogovchenko and Yu. Rogovchenko [12], Rogovchenko [13], Tong [15] and Trench [16], proved to be successful for Eq. (12). However, the problem remains open for other classes of nonlinear differential equations.

Remark 10. As opposed to related results reported in the literature, the region of linear-like behavior in the (u, u')-plane defined by the condition (30) contains unbounded strip. Study of the relationship between unbounded connected domains of $\boldsymbol{R}^2$ having smooth boundary and linear-like solutions of nonlinear differential equations, if any, is another interesting open problem.

Remark 11. Following Kusano and Trench [8, Example 2] and Zhang [19, Corollary 3], let us focus attention on the role played by the terms u/t and $u' - u/t$ in Eqs. (1) and (12) in connection with the problem of existence of radial solutions to nonlinear elliptic equations. Consider equation

$$\triangle u + f(|x|, u, |\nabla u|) = 0, \qquad x \in \Omega_{t_0}, \tag{43}$$

where $\Omega_{t_0} = \{x \in \boldsymbol{R}^3 : |x| > t_0\}$.

If $u = u(|x|)$ is a radially symmetric solution of Eq. (43), the function $y(t) = tu(t)$ satisfies the ordinary differential equation

$$y'' + tf\left(t, \frac{y}{t}, \frac{1}{t}\left|y' - \frac{y}{t}\right|\right) = 0, \qquad t > t_0.$$

In this case, we have

$$\triangle u = t^{-1}y''(t), \qquad u' = t^{-1}z(t) \qquad \text{and} \qquad |\nabla u| = t^{-1}|z(t)|,$$

where $z(t) = y'(t) - t^{-1}y(t)$. It is straightforward now to restate Theorem 6 for Eq. (43).

Finally, we observe that property (L) can be generalized as follows: for a continuable solution $u(t)$ of Eq. (1), there exists a real constant a such that

$$\lim_{t\to\infty}\left[u'(t) - \frac{u(t)}{t}\right] = a. \tag{44}$$

It is easy to see that (44) implies

$$\lim_{t\to\infty}\frac{u(t)}{t \ln t} = a.$$

We believe that the general condition (44) is of interest for the study of the problems where property (L) has been considered (for instance, nonoscillation of solutions of ordinary differential equations or asymptotic behavior of solutions of elliptic equations in exterior domains). To the best of our knowledge, no attempts in this direction have been made yet.

To show the potential of this generalization, consider a perturbation of Eq. (1), namely, nonlinear differential equation

$$u'' + f(t, u, u') = b(t), \qquad t \geq t_0, \tag{45}$$

where $b(t)$ is a continuous function satisfying

$$\lim_{t\to\infty} \frac{1}{t}\int_{t_0}^{t} sb(s)ds = a \in \boldsymbol{R}.$$

Elementary examples of such functions are, for instance, $b(t) = t^{-1}$ and $b(t) = t^{-1}(2^{-1} - \cos t)$, where $t \geq t_0 \geq 1$.

Assume, in addition, that

$$|f(t,u,u')| \leq h(t)\left|u' - \frac{u}{t}\right|,$$

where $h(t)$ is a continuous, nonnegative function such that

$$\int_{t_0}^{\infty} sh(s)ds < +\infty. \tag{46}$$

Theorem 12. *Let* $u_0 \in \boldsymbol{R}$. *Then there exists a solution* $u(t)$ *of the problem* (45), (23), *defined on* $[t_0, +\infty)$, *which satisfies*

$$\lim_{t\to\infty}\left[u'(t) - \frac{u(t)}{t}\right] = a.$$

Proof. Let $u(t)$ be a solution of (45), (23). Using the transformation $v = tu' - u$, we obtain

$$v' + g\left(t, v(t), \int_{t_0}^{t} s^{-2}v(s)ds\right) = tb(t), \qquad t \geq t_0,$$

where

$$g(t,v,w) = tf\left(t, t\left(\frac{u_0}{t_0} + w\right), \frac{u_0}{t_0} + w + \frac{v}{t}\right).$$

We have

$$\left|g\left(t, v(t), \int_{t_0}^{t} s^{-2}v(s)ds\right)\right| \leq h(t)|v(t)|.$$

Introduce the space $L(t_0)$ of all continuous real-valued functions $v(t)$ defined on $[t_0, +\infty)$ with the property

$$\lim_{t\to\infty} \frac{v(t)}{t} \in \boldsymbol{R}.$$

Being endowed with the Chebyshev norm

$$\|v\| = \sup_{t \geq t_0} \frac{|v(t)|}{t},$$

$L(t_0)$ becomes a Banach space.

Define the integral operator $\mathscr{T} : L(t_0) \to L(t_0)$ by the formula

$$(\mathscr{T} v)(t) = \int_{t_0}^{t} sb(s)ds + \int_{t}^{\infty} g\left(s, v(s), \int_{t_0}^{s} \tau^{-2} v(\tau)d\tau\right)ds, \qquad t \geq t_0.$$

Then, using the technique developed by the authors in [10], we conclude that $\mathscr{T}$ is completely continuous and has a fixed point $v_0(t)$ in $L(t_0)$. Let

$$u(t) = t\left(\frac{u_0}{t_0} + \int_{t_0}^{t} s^{-2} v_0(s)ds\right).$$

Since $v_0(t)$ is *sublinear* in the sense that $|v_0(t)| \leq \|v_0\| t$ for all $t \geq t_0$,

$$\begin{aligned} \left| \left[u'(t) - \frac{u(t)}{t} \right] - \frac{1}{t}\int_{t_0}^{t} sb(s)ds \right| &= \left| \frac{(\mathscr{T} v_0)(t)}{t} - \frac{1}{t}\int_{t_0}^{t} sb(s)ds \right| \\ &\leq \|v_0\| \frac{1}{t}\int_{t}^{\infty} sh(s)ds, \qquad t \geq t_0. \end{aligned}$$

Passing to the limit as $t \to \infty$, we complete the proof of the theorem. ■

Remark 13. Clearly, $a = 0$ whenever the function $t|b(t)|$ is integrable, and one can use the technique developed by the authors in [10, Lemma 7] to prove that assumptions (4) and (6) hold for Eq. (45) with the choice $p_i(w) = w + 1/2$, $i = 1, 2$. According to [10, Theorem 6], for any pair of real numbers A, B there exists a solution $u(t)$ of Eq. (45) defined on $[t_0, +\infty)$ with the asymptotic representation $u(t) = At + B + o(1)$ as $t \to +\infty$. This result and the fact that for $a = 0$ condition (44) does not provide enough information on the behavior of $u(t)$ for large t stimulate particular interest to the case $a \neq 0$. In fact, assume that $a > 0$ and $b(t) \geq 0$ for all $t \geq t_0$. Then

$$\int_{t_0}^{\infty} b(s)ds = +\infty. \tag{47}$$

To prove (47), note that

$$0 < a = \lim_{t \to +\infty} \frac{1}{t}\int_{t_0}^{t} sb(s)ds = \lim_{t \to +\infty} \frac{1}{t}\int_{x}^{t} sb(s)ds \leq \int_{x}^{\infty} b(s)ds$$

for all $x \geq t_0$, which yields the conclusion. Condition (47) does not allow application of the results reported by the authors in [10], emphasizing thus importance of Theorem 12.

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