

# The Landau-Lifshitz Flow of Maps into the Lobachevsky Plane

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**Abstract.** The global existence of weak and strong solutions to the Cauchy problem for the Landau-Lifshitz flow of maps on the one-dimensional torus into the Lobachevsky plane is proved by the method of higher order parabolic regularization and energy estimates.

*Key Words and Phrases.* Noncompact Landau-Lifshitz flow, Cauchy problem, Global existence results.

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## I. Introduction

It is well known that the classical  $SU(2)/U(1)$  Heisenberg ferromagnet model in the continuum limit is described by the  $O(3)$  isotropic Landau-Lifshitz equation and is gauge equivalent to the attractive nonlinear Schrödinger equation (see [5]). It was shown in [9] that a classical version of the Heisenberg spin model on the noncompact  $SU(1,1)/U(1)$  manifold is gauge-equivalent to the repulsive nonlinear Schrödinger equation. The corresponding  $O(2,1)$  isotropic Landau-Lifshitz equation has been presented in [4]. All those equations are completely integrable and can be solved by the inverse spectral-transform method.

As to the initial or initial-boundary value problem for the compact Landau-Lifshitz equations with or without damping has been investigated by many authors ([1], [2], [3], [7], [10], [14]). In this paper we consider the initial value problem for the noncompact Landau-Lifshitz equation on the one-dimensional torus  $T^1$ .

The one-dimensional noncompact Heisenberg model is described by the isotropic Landau-Lifshitz equation

$$(1) \quad S_t = \frac{1}{2i}[S, S_{xx}]$$

where

$$(2) \quad S = \begin{pmatrix} u_3 & -u_1 + iu_2 \\ u_1 + iu_2 & -u_3 \end{pmatrix} \in \mathfrak{su}(1,1)$$

and

$$(3) \quad \det S = -1, \quad S^2 = I.$$

In order to write down the corresponding  $O(2, 1)$  isotropic Landau-Lifshitz equation in an explicit form, we prepare the following:

Let  $\mathbf{R}^{2+1}$  be the 3-dimensional Minkowski space endowed with the metric

$$g = (g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

The upper Lovachevsky plane  $\mathfrak{H} \subset \mathbf{R}^{2+1}$  is defined by

$$(4) \quad \mathfrak{H} = \{\mathbf{u} = (u_1, u_2, u_3) : |u_1|^2 + |u_2|^2 - |u_3|^2 = -1, u_3 > 0\}.$$

The pseudo cross product  $\dot{\times}$  and the pseudo scalar product  $[\cdot, \cdot]$  in  $\mathbf{R}^{2+1}$  are given by

$$(5) \quad \mathbf{a} \dot{\times} \mathbf{b} = (a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, -(a_1b_2 - a_2b_1)).$$

and

$$(6) \quad [\mathbf{a}, \mathbf{b}] = \sum_{j=1}^3 g_{jj}a_jb_j = a_1b_1 + a_2b_2 - a_3b_3,$$

respectively. Then, (1) can be rewritten as

$$(7) \quad \mathbf{u}_t = \mathbf{u} \dot{\times} \mathbf{u}_{xx}$$

where  $\mathbf{u}$  takes values in  $\mathfrak{H}$  (see [4]). We call (7) the  $O(2, 1)$  isotropic Landau-Lifshitz equation, or equivalently say, the Landau-Lifshitz flow of maps into the Lobachevsky plane.

Motivated by the Gilbert damping term in the compact Landau-Lifshitz equation, we extend (7) to the following from:

$$(8) \quad \mathbf{u}_t = -\lambda \mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xx}) + \mu \mathbf{u} \dot{\times} \mathbf{u}_{xx}$$

where  $\lambda \geq 0$  is a damping parameter and  $\mu \in \mathbf{R}$  (see [8]).

Our aim of this paper is to establish the global existence of weak or strong solutions to the Cauchy problem for (8) in  $\mathbf{T}^1 \times [0, \infty)$  with initial data

$$(9) \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad x \in \mathbf{T}^1.$$

For a solution  $\mathbf{u} = (u^1, u^2, u^3)$  of (8)–(9) let us introduce the pseudo energy density  $e(\mathbf{u})$  by

$$e(\mathbf{u}) = [\mathbf{u}_x, \mathbf{u}_x] = |u_{1x}|^2 + |u_{2x}|^2 - |u_{3x}|^2.$$

If  $\mathbf{u}$  is sufficiently smooth, we have the energy identity

$$\int_{\mathbf{T}^1} e(\mathbf{u}(x, t))dx + \lambda \int_0^t \int_{\mathbf{T}^1} [\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \mathbf{u} \dot{\times} \mathbf{u}_{xx}](x, s)dxds = \int_{\mathbf{T}^1} e(\mathbf{u}_0(x))dx.$$

At first glance the integrands  $e(\mathbf{u})$  and  $[\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \mathbf{u} \dot{\times} \mathbf{u}_{xx}]$  in the energy identity are not positive definite and cannot be handled well at all. Under the constraint  $\mathbf{u} \in \mathfrak{S}$  we, however, see that they are nonnegative. Indeed, the direct computation yields

$$\begin{aligned} e(\mathbf{u}) &= |u_{1x}|^2 + |u_{2x}|^2 - \frac{1}{|u_3|^2} |u_1 u_{1x} + u_2 u_{2x}|^2 \\ &= \frac{1}{|u_3|^2} (|u_{1x}|^2 + |u_{2x}|^2 + |u_2 u_{1x} - u_1 u_{2x}|^2) \geq 0 \end{aligned}$$

since  $|u_3|^2 = |u_1|^2 + |u_2|^2 + 1$ . As to the nonnegativity of the other quantity, it will be shown in the next section. Making use of the energy identity in an essential manner, we have

**Theorem 1.** *Let  $\lambda \geq 0$  and  $\mu \in \mathbf{R}$ . For every  $\mathbf{u}_0 \in H^1(\mathbf{T}^1)$  such that  $\mathbf{u}_0(x) \in \mathfrak{S}$  ( $x \in \mathbf{T}^1$ ), there exists a global weak solution  $\mathbf{u}$  of (8)–(9), satisfying for any  $T > 0$*

$$(10) \quad \mathbf{u}(x, t) \in \mathfrak{S}, \quad (x, t) \in \mathbf{T}^1 \times (0, \infty),$$

$$(11) \quad \mathbf{u} \in L^\infty([0, T], H^1(\mathbf{T}^1)) \cap C([0, T], L^2(\mathbf{T}^1)),$$

$$(12) \quad \begin{aligned} &\int_{\mathbf{T}^1} (\langle \mathbf{u}(T), \varphi(T) \rangle - \langle \mathbf{u}(0), \varphi(0) \rangle) dx \\ &= \int_0^T \int_{\mathbf{T}^1} (\langle u, \varphi_t \rangle - \lambda \langle \mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_x)_x, \varphi \rangle + \mu \langle \mathbf{u} \dot{\times} \mathbf{u}_x, \varphi_x \rangle) dxdt \end{aligned}$$

for all  $\varphi \in C^\infty(\mathbf{T}^1 \times [0, T], \mathbf{R}^3)$  and

$$(13) \quad \int_{\mathbf{T}^1} e(\mathbf{u}(x, t))dx \leq \int_{\mathbf{T}^1} e(\mathbf{u}_0(x))dx \quad \forall t \in [0, T].$$

**Theorem 2.** *Let  $\lambda \geq 0$  and  $\mu \in \mathbf{R}$ . Given initial function  $\mathbf{u}_0 \in H^2(\mathbf{T}^1)$  with  $\mathbf{u}_0(x) \in \mathfrak{S}$  ( $x \in \mathbf{T}^1$ ), there exists a global strong solution  $\mathbf{u}$  of (8)–(9), such that any  $T > 0$*

$$(14) \quad \mathbf{u} \in L^\infty([0, T], H^2(\mathbf{T}^1)),$$

$$(15) \quad \mathbf{u}_t \in L^\infty([0, T], H^2(\mathbf{T}^1))$$

and satisfies the energy inequality (13).

The proofs of Theorems 1 and 2 are based on the method of higher order parabolic regularization, which was shown to be useful for solving a class of nonlinear partial differential equations (see [12], [13]). The reason why we employ the higher order parabolic regularization is as follows: Firstly, under the condition  $\mathbf{u} \in \mathfrak{S}$ , (8) is formally equivalent to

$$\mathbf{u}_t = \lambda(\mathbf{u}_{xx} + [\mathbf{u}_x, \mathbf{u}_x]\mathbf{u}) + \mu\mathbf{u} \dot{\times} \mathbf{u}_{xx}$$

(see below in the next section). It is a fully nonlinear parabolic equation if  $\mu \neq 0$ . Secondly, noncompactness of the constraint  $\mathbf{u} \in \mathfrak{S}$  gives difficulties to obtain higher order energy estimates. Besides without the constraint you cannot obtain energy estimates and it is very difficult to construct approximate smooth solutions which have the constraint.

## II. Preliminaries

As to the properties of the pseudo cross product  $\dot{\times}$ , we have

**Lemma 1.** *The product “ $\dot{\times}$ ” and  $[\cdot, \cdot]$  on  $\mathfrak{S}$  satisfy the following identities*

$$(16) \quad \mathbf{a} \dot{\times} \mathbf{b} = -\mathbf{b} \dot{\times} \mathbf{a},$$

$$(17) \quad \mathbf{a} \dot{\times} (\mathbf{b} \dot{\times} \mathbf{c}) = [\mathbf{a}, \mathbf{b}]\mathbf{c} - [\mathbf{a}, \mathbf{c}]\mathbf{b},$$

$$(18) \quad [\mathbf{a}, \mathbf{b} \dot{\times} \mathbf{c}] = [\mathbf{b}, \mathbf{c} \dot{\times} \mathbf{a}] = [\mathbf{c}, \mathbf{a} \dot{\times} \mathbf{b}].$$

From (16)–(18) we get

$$(19) \quad [\mathbf{a}, \mathbf{a} \dot{\times} \mathbf{b}] = 0,$$

$$(20) \quad [\mathbf{a} \dot{\times} \mathbf{b}, \mathbf{c} \dot{\times} \mathbf{d}] = [\mathbf{a}, \mathbf{b} \dot{\times} (\mathbf{c} \dot{\times} \mathbf{d})] = [\mathbf{a}, \mathbf{d}][\mathbf{b}, \mathbf{c}] - [\mathbf{a}, \mathbf{c}][\mathbf{b}, \mathbf{d}],$$

In order to obtain appropriate a priori estimates of solutions, we prepare the following.

**Lemma 2.** *For a smooth map  $\mathbf{u} = (u_1, u_2, u_3) : T^1 \rightarrow \mathfrak{S}$  it holds that*

$$(21) \quad [\mathbf{u}, \mathbf{u}_x] = 0,$$

$$(22) \quad [\mathbf{u}, \mathbf{u}_{xx}] = -[\mathbf{u}_x, \mathbf{u}_x] = -e(\mathbf{u})$$

$$(23) \quad [\mathbf{u}, \mathbf{u}_{xxx}] = -3[\mathbf{u}_x, \mathbf{u}_{xx}] = -\frac{3}{2}e(\mathbf{u})_x,$$

$$(24) \quad [\mathbf{u}, \mathbf{u}_{xxxx}] = -4[\mathbf{u}_x, \mathbf{u}_{xxx}] - 3[\mathbf{u}_{xx}, \mathbf{u}_{xx}] = -2e(\mathbf{u})_{xx} + [\mathbf{u}_{xx}, \mathbf{u}_{xx}]$$

$$(25) \quad [\mathbf{u}, \mathbf{u}_{xxxxx}] = -10[\mathbf{u}_{xx}, \mathbf{u}_{xxx}] - 5[\mathbf{u}_x, \mathbf{u}_{xxxx}] = -\frac{5}{2}e(\mathbf{u})_{xxx} - \frac{5}{2}[\mathbf{u}_{xx}, \mathbf{u}_{xx}]_x.$$

*Proof.* Since  $[\mathbf{u}, \mathbf{u}] = -1$ , we have (21). Differentiating the both sides of (21) in  $x$ , we get

$$(26) \quad [\mathbf{u}, \mathbf{u}_{xx}] + [\mathbf{u}_x, \mathbf{u}_x] = 0$$

from which (22) follows. Differentiation of (26) gives

$$[\mathbf{u}, \mathbf{u}_{xxx}] = -3[\mathbf{u}_x, \mathbf{u}_{xx}] = -\frac{3}{2}[\mathbf{u}_x, \mathbf{u}_x]_x$$

which implies (23). In the same manner we obtain (24) and (25).  $\blacksquare$

**Proposition 1.** For any  $\mathbf{u} = (u_1, u_2, u_3) \in \mathfrak{S}$  and  $\mathbf{v} = (v_1, v_2, v_3) \in \mathbf{R}^3$ , the quantity  $[\mathbf{u} \dot{\times} \mathbf{v}, \mathbf{u} \dot{\times} \mathbf{v}]$  ( $j \in N$ ) is nonnegative. Moreover, for any  $\alpha \geq 0$

$$(27) \quad [\mathbf{u} \dot{\times} \mathbf{v}, \mathbf{u} \dot{\times} \mathbf{v}] + \alpha[\mathbf{u}, \mathbf{v}]^2 \geq \alpha \frac{[\mathbf{u}, \mathbf{v}]^2}{|u_3|^2} + \frac{\alpha}{1 + \alpha} \frac{|v_1|^2 + |v_2|^2}{|u_3|^2}.$$

*Proof.* By definition we have

$$(28) \quad v_3 = \frac{u_1 v_1 + u_2 v_2 - [\mathbf{u}, \mathbf{v}]}{u_3}$$

and

$$[\mathbf{u} \dot{\times} \mathbf{v}, \mathbf{u} \dot{\times} \mathbf{v}] = [\mathbf{v}, \mathbf{v}] + [\mathbf{u}, \mathbf{v}]^2$$

since  $\mathbf{u} \in \mathfrak{S}$ . Hence,

$$\begin{aligned} & [\mathbf{u} \dot{\times} \mathbf{v}, \mathbf{u} \dot{\times} \mathbf{v}] + \alpha[\mathbf{u}, \mathbf{v}]^2 \\ &= |v_1|^2 + |v_2|^2 - \frac{|u_1 v_1 + u_2 v_2 - [\mathbf{u}, \mathbf{v}]|^2}{|u_3|^2} + (1 + \alpha)[\mathbf{u}, \mathbf{v}]^2 \\ &= \frac{1}{|u_3|^2} \left\{ \left( \frac{1}{\sqrt{1 + \alpha}} v_1 + \sqrt{1 + \alpha} u_1 [\mathbf{u}, \mathbf{v}] \right)^2 + \left( \frac{1}{\sqrt{1 + \alpha}} v_2 + \sqrt{1 + \alpha} u_2 [\mathbf{u}, \mathbf{v}] \right)^2 \right. \\ &\quad \left. + \alpha[\mathbf{u}, \mathbf{v}]^2 + \frac{\alpha}{1 + \alpha} (|v_1|^2 + |v_2|^2) + |u_2 v_1 - u_1 v_2|^2 \right\} \end{aligned}$$

from which (27) follows.  $\blacksquare$

**Corollary 1.** For a smooth map  $\mathbf{u} = (u_1, u_2, u_3) : \mathbf{T}^1 \rightarrow \mathfrak{S}$ , the functions  $x \rightarrow [\mathbf{u} \dot{\times} D_x^j \mathbf{u}, \mathbf{u} \dot{\times} D_x^j \mathbf{u}](x)$  are nonnegative for any  $j \in N$ . In particular,  $[\mathbf{u}_{xx}, \mathbf{u}_{xx}] + e(\mathbf{u})^2$  is nonnegative. Moreover, for any  $\alpha \geq 0$

$$(29) \quad e(\mathbf{u}) \geq \frac{\alpha}{1 + \alpha} \frac{|u_{1x}|^2 + |u_{2x}|^2}{|u_3|^2},$$

$$(30) \quad [\mathbf{u} \dot{\times} D_x^j \mathbf{u}, \mathbf{u} \dot{\times} D_x^j \mathbf{u}] + \alpha [\mathbf{u}, D_x^j \mathbf{u}]^2 = [D_x^j \mathbf{u}, D_x^j \mathbf{u}] + (1 + \alpha) [\mathbf{u}, D_x^j \mathbf{u}]^2 \\ \geq \alpha \frac{[\mathbf{u}, D_x^j \mathbf{u}]^2}{|u_3|^2} + \frac{\alpha}{1 + \alpha} \frac{|D_x^j u_1|^2 + |D_x^j u_2|^2}{|u_3|^2}.$$

*Proof.* Take  $\mathbf{v} = D_x^j \mathbf{u}$  in Proposition 1 to obtain (30). From (22) we have

$$[\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \mathbf{u} \dot{\times} \mathbf{u}_{xx}] = [\mathbf{u}_{xx}, \mathbf{u}_{xx}] + [\mathbf{u}, \mathbf{u}_{xx}]^2 = [\mathbf{u}_{xx}, \mathbf{u}_{xx}] + e(\mathbf{u})^2 \geq 0. \quad \blacksquare$$

The norm in a normed space  $X$  will be denoted by  $\|\cdot\|_X$ . The inner product in  $L^2(\mathbf{T}^1)$  will be denoted by  $(\cdot, \cdot)$ .

Finally we shall use the following well known results.

**Lemma 3** (The Gagliardo-Nirenberg inequality cf. [6]). *Let  $1 \leq p \leq q \leq +\infty$ ,  $r \geq 1$  and  $0 \leq j < s$ . For all  $u \in H^{s,p}(\mathbf{T}^1) \cap L^r(\mathbf{T}^1)$ , we have*

$$(31) \quad \|D^j u\|_{L^q(\mathbf{T}^1)} \leq C \|u\|_{H^{s,p}(\mathbf{T}^1)}^a \|u\|_{L^r(\mathbf{T}^1)}^{1-a}$$

with a constant  $C > 0$ , independent of  $u$ , and  $a$  given by

$$(32) \quad \frac{1}{q} - j = a \left( \frac{1}{p} - s \right) + (1 - a) \frac{1}{r}.$$

provided that  $s - j - 1/p$  is not a nonnegative integer. If  $s - j - 1/p$  is a nonnegative integer, then (31) holds for  $a = j/s$ .

**Lemma 4** (cf. [11]). *For any  $u, v \in H^{s,2}(\mathbf{T}^1)$  ( $s \in \mathbf{N}$ ), we have*

$$(33) \quad \|uv\|_{H^{s,2}(\mathbf{T}^1)} \leq C (\|u\|_{L^\infty(\mathbf{T}^1)} \|v\|_{H^{s,2}(\mathbf{T}^1)} + \|u\|_{H^{s,2}(\mathbf{T}^1)} \|v\|_{L^\infty(\mathbf{T}^1)})$$

### III. Higher order parabolic regularization

For any  $\varepsilon > 0$  we consider the Cauchy problem of the higher order parabolic regularization of (8):

$$(34) \quad \mathbf{u}_t - \varepsilon \mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xx})_{xx} = -\lambda \mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xx}) + \mu \mathbf{u} \dot{\times} \mathbf{u}_{xx}, \quad (x, t) \in \mathbf{T}^1 \times [0, \infty),$$

with

$$(35) \quad \mathbf{u}(x, 0) = \mathbf{u}_0(x) \in \mathfrak{H}, \quad x \in \mathbf{T}^1.$$

**Lemma 5.**  *$\mathbf{u}$  is a regular solution of (34)–(35) if and only if  $\mathbf{u}$  is a regular solution of the equation*

$$(36) \quad \mathbf{u}_t + \varepsilon (\mathbf{u}_{xxxx} + 2[\mathbf{u}, \mathbf{u}_{xxx}] \mathbf{u}_x + [\mathbf{u}, \mathbf{u}_{xxxx}] \mathbf{u}) \\ = -\lambda \mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xx}) + \mu \mathbf{u} \dot{\times} \mathbf{u}_{xx}, \quad (x, t) \in \mathbf{T}^1 \times [0, \infty)$$

with initial condition (35).

*Proof.* Let  $\mathbf{u}$  be a regular solution of (34)–(35). By virtue of (19) we have

$$\frac{1}{2} \frac{\partial}{\partial t} [\mathbf{u}, \mathbf{u}] = [\mathbf{u}, \mathbf{u}_t] = 0$$

from which we obtain  $[\mathbf{u}(x, t), \mathbf{u}(x, t)] = -1$ , that is,  $\mathbf{u}(x, t) \in \mathfrak{S}$ ,  $(x, t) \in \mathbf{T}^1 \times (0, \infty)$ .

Then, making use of (17) and (21)–(24), we have

$$\begin{aligned} (37) \quad \mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xx})_{xx} &= 2\mathbf{u} \dot{\times} (\mathbf{u}_x \dot{\times} \mathbf{u}_{xxx}) + \mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xxxx}) \\ &= 2[\mathbf{u}, \mathbf{u}_x] \mathbf{u}_{xxx} - 2[\mathbf{u}, \mathbf{u}_{xxx}] \mathbf{u}_x + [\mathbf{u}, \mathbf{u}] \mathbf{u}_{xxxx} - [\mathbf{u}, \mathbf{u}_{xxxx}] \mathbf{u} \\ &= -\mathbf{u}_{xxxx} - 2[\mathbf{u}, \mathbf{u}_{xxx}] \mathbf{u}_x - [\mathbf{u}, \mathbf{u}_{xxxx}] \mathbf{u} \end{aligned}$$

Hence,  $\mathbf{u}$  satisfies (36). Conversely, suppose that  $\mathbf{u}$  is the solution of (36) with (35). Setting

$$U(x, t) = \frac{1}{2} [\mathbf{u}(x, t), \mathbf{u}(x, t)] + 1,$$

we have

$$(38) \quad U_t + \varepsilon U_{xxxx} = -2\varepsilon [\mathbf{u}, \mathbf{u}_{xxx}] U_x - \varepsilon [\mathbf{u}, \mathbf{u}_{xxxx}] U$$

with

$$(39) \quad U(x, 0) = 0, \quad x \in \mathbf{T}^1.$$

Then, integrating by parts and making use of Lemma 2, we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbf{T}^1} |U|^2 dx + \varepsilon \int_{\mathbf{T}^1} |U_{xx}|^2 dx \leq C \int_{\mathbf{T}^1} |U|^2 dx$$

where  $C$  is a positive constant depending on  $\|\mathbf{u}\|_{C([0, T]; C^3(\mathbf{T}^1))}$  for any  $T > 0$ . Hence, we have

$$\int_{\mathbf{T}^1} |U(x, t)|^2 dx \leq e^{Ct} \int_{\mathbf{T}^1} |U(x, 0)|^2 dx = 0,$$

which implies that  $U(x, t) \equiv 0$ ,  $(x, t) \in \mathbf{T}^1 \times (0, \infty)$ . This and the continuity of  $\mathbf{u}$  yield that  $\mathbf{u}(x, t) \in \mathfrak{S} \quad \forall (x, t) \in \mathbf{T}^1 \times (0, \infty)$  since  $\mathbf{u}_0 \in \mathfrak{S}$ . Then, in view of (37) we conclude that  $\mathbf{u}$  solves (34).  $\blacksquare$

We now establish the local existence and uniqueness of smooth solutions of the Cauchy problem (34)–(35).

**Theorem 3.** *Let  $\varepsilon > 0$  be arbitrary. For every initial condition  $\mathbf{u}_0 \in H^s(\mathbf{T}^1)$  ( $s \geq 2$ ), there exists a time  $T^* > 0$  depending only on the size of  $\|\mathbf{u}_0\|_{H^1(\mathbf{T}^1)}$  and there exists a unique  $\mathbf{u}$  such that  $\mathbf{u}(x, t) \in \mathfrak{S} \forall (x, t) \in \mathbf{T}^1 \times (0, \infty)$ ,  $\mathbf{u} \in C([0, T]; H^s(\mathbf{T}^1)) \cap L^2([0, T]; H^{s+2}(\mathbf{T}^1))$  for all  $T \in [0, T^*)$ ,  $\mathbf{u}$  satisfies (34)–(35). If  $\mathbf{u}_0 \in H^\infty(\mathbf{T}^1)^3 \equiv \bigcap_{s \in \mathbb{N}} H^s(\mathbf{T}^1)^3$ , then  $\mathbf{u}_\varepsilon \in C^\infty(\mathbf{T}^1 \times [0, T^*))^3$ . Moreover, the following alternative holds*

- (1)  $T^* = +\infty$ ,
- (2)  $\limsup_{t \rightarrow T^*} \|\mathbf{u}\|_{H^1(\mathbf{T}^1)} < \infty$ .

*Proof.* Defining

$$(40) \quad P_N g = \sum_{|n| \leq N} \hat{g}(n) e^{inx},$$

we consider the truncated equation

$$(41) \quad \begin{aligned} \mathbf{u}_t^N + \varepsilon \{ \mathbf{u}_{xxxx}^N + P_N(2[\mathbf{u}^N, \mathbf{u}_{xxx}^N] \mathbf{u}_x^N + [\mathbf{u}^N, \mathbf{u}_{xxxx}^N] \mathbf{u}^N) \} \\ = -\lambda P_N(\mathbf{u}^N \dot{\times} (\mathbf{u}^N \dot{\times} \mathbf{u}_{xx}^N)) + \mu P_N(\mathbf{u}^N \dot{\times} \mathbf{u}_{xx}^N), \quad (x, t) \in \mathbf{T}^1 \times [0, \infty) \end{aligned}$$

with

$$(42) \quad \mathbf{u}(x, 0) = P_N \mathbf{u}_0(x), \quad x \in \mathbf{T}^1.$$

Thanks to the Cauchy-Lipschitz theorem, there exists a unique solution of (41)–(42) defined on  $[0, T_N)$  for some  $T_N > 0$ .

Taking the inner product in  $L^2(\mathbf{T}^1)^3$  of (41) with  $D^{2r} \mathbf{u}^N$  ( $r \geq 0$ ), we obtain

$$\frac{1}{2} \frac{d}{dt} \|D^r \mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3}^2 + \varepsilon \|D^{r+2} \mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3}^2 = \mathbf{I}_1^r + \mathbf{I}_2^r$$

with

$$\mathbf{I}_1^r = -\varepsilon (2[\mathbf{u}^N, \mathbf{u}_{xxx}^N] \mathbf{u}_x^N + [\mathbf{u}^N, \mathbf{u}_{xxxx}^N] \mathbf{u}^N, D^{2r} \mathbf{u}^N)_{L^2(\mathbf{T}^1)^3}$$

and

$$\mathbf{I}_2^r = -\lambda (\mathbf{u}^N \dot{\times} (\mathbf{u}^N \dot{\times} \mathbf{u}_{xx}^N), D^{2r} \mathbf{u}^N)_{L^2(\mathbf{T}^1)^3} + \mu (\mathbf{u}^N \dot{\times} \mathbf{u}_{xx}^N, D^{2r} \mathbf{u}^N)_{L^2(\mathbf{T}^1)^3}.$$

We first estimate each term when  $r = 0, 1$ . By virtue of Lemma 2, integrating by parts, we have

$$\begin{aligned} \mathbf{I}_1^0 = & -\varepsilon \{ (2[\mathbf{u}^N, \mathbf{u}_{xxx}^N] \mathbf{u}_x^N, \mathbf{u}^N)_{L^2(\mathbf{T}^1)^3} + 4(e(\mathbf{u}^N)_x \mathbf{u}^N, \mathbf{u}_x^N)_{L^2(\mathbf{T}^1)^3} \\ & - ([\mathbf{u}_{xx}^N, \mathbf{u}_{xx}^N] \mathbf{u}^N, \mathbf{u}^N)_{L^2(\mathbf{T}^1)^3} \} \end{aligned}$$



Using the well-known Gagliardo-Nirenberg inequality, we have

$$\begin{aligned} |\mathbf{I}_1^0| &\leq C\varepsilon(\|\mathbf{u}^N\|_{L^\infty(\mathcal{T}^1)^3}^2\|\mathbf{u}_x^N\|_{L^2(\mathcal{T}^1)^3}\|\mathbf{u}_{xxx}^N\|_{L^2(\mathcal{T}^1)^3} + \|\mathbf{u}^N\|_{L^\infty(\mathcal{T}^1)^3}^2\|\mathbf{u}_{xx}^N\|_{L^2(\mathcal{T}^1)^3}^2) \\ &\leq \frac{1}{8}\varepsilon\|\mathbf{u}_{xxx}^N\|_{L^2(\mathcal{T}^1)^3}^2 + C(\varepsilon)\|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^6. \end{aligned}$$

where and in the sequel of this section  $C(\varepsilon)$  denotes various positive constants depending on  $\varepsilon$ . Analogously,

$$|\mathbf{I}_2^0| \leq \frac{1}{8}\varepsilon\|\mathbf{u}_{xxx}^N\|_{L^2(\mathcal{T}^1)^3}^2 + C(\varepsilon)(\|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^4 + \|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^6).$$

In much the same manner, when  $r = 1$ , we have

$$\begin{aligned} |\mathbf{I}_1^1| &\leq C\varepsilon(\|\mathbf{u}^N\|_{L^\infty(\mathcal{T}^1)^3}\|\mathbf{u}_x^N\|_{L^4(\mathcal{T}^1)^3}\|\mathbf{u}_{xx}^N\|_{L^4(\mathcal{T}^1)^3}\|\mathbf{u}_{xxx}^N\|_{L^2(\mathcal{T}^1)^3} \\ &\quad + \|\mathbf{u}\|_{L^\infty(\mathcal{T}^1)^3}\|\mathbf{u}_{xx}^N\|_{L^3(\mathcal{T}^1)^3}^3) \\ &\leq \frac{1}{8}\varepsilon\|\mathbf{u}_{xxx}^N\|_{L^2(\mathcal{T}^1)^3}^2 + C(\varepsilon)\|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^6 \end{aligned}$$

and

$$\begin{aligned} |\mathbf{I}_2^1| &\leq \lambda\|\mathbf{u}^N\|_{L^\infty(\mathcal{T}^1)^3}^2\|\mathbf{u}_{xx}^N\|_{L^2(\mathcal{T}^1)^3}^2 + |\mu|\|\mathbf{u}^N\|_{L^\infty(\mathcal{T}^1)^3}\|\mathbf{u}_{xx}^N\|_{L^2(\mathcal{T}^1)^3}^2 \\ &\leq \frac{1}{8}\varepsilon\|\mathbf{u}_{xxx}^N\|_{L^2(\mathcal{T}^1)^3}^2 + C(\varepsilon)(\|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^4 + \|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^6). \end{aligned}$$

Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^2 + \frac{1}{2}\varepsilon\|\mathbf{u}^N\|_{H^3(\mathcal{T}^1)^3}^2 \leq C(\varepsilon)(\|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^4 + \|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^6)$$

from which we conclude that there exists a time  $T^* > 0$ , depending only on the size of  $\|\mathbf{u}_0\|_{H^1(\mathcal{T}^1)}$ , such that for any  $T \in (0, T^*)$

$$(43) \quad \sup_{0 \leq t \leq T} \|\mathbf{u}^N\|_{H^1(\mathcal{T}^1)^3}^2 + \varepsilon \int_0^T \|\mathbf{u}(\mathbf{t})\|_{H^3(\mathcal{T}^1)^3}^2 dt \leq C(\|\mathbf{u}_0\|_{H^1(\mathcal{T}^1)^3}, T^*, \varepsilon)$$

where  $C(\|\mathbf{u}_0\|_{H^1(\mathcal{T}^1)}, T^*, \varepsilon)$  is a positive constant depending on the indicated variables, but not on  $N$ .

We next bound the terms when  $2 \leq r \leq s$ . By virtue of Lemma 3 and Lemma 4 we have

$$\begin{aligned}
|I_1^r| &\leq \varepsilon \|D_x^{r-2}(2[\mathbf{u}^N, \mathbf{u}_{xxx}^N])\mathbf{u}_x^N + [\mathbf{u}^N, \mathbf{u}_{xxx}^N]\mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3} \|D_x^{r+2}\mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3} \\
&\leq C\varepsilon \{ \|\mathbf{u}_x^N, \mathbf{u}_{xxx}^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}^N\|_{H^{r-2}(\mathbf{T}^1)^3} + \|\mathbf{u}_x^N, \mathbf{u}_{xxx}^N\|_{H^{r-2}(\mathbf{T}^1)^3} \|\mathbf{u}^N\|_{L^\infty(\mathbf{T}^1)^3} \\
&\quad + \|\mathbf{u}_{xx}^N, \mathbf{u}_{xx}^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}^N\|_{H^{r-2}(\mathbf{T}^1)^3} \\
&\quad + \|\mathbf{u}_{xx}^N, \mathbf{u}_{xx}^N\|_{H^{r-2}(\mathbf{T}^1)^3} \|\mathbf{u}^N\|_{L^\infty(\mathbf{T}^1)^3} \} \|D_x^{r+2}\mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3} \\
&\leq C\varepsilon \{ \|\mathbf{u}_x^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}_{xxx}^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}^N\|_{H^{r-2}(\mathbf{T}^1)^3} \\
&\quad + (\|\mathbf{u}_x^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}_{xxx}^N\|_{H^{r-2}(\mathbf{T}^1)^3} + \|\mathbf{u}_x^N\|_{H^{r-2}(\mathbf{T}^1)^3} \|\mathbf{u}_{xxx}^N\|_{L^\infty(\mathbf{T}^1)^3}) \|\mathbf{u}^N\|_{L^\infty(\mathbf{T}^1)^3} \\
&\quad + \|\mathbf{u}_{xx}^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}_{xx}^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}^N\|_{H^{r-2}(\mathbf{T}^1)^3} \\
&\quad + \|\mathbf{u}_{xx}^N\|_{L^\infty(\mathbf{T}^1)^3} \|\mathbf{u}_{xx}^N\|_{H^{r-2}(\mathbf{T}^1)^3} \|\mathbf{u}^N\|_{L^\infty(\mathbf{T}^1)^3} \} \|D_x^{r+2}\mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3} \\
&\leq \frac{1}{4} \varepsilon \|D_x^{r+2}\mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3}^2 + C(\varepsilon) \|\mathbf{u}^N\|_{H^1(\mathbf{T}^1)^3}^6.
\end{aligned}$$

In the same manner we get

$$|I_2^r| \leq \frac{1}{\varepsilon} \varepsilon \|D_x^{r+2}\mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3}^2 + C(\varepsilon) (\|\mathbf{u}^N\|_{H^1(\mathbf{T}^1)^3}^4 + \|\mathbf{u}^N\|_{H^1(\mathbf{T}^1)^3}^6).$$

Hence we obtain

$$\frac{1}{2} \frac{d}{dt} \|D_x^r \mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3}^2 + \frac{1}{2} \varepsilon \|D_x^{r+2}\mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3}^2 \leq C(\varepsilon) (\|\mathbf{u}^N\|_{H^1(\mathbf{T}^1)^3}^4 + \|\mathbf{u}^N\|_{H^1(\mathbf{T}^1)^3}^6).$$

from which we conclude that for any  $T \in (0, T^*)$

$$(44) \quad \sup_{0 \leq t \leq T} \|D_x^r \mathbf{u}^N\|_{L^2(\mathbf{T}^1)^3}^2 + \int_0^T \|D_x^{r+2}\mathbf{u}(t)^N\|_{L^2(\mathbf{T}^1)^3}^2 dt \leq C(\|\mathbf{u}_0\|_{H^r(\mathbf{T}^1)}, T^*, \varepsilon)$$

where  $C(\|\mathbf{u}_0\|_{H^r(\mathbf{T}^1)}, T^*)$  is a positive constant depending on the indicated variables, but not on  $N$ . In view of (41) we have

$$(45) \quad \|\mathbf{u}_t\|_{L^2(0, T; L^2(\mathbf{T}^1)^3)} \leq C(\|\mathbf{u}_0\|_{H^r(\mathbf{T}^1)}, T^*, \varepsilon).$$

From (43)–(45) we conclude that there exists a subsequence  $\{\mathbf{u}^{N_k}\}$  and a function  $\mathbf{u}$  such that

$$\begin{aligned}
\mathbf{u}^{N_k} &\rightarrow \mathbf{u} && \text{in } L^\infty(0, T; H^s(\mathbf{T}^1)^3) \text{ weakly star,} \\
\mathbf{u}^{N_k} &\rightarrow \mathbf{u} && \text{in } L^2(0, T; H^{s+2}(\mathbf{T}^1)^3) \text{ weakly,} \\
\mathbf{u}_t^{N_k} &\rightarrow \mathbf{u}_t && \text{in } L^2(0, T; L^2(\mathbf{T}^1)^3) \text{ weakly.}
\end{aligned}$$

Moreover, by well-known Aubin's compactness theorem we can assume that

$$\mathbf{u}^{N_k} \rightarrow \mathbf{u} \quad \text{in } L^2(0, T; H^{s+1}(\mathbf{T}^1)^3) \text{ strongly.}$$

Therefore we can take the limit in (41) and we see that  $\mathbf{u}$  satisfies

$$\begin{aligned} \mathbf{u}_t + \varepsilon(\mathbf{u}_{xxxx} + 2[\mathbf{u}, \mathbf{u}_{xxx}]\mathbf{u}_x + [\mathbf{u}, \mathbf{u}_{xxx}]\mathbf{u}) \\ = -\lambda\mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xx}) + \mu\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \quad (x, t) \in \mathbf{T}^1 \times [0, T^*) \end{aligned}$$

and the initial condition (35).

Uniqueness assertion is established by the standard argument. This completes the proof.

#### IV. Proof of Theorem 1

For any  $0 < \varepsilon \leq \varepsilon_0$  let  $\mathbf{u}_0^\varepsilon$  be such that  $\mathbf{u}_0^\varepsilon \in H^\infty(\mathbf{T}^1)^3$ ,  $\mathbf{u}_0^\varepsilon(x) \in \mathfrak{S}$ , ( $x \in \mathbf{T}^1$ ) and

$$(46) \quad \mathbf{u}_0^\varepsilon \rightarrow \mathbf{u}_0 \quad \text{in } H^1(\mathbf{T}^1)^3 \text{ strongly.}$$

Let  $\mathbf{u}^\varepsilon$  be the solution of (34) with initial condition

$$(47) \quad \mathbf{u}^\varepsilon(x, 0) = \mathbf{u}_0^\varepsilon(x), \quad x \in \mathbf{T}^1.$$

Theorem 3 yields that  $\mathbf{u}^\varepsilon \in C^\infty(\mathbf{T}^1 \times [0, T^*))^3$  and  $[\mathbf{u}^\varepsilon(x, t), \mathbf{u}^\varepsilon(x, t)] = -1$ ,  $\forall (x, t) \in \mathbf{T}^1 \times (0, \infty)$ . Therefore, we may use freely Lemma 2 and Corollary 1 to obtain higher order energy estimates.

Put

$$E_0 = \int_{\mathbf{T}^1} e(\mathbf{u}_0) dx.$$

**Lemma 6.** *There exists positive constants  $\hat{T} = \hat{T}(E_0)$  and  $C = C(E_0, \hat{T})$ , independent of  $\varepsilon$  such that*

$$(48) \quad 0 \leq \int_{\mathbf{T}^1} e(\mathbf{u}^\varepsilon(x, t)) dx dt \leq C(E_0), \quad \forall t \in [0, \hat{T}],$$

$$(49) \quad \varepsilon \int_0^{\hat{T}} \int_{\mathbf{T}^1} |e(\mathbf{u}^\varepsilon(x, t))_{,x}|^2 dx dt \leq C(E_0),$$

$$(50) \quad \varepsilon \int_0^{\hat{T}} \int_{\mathbf{T}^1} \frac{|u_{1xxx}^\varepsilon|^2 + |u_{2xxx}^\varepsilon|^2}{|u_3^\varepsilon|^2} dx dt \leq C(E_0).$$

*Proof.* For simplicity we abbreviate the subscript  $\varepsilon$  off and on. Using (34) (or equivalently, of (36)) we have

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \int_{T^1} e(\mathbf{u}^\varepsilon) dx &= \int_{T^1} [\mathbf{u}_x, \mathbf{u}_{xt}] dx = - \int_{T^1} [\mathbf{u}_t, \mathbf{u}_{xx}] dx \\
&= \varepsilon \int_{T^1} [\mathbf{u}_{xxxx}, \mathbf{u}_{xx}] dx + 2\varepsilon \int_{T^1} [\mathbf{u}, \mathbf{u}_{xxx}] [\mathbf{u}_{xx}, \mathbf{u}_x] dx \\
&\quad + \varepsilon \int_{T^1} [\mathbf{u}, \mathbf{u}_{xxx}] [\mathbf{u}, \mathbf{u}_{xx}] dx - \lambda \int_{T^1} [\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \mathbf{u} \dot{\times} \mathbf{u}_{xx}] dx \\
&= -\varepsilon \int_{T^1} [\mathbf{u}_{xxx}, \mathbf{u}_{xxx}] dx - \frac{2}{3} \varepsilon \int_{T^1} [\mathbf{u}, \mathbf{u}_{xxx}]^2 dx \\
&\quad + \varepsilon \int_{T^1} (2e(\mathbf{u})_{xx} e(\mathbf{u}) - e(\mathbf{u}) [\mathbf{u}_{xx}, \mathbf{u}_{xx}]) dx - \lambda \int_{T^1} [\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \mathbf{u} \dot{\times} \mathbf{u}_{xx}] dx \\
&= -\varepsilon \int_{T^1} [\mathbf{u}_{xxx}, \mathbf{u}_{xxx}] dx - \frac{7}{2} \varepsilon \int_{T^1} (e(\mathbf{u})_x)^2 dx - \varepsilon \int_{T^1} e(\mathbf{u}) [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx \\
&\quad - \lambda \int_{T^1} [\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \mathbf{u} \dot{\times} \mathbf{u}_{xx}] dx.
\end{aligned}$$

Here we have used (21)–(24). Noting  $[\mathbf{u}_{xxx}, \mathbf{u}_{xxx}] = [\mathbf{u} \dot{\times} \mathbf{u}_{xxx}, \mathbf{u} \dot{\times} \mathbf{u}_{xxx}] - [\mathbf{u}, \mathbf{u}_{xxx}]^2$  and applying (30) with  $\alpha = 1/3$ , we have

$$\begin{aligned}
(51) \quad &\frac{1}{2} \frac{d}{dt} \int_{T^1} e(\mathbf{u}) dx + \frac{1}{2} \varepsilon \int_{T^1} (e(\mathbf{u})_x)^2 dx + \frac{3}{4} \varepsilon \int_{T^1} \frac{(e(\mathbf{u})_x)^2}{u_3^2} dx \\
&\quad + \frac{1}{4} \varepsilon \int_{T^1} \frac{|u_{1xxx}|^2 + |u_{2xxx}|^2}{|u_3|^2} dx \\
&\quad + \varepsilon \int_{T^1} ([\mathbf{u}_{xx}, \mathbf{u}_{xx}] + e(\mathbf{u})^2) e(\mathbf{u}) dx + \lambda \int_{T^1} [\mathbf{u} \dot{\times} \mathbf{u}_{xx}, \mathbf{u} \dot{\times} \mathbf{u}_{xx}] dx \\
&\leq \varepsilon \int_{T^1} e(\mathbf{u})^3 dx.
\end{aligned}$$

By Lemma 3 we get

$$\int_{T^1} e(\mathbf{u})^3 dx \leq C \|e(\mathbf{u})_x\|_2^{2/3} \|e(\mathbf{u})\|_1^{5/3} \leq \frac{1}{4} \int_{T^1} (e(\mathbf{u})_x)^2 dx + C \left( \int_{T^1} e(\mathbf{u}) dx \right)^5.$$

Hence, from (51) we deduce that

$$\frac{d}{dt} \int_{T^1} e(\mathbf{u}) dx \leq C \left( \int_{T^1} e(\mathbf{u}) dx \right)^5$$

from which it follows that

$$\int_{T^1} e(\mathbf{u})dx \leq \frac{1}{((\int_{T^1} e(\mathbf{u}_0)dx)^{-4} - 4Ct)^{1/4}}.$$

Put  $\hat{T} = (\int_{T^1} e(\mathbf{u}_0)dx)^{-4}/8C$ . Then,

$$(52) \quad \int_{T^1} e(\mathbf{u}(x, t))dx \leq 2^{1/4} \int_{T^1} e(\mathbf{u}_0)dx, \quad 0 \leq t \leq \hat{T}.$$

From (51) and (52) we obtain (49) and (50). ■

**Lemma 7.** For any  $0 \leq \varepsilon \leq \varepsilon_0$

$$(53) \quad \|\mathbf{u}^\varepsilon\|_{L^\infty(T^1 \times [0, \hat{T}])^3} \leq C(\|u_{03}\|_{L^\infty(T^1)}, E_0, \varepsilon_0, \hat{T})$$

$$(54) \quad \|\mathbf{u}^\varepsilon\|_{L^\infty(0, \hat{T}; H^1(T^1)^3)} \leq C(\|u_{03}\|_{L^\infty(T^1)}, E_0, \varepsilon_0, \hat{T})$$

where  $C(\cdot, \cdot, \cdot)$  is a positive constant depending on the indicated variables, but, not depending on  $\varepsilon$ .

*Proof.* Estimate (48) and (29) yield

$$(55) \quad \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \frac{|u_{1x}|^2 + |u_{2x}|^2}{|u_3|^2} dx \leq C(E_0).$$

Then, we get

$$(56) \quad \begin{aligned} \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \frac{|u_{3x}|^2}{|u_3|^2} dx &\leq \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \frac{1}{u_3^2} \left( \frac{u_1 u_{1x} + u_2 u_{2x}}{u_3} \right)^2 dx \\ &\leq 2 \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \frac{(u_1^2 + u_2^2)|u_{1x}|^2 + |u_{2x}|^2}{|u_3|^4} dx \\ &\leq 4 \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \frac{|u_{1x}|^2 + |u_{2x}|^2}{|u_3|^2} dx \leq C(E_0). \end{aligned}$$

since  $u_3 \geq 1$ . Analogously, from (49) and (49) we have

$$(57) \quad \begin{aligned} \varepsilon \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \frac{|u_{3xxx}|^2}{|u_3|^2} dx &\leq \varepsilon \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \frac{1}{|u_3|^2} \frac{|u_1 u_{1xxx} + u_2 u_{2xxx} + \frac{3}{2} e(\mathbf{u})_x|^2}{|u_3|^2} dx \\ &\leq C\varepsilon \sup_{0 \leq t \leq \hat{T}} \int_{T^1} \left( \frac{|u_{1xxx}|^2 + |u_{2xxx}|^2}{|u_3|^2} + \frac{|(e(\mathbf{u}))_x|^2}{|u_3|^4} \right) dx \\ &\leq C(E_0). \end{aligned}$$

From (34) or (36) we see that

$$u_{3t} = -\varepsilon(u_{3xxxx} + 2[\mathbf{u}, \mathbf{u}_{xxx}]u_{3x} + [\mathbf{u}, \mathbf{u}_{xxxx}]u_3) \\ + \lambda(u_{3xx} - e(\mathbf{u})u_3) + \mu(u_2u_{1xx} - u_1u_{2xx}).$$

Hence,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{T^1} \log(|u_3|^2) dx &= \int_{T^1} \frac{u_{3t}}{u_3} dx \\ &= -\varepsilon \int_{T^1} \frac{u_{3x}u_{3xxx}}{u_3^2} dx + 3\varepsilon \int_{T^1} \frac{u_{3x}e(\mathbf{u})_x}{u_3} dx \\ &\quad - \varepsilon \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx + \lambda \int_{T^1} \frac{|u_{3x}|^2}{|u_3|^2} dx - \lambda \int_{T^1} e(\mathbf{u}) dx \\ &\quad - \mu \int_{T^1} \left( \frac{u_2u_{3x}u_{1x}}{|u_3|^2} + \frac{u_1u_{2x}u_{3x}}{|u_3|^2} \right) dx \\ &= \sum_{k=1}^6 I_k. \end{aligned}$$

We estimate

$$\begin{aligned} I_1 &= -\varepsilon \int_{T^1} \frac{u_{3x}u_{3xxx}}{|u_3|^2} dx \leq 2\varepsilon \int_{T^1} \frac{|u_{3x}|^2}{|u_3|^2} dx + 2\varepsilon \int_{T^1} \frac{|u_{3xxx}|^2}{|u_3|^2} dx \\ &\leq C(E_0) + 2\varepsilon \int_{T^1} \frac{|u_{3xxx}|^2}{|u_3|^2} dx; \\ I_2 &= 3\varepsilon \int_{T^1} \frac{u_{3x}e(\mathbf{u})_x}{|u_3|} dx \leq 6\varepsilon \int_{T^1} \frac{|u_{3x}|^2}{|u_3|^2} dx + 6\varepsilon \int_{T^1} |e(\mathbf{u})_x|^2 dx \\ &\leq C(E_0) + 6\varepsilon \int_{T^1} |e(\mathbf{u})_x|^2 dx; \\ I_3 &= -\varepsilon \int_{T^1} ([\mathbf{u}_{xx}, \mathbf{u}_{xx}] + e(\mathbf{u})^2) dx + \varepsilon \int_{T^1} e(\mathbf{u})^2 dx \\ &\leq \varepsilon C \int_{T^1} |e(\mathbf{u})_x|^2 dx + \varepsilon_0 C(E_0); \\ I_4 &= \lambda \int_{T^1} \frac{|u_{3x}|^2}{|u_3|^2} dx \leq C(E_0); \\ I_5 &= -\lambda \int_{T^1} e(\mathbf{u}) dx \leq 0; \end{aligned}$$

$$\begin{aligned}
 I_6 &= \mu \int_{T^1} \left( \frac{u_2 u_{3x} u_{1x}}{|u_3|^2} - \frac{u_1 u_{2x} u_{3x}}{|u_3|^2} \right) dx = \mu \int_{T^1} \left( \frac{u_2 u_{1x} - u_1 u_{2x}}{|u_3|^2} \right) \left( \frac{u_1 u_{1x} + u_2 u_{2x}}{u_3} \right) dx \\
 &\leq \frac{\mu}{2} \left( \int_{T^1} \left( \frac{|u_2 u_{1x} - u_1 u_{2x}|^2}{|u_3|^2} dx + 2 \int_{T^1} \frac{|u_1|^2 |u_{1x}|^2 + |u_2|^2 |u_{2x}|^2}{|u_3|^4} dx \right) \right) \\
 &\leq C(E_0) + C \int_{T^1} \frac{|u_{1x}|^2 + |u_{2x}|^2}{|u_3|^2} dx \leq C(E_0).
 \end{aligned}$$

Hence,

$$\begin{aligned}
 \int_{T^1} \log(|u_3|^2(x, t)) dx &\leq \int_{T^1} \log(|u_{03}|^2(x)) dx + C\varepsilon \int_0^{\hat{T}} \int_{T^1} \frac{|u_{3xxx}|^2}{|u_3|^2} dx dt \\
 &\quad + C\varepsilon \int_0^{\hat{T}} \int_{T^1} |e(\mathbf{u})_x|^2 dx dt + C(E_0)\hat{T}.
 \end{aligned}$$

Putting  $w^\varepsilon(x, t) = \log(|u_3|^2(x, t))$ , we have

$$(58) \quad \|w^\varepsilon\|_{L^\infty(0, T; L^1(T^1))} \leq C(\|u_{03}\|_{L^\infty(T^1)}, E_0, \hat{T}).$$

Since

$$\int_{T^1} |w_x^\varepsilon(x, t)|^2 dx = \int_{T^1} \frac{|u_{3x}|^2}{|u_3|^2} dx \leq C(E_0),$$

we have

$$\|w_x^\varepsilon\|_{L^\infty(0, \hat{T}; L^2(T^1))} \leq C(E_0).$$

By the Gagliardo-Nirenberg inequality, we get

$$\|w^\varepsilon\|_{L^\infty(T^1)} \leq C \|w_x^\varepsilon\|_{L^2(T^1)}^{2/3} \|w^\varepsilon\|_{L^1(T^1)}^{1/3}$$

from which it follows that

$$(59) \quad \|u_3^\varepsilon\|_{L^\infty(T^1 \times [0, \hat{T}])} \leq C(\|u_{03}\|_{L^\infty(T^1)}, E_0, \hat{T}).$$

Hence, we have (53). Combining estimates (48), (55), (56) and (59), we obtain

$$\|\mathbf{u}^\varepsilon\|_{L^\infty(0, \hat{T}; H^1(T^1)^3)} \leq C(\|u_{03}\|_{L^\infty(T^1)^3}, E_0, \hat{T}, \varepsilon_0). \quad \blacksquare$$

**Lemma 8.** *For  $0 \leq \varepsilon \leq \varepsilon_0$ , there exists a positive constant  $C$ , independent of  $\varepsilon$ , such that*

$$(60) \quad \varepsilon^{1/2} \|\mathbf{u}_{xxx}^\varepsilon\|_{L^2(0, \hat{T}; L^2(T^1)^3)} \leq C,$$

$$(61) \quad \varepsilon^{1/2} \|[\mathbf{u}^\varepsilon, \mathbf{u}_{xxx}^\varepsilon] \mathbf{u}_x^\varepsilon\|_{L^2(0, \hat{T}; L^1(T^1)^3)} \leq C$$

and

$$(62) \quad \varepsilon^{1/2} \| [\mathbf{u}^\varepsilon, \mathbf{u}_{xxxx}^\varepsilon] \mathbf{u}^\varepsilon \|_{L^2(0, \hat{T}; L^1(\mathbf{T}^1)^3)} \leq C,$$

$$(63) \quad \| \mathbf{u}_t^\varepsilon \|_{L^2(0, \hat{T}; H^{-1}(\mathbf{T}^1)^3)} \leq C.$$

*Proof.* From (50), (54) and (56) we obtain (60). Then,

$$(64) \quad \varepsilon^{1/2} \| \mathbf{u}_{xxxx}^\varepsilon \|_{L^2(0, \hat{T}; H^{-1}(\mathbf{T}^1)^3)} \leq C.$$

We also have

$$(65) \quad \begin{aligned} & \varepsilon^{1/2} \| [\mathbf{u}^\varepsilon, \mathbf{u}_{xxx}^\varepsilon] \mathbf{u}_x^\varepsilon \|_{L^2(0, \hat{T}; L^1(\mathbf{T}^1)^3)} \\ & \leq C \varepsilon^{1/2} \| e(\mathbf{u}^\varepsilon)_x \|_{L^2(0, \hat{T}; L^2(\mathbf{T}^1)^3)} \| \mathbf{u}^\varepsilon \|_{L^\infty(0, \hat{T}; H^1(\mathbf{T}^1)^3)} \\ & \leq C \end{aligned}$$

and

$$(66) \quad \begin{aligned} & \varepsilon^{1/2} \| [\mathbf{u}^\varepsilon, \mathbf{u}_{xxxx}^\varepsilon] \mathbf{u}^\varepsilon \|_{L^2(0, \hat{T}; L^1(\mathbf{T}^1)^3)} \\ & \leq C \varepsilon^{1/2} \| \mathbf{u}_{xxx}^\varepsilon \|_{L^2(0, \hat{T}; L^2(\mathbf{T}^1)^3)} \| \mathbf{u}_x^\varepsilon \|_{L^\infty(0, \hat{T}; H^1(\mathbf{T}^1)^3)} \\ & \leq C. \end{aligned}$$

Noting  $L^1(\mathbf{T}^1) \subset H^{-1}(\mathbf{T}^1)$  and applying all the estimates obtained above to (36), we get (63).  $\blacksquare$

We now take the limit as  $\varepsilon$  goes to zero. From (54) and (63) we see that  $\{\mathbf{u}^\varepsilon\}$  is bounded in  $L^\infty(0, \hat{T}; H^1(\mathbf{T}^1)^3)$  and  $\{\mathbf{u}_t^\varepsilon\}$  is bounded in  $L^2(0, \hat{T}; H^{-1}(\mathbf{T}^1)^3)$ . Hence, up to subsequences, we deduce that

$$(67) \quad \mathbf{u}^\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } L^\infty(0, \hat{T}; H^1(\mathbf{T}^1)^3) \text{ weakly star}$$

and

$$(68) \quad \mathbf{u}_t^\varepsilon \rightharpoonup \mathbf{u} \quad \text{in } L^2(0, \hat{T}; H^{-1}(\mathbf{T}^1)^3) \text{ weakly.}$$

Moreover, from (60)–(62), we see that

$$(69) \quad \varepsilon \mathbf{u}_{xxxx}^\varepsilon \rightarrow 0 \quad \text{in } L^2(0, \hat{T}; H^{-1}(\mathbf{T}^1)^3),$$

$$(70) \quad \varepsilon [\mathbf{u}^\varepsilon, \mathbf{u}_{xxx}^\varepsilon] \mathbf{u}_x^\varepsilon \rightarrow 0 \quad \text{in } L^2(0, \hat{T}; L^1(\mathbf{T}^1)^3),$$

and

$$(71) \quad \varepsilon [\mathbf{u}^\varepsilon, \mathbf{u}_{xxxx}^\varepsilon] \mathbf{u}^\varepsilon \rightarrow 0 \quad \text{in } L^2(0, \hat{T}; L^1(\mathbf{T}^1)^3).$$



Since the embedding from  $H^1(\mathbf{T}^1)$  into  $L^p(\mathbf{T}^1)$  is compact for any  $p \geq 1$ , by Aubin's compactness theorem we can assume that

$$(72) \quad \mathbf{u}^\varepsilon \rightarrow \mathbf{u} \quad \text{in } L^r(0, \hat{T}; L^p(\mathbf{T}^1)^3) \text{ strongly and a.e. in } \mathbf{T}^1 \times [0, \hat{T}]$$

where  $1 < p, r < \infty$ . Hence, for any  $\varphi \in L^2(0, \hat{T}; H^1(\mathbf{T}^1)^3)$  we have

$$(73) \quad \begin{aligned} & \int_0^{\hat{T}} \int_{\mathbf{T}^1} \langle \mathbf{u}^\varepsilon \times \mathbf{u}_{xx}^\varepsilon, \varphi \rangle dx dt \\ &= \int_0^{\hat{T}} \int_{\mathbf{T}^1} \langle (\mathbf{u}^\varepsilon \times \mathbf{u}_x^\varepsilon)_x, \varphi \rangle dx dt \\ &= - \int_0^{\hat{T}} \int_{\mathbf{T}^1} \langle \mathbf{u}^\varepsilon \times \mathbf{u}_x^\varepsilon, \varphi_x \rangle dx dt \rightarrow - \int_0^{\hat{T}} \int_{\mathbf{T}^1} \langle \mathbf{u} \times \mathbf{u}_x, \varphi_x \rangle dx dt. \end{aligned}$$

When  $\lambda > 0$ , from (51) we infer that  $\{\mathbf{u}^\varepsilon \times \mathbf{u}_{xx}^\varepsilon\}$  is bounded in  $L^2(0, \hat{T}; L^2(\mathbf{T}^1)^3)$ . Hence, we can assume that

$$(74) \quad \mathbf{u}^\varepsilon \times \mathbf{u}_{xx}^\varepsilon \rightarrow (\mathbf{u} \times \mathbf{u}_x)_x \quad \text{in } L^2(0, \hat{T}; L^2(\mathbf{T}^1)^3) \text{ weakly}$$

from which we conclude that

$$(75) \quad \mathbf{u}^\varepsilon \times \mathbf{u}^\varepsilon \times \mathbf{u}_{xx}^\varepsilon \rightarrow \mathbf{u} \times (\mathbf{u} \times \mathbf{u}_x)_x \quad \text{in } L^1(0, \hat{T}; L^1(\mathbf{T}^1)^3) \text{ weakly.}$$

Hence, considering (67)–(75), we can pass to the limit in each terms of the equation (34) to find that  $\mathbf{u}$  satisfies (12) with  $T = \hat{T}$ . Thus we have the local existence of weak solutions to the problem (8)–(9).

We now discuss the continuation of solutions in the large. Firstly, from (51) we have

$$\begin{aligned} \int_{\mathbf{T}^1} e(\mathbf{u}^\varepsilon(x, t)) dx &\leq \int_{\mathbf{T}^1} e(\mathbf{u}_0^\varepsilon(x)) dx + 2\varepsilon \int_0^{\hat{T}} \int_{\mathbf{T}^1} e(\mathbf{u}^\varepsilon)^3(x, t) dx dt \\ &\leq \int_{\mathbf{T}^1} e(\mathbf{u}_0^\varepsilon(x)) dx + C\varepsilon \int_0^{\hat{T}} \|e(\mathbf{u}^\varepsilon)_x\|_{L^2(\mathbf{T}^1)^3}^{4/3} \|e(\mathbf{u}^\varepsilon)\|_{L^1(\mathbf{T}^1)^3}^{5/3} dt \\ &\leq \int_{\mathbf{T}^1} e(\mathbf{u}_0^\varepsilon(x)) dx \\ &\quad + C\varepsilon^{1/3} \left( \varepsilon \int_0^{\hat{T}} \|e(\mathbf{u}^\varepsilon)_x\|_{L^2(\mathbf{T}^1)^3}^2 dt \right)^{2/3} \|e(\mathbf{u}^\varepsilon)\|_{L^\infty(0, \hat{T}; L^1(\mathbf{T}^1)^3)}^{5/3} \hat{T}^{1/3}. \end{aligned}$$

Hence, letting  $\theta(t) \in C([0, \hat{T}])$ ,  $\theta \geq 0$ , we have

$$\liminf_{\varepsilon \rightarrow 0} \int_0^{\hat{T}} \int_{T^1} e(\mathbf{u}^\varepsilon(x, t)) dx \theta(t) dt \leq \int_0^{\hat{T}} \int_{T^1} e(\mathbf{u}_0(x)) dx \theta(t) dt.$$

Note that

$$e(\mathbf{u}^\varepsilon) = \left| \frac{u_{1x}^\varepsilon}{u_3^\varepsilon} \right|^2 + \left| \frac{u_{2x}^\varepsilon}{u_3^\varepsilon} \right|^2 + \left| \frac{u_2^\varepsilon u_{1x}^\varepsilon - u_1^\varepsilon u_{2x}^\varepsilon}{u_3^\varepsilon} \right|^2.$$

Since  $\{u_{1x}^\varepsilon/u_3^\varepsilon\}$  is bounded in  $L^2(T^1 \times [0, \hat{T}])$ , up to the subsequences, we deduce that

$$\frac{u_{1x}^\varepsilon}{u_3^\varepsilon} \rightarrow v \quad \text{in } L^2(T^1 \times [0, \hat{T}]) \text{ weakly.}$$

Noting  $1 \leq u_3^\varepsilon \leq C$ , from (67) and (72) we may assume that

$$\frac{u_{1x}^\varepsilon}{u_3^\varepsilon} \rightarrow \frac{u_{1x}}{u_3} \quad \text{in } L^{4/3}(T^1 \times [0, \hat{T}]) \text{ weakly}$$

Hence  $v = u_{1x}/u_3$ . Analogously we have, up to the subsequences,

$$\frac{u_{2x}^\varepsilon}{u_3^\varepsilon} \rightarrow \frac{u_{2x}}{u_3} \quad \text{in } L^2(T^1 \times [0, \hat{T}]) \text{ weakly}$$

and

$$\frac{u_2^\varepsilon u_{1x}^\varepsilon - u_1^\varepsilon u_{2x}^\varepsilon}{u_3^\varepsilon} \rightarrow \frac{u_2 u_{1x} - u_1 u_{2x}}{u_3} \quad \text{in } L^2(T^1 \times [0, \hat{T}]) \text{ weakly.}$$

Lower semicontinuity of the  $L^2$ -norm for the weak topology yields that

$$\begin{aligned} & \int_0^{\hat{T}} \int_{T^1} e(\mathbf{u}(x, t)) dx \theta(t) dt \\ & \leq \liminf_{\varepsilon \rightarrow 0} \int_0^{\hat{T}} \int_{T^1} e(\mathbf{u}^\varepsilon(x, t)) dx \theta(t) dt \leq \int_0^{\hat{T}} \int_{T^1} e(\mathbf{u}_0(x)) dx \theta(t) dt. \end{aligned}$$

Since  $\int_{T^1} e(\mathbf{u}(x, t)) dx \in L^\infty(0, \hat{T})$  and  $C([0, \hat{T}])$  is dense in  $L^1(0, \hat{T})$ , we have

$$(76) \quad \int_{T^1} e(\mathbf{u}(x, t)) dx \leq \int_{T^1} e(\mathbf{u}_0(x)) dx \quad \forall t \in [0, \hat{T}].$$

Let  $T^* > 0$  be the supremum of  $T$  such that on  $[0, T]$  the Cauchy problem (8)–(9) with (13) has a weak solution  $\mathbf{u}$  mentioned in Theorem 1. Clearly  $T^* > \hat{T}$ . Suppose that  $T^* < +\infty$ . Let  $\delta$  be arbitrary small fixed number so that  $\hat{T}/2 > \delta$ .

Let  $\mathbf{v}_0^\varepsilon$  be such that  $\mathbf{v}_0^\varepsilon \in H^\infty(\mathbf{T}^1)^3$ ,  $\mathbf{v}_0^\varepsilon(x) \in \mathfrak{S}$  ( $x \in \mathbf{T}^1$ ) and

$$\mathbf{v}_0^\varepsilon \rightarrow \mathbf{u}(\cdot, T^* - \delta) \quad \text{in } H^1(\mathbf{T}^1)^3 \text{ strongly.}$$

Let  $\mathbf{v}^\varepsilon$  be the solution of (34) with initial condition

$$\mathbf{v}^\varepsilon(x, 0) = \mathbf{v}_0^\varepsilon(x), \quad x \in \mathbf{T}^1.$$

From (13) we may assume that

$$\int_{\mathbf{T}^1} e(\mathbf{v}_0^\varepsilon(x)) dx \leq \int_{\mathbf{T}^1} e(\mathbf{u}_0(x)) dx.$$

Then, the above mentioned argument on local existence showed that there exists a subsequence of  $\{\mathbf{v}^\varepsilon\}$  whose limit  $\mathbf{v}$  is a weak solution to the problem (8)–(9) with (13). The existence time of  $\mathbf{v}$  is not less than  $\hat{T}$  and independent of  $\delta$ . Set

$$\mathbf{u}^*(x, t) = \begin{cases} \mathbf{u}(x, t), & \text{for } 0 \leq t \leq T^* - \delta, \\ \mathbf{v}(x, t - T^* + \delta), & \text{for } T^* - \delta \leq t \leq T^* - \delta + \hat{T} \end{cases}$$

Then,  $\mathbf{u}^*$  is a weak solution of (8)–(9) with (13) on  $[0, T^* - \delta + \hat{T}]$ , which contradicts the definition of  $T^*$ . Hence  $T^* = +\infty$ . This completes the proof.

### V. Proof of Theorem 2

As in Section 1, we construct the strong solution of (8)–(9) with (13) as limits of the sequence  $\{\mathbf{u}^\varepsilon\}$  satisfying

$$(77) \quad \mathbf{u}_t + \varepsilon(\mathbf{u}_{xxxx} + 2[\mathbf{u}, \mathbf{u}_{xxx}]\mathbf{u}_x + [\mathbf{u}, \mathbf{u}_{xxx}]\mathbf{u}) = -\lambda\mathbf{u} \times (\mathbf{u} \times \mathbf{u}_{xx}) + \mu\mathbf{u} \times \mathbf{u}_{xx},$$

$$(x, t) \in \mathbf{T}^1 \times [0, T^*]$$

$$(78) \quad \mathbf{u}^\varepsilon(x, 0) = \mathbf{u}_0^\varepsilon(x), \quad x \in \mathbf{T}^1$$

when  $\varepsilon$  tends to zero. Below, for simplicity we again abbreviate the subscript  $\varepsilon$  off and on.

#### Lemma 9.

$$(79) \quad \int_{\mathbf{T}^1} [\mathbf{u}_{xxx}, \mathbf{u}_x \times \mathbf{u}_{xx}] dx = \frac{5}{2} \int_{\mathbf{T}^1} e(\mathbf{u})[\mathbf{u}_x, \mathbf{u} \times \mathbf{u}_{xxx}] dx.$$

*Proof.* Note that if  $[\mathbf{u}, \mathbf{u}] = -1$ , then

$$\mathbf{u}_{xxx} = -\mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xxx}) - [\mathbf{u}, \mathbf{u}_{xxx}]\mathbf{u}.$$

Using this and integrating by parts, we obtain

$$\begin{aligned} & \int_{T^1} [\mathbf{u}_{xxx}, \mathbf{u}_x \dot{\times} \mathbf{u}_{xx}] dx \\ &= - \int_{T^1} [\mathbf{u} \dot{\times} (\mathbf{u} \dot{\times} \mathbf{u}_{xxx}) + [\mathbf{u}, \mathbf{u}_{xxx}]\mathbf{u}, \mathbf{u}_x \dot{\times} \mathbf{u}_{xx}] dx \\ &= - \int_{T^1} ([\mathbf{u}, \mathbf{u}_{xx}][\mathbf{u} \dot{\times} \mathbf{u}_{xxx}, \mathbf{u}_x] - [\mathbf{u}, \mathbf{u}_x][\mathbf{u} \dot{\times} \mathbf{u}_{xxx}, \mathbf{u}_{xx}]) dx \\ &\quad + \frac{3}{2} \int_{T^1} e(\mathbf{u})_x [\mathbf{u}, \mathbf{u}_x \dot{\times} \mathbf{u}_{xx}] dx \\ &= \frac{5}{2} \int_{T^1} e(\mathbf{u}) [\mathbf{u}_x, \mathbf{u} \dot{\times} \mathbf{u}_{xxx}] dx. \end{aligned} \quad \blacksquare$$

**Lemma 10.**

$$\begin{aligned} (80) \quad & \frac{d}{dt} \left( \frac{1}{4} \int_{T^1} e(\mathbf{u})^2 dx + \frac{1}{5} \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx \right) \\ &= -\frac{17}{2} \varepsilon \int_{T^1} (e(\mathbf{u})_x)^2 e(\mathbf{u}) dx - \frac{11}{10} \varepsilon \int_{T^1} (e(\mathbf{u})_{xx})^2 dx \\ &\quad + \frac{6}{5} \varepsilon \int_{T^1} e(\mathbf{u})_{xx} [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx - \varepsilon \int_{T^1} e(\mathbf{u})^2 [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx \\ &\quad - \varepsilon \int_{T^1} e(\mathbf{u}) [\mathbf{u}_{xxx}, \mathbf{u}_{xxx}] dx - \frac{2}{5} \varepsilon \int_{T^1} [\mathbf{u}_{xxx}, \mathbf{u}_{xxx}] dx \\ &\quad - \frac{2}{5} \varepsilon \int_{T^1} [\mathbf{u}, \mathbf{u}_{xxx}]^2 dx - \frac{13}{10} \lambda \int_{T^1} (e(\mathbf{u})_x)^2 dx - \frac{2}{5} \lambda \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx \\ &\quad - \frac{7}{5} \lambda \int_{T^1} e(\mathbf{u}) [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx - \lambda \int_{T^1} e(\mathbf{u})^3 dx. \end{aligned}$$

*Proof.* Use Lemma 9. Simple but tedious calculation gives (80). \blacksquare

Put

$$\begin{aligned} (81) \quad I &= \frac{6}{5} \varepsilon \int_{T^1} e(\mathbf{u})_{xx} [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx - \frac{2}{5} \varepsilon \int_{T^1} [\mathbf{u}_{xxx}, \mathbf{u}_{xxx}] dx \\ &\quad - \frac{2}{5} \varepsilon \int_{T^1} [\mathbf{u}, \mathbf{u}_{xxx}]^2 dx. \end{aligned}$$

Observing that

$$e(\mathbf{u})_{xx}[\mathbf{u}_{xx}, \mathbf{u}_{xx}] = (e(\mathbf{u})_{xx})^2 + \frac{1}{4}[\mathbf{u}_{xx}, \mathbf{u}_{xx}]^2 - \frac{1}{4}[\mathbf{u}, \mathbf{u}_{xxxx}]^2,$$

we have

$$\begin{aligned} (82) \quad I &= \left(\frac{6}{5} - \gamma\right) \varepsilon \int_{T^1} e(\mathbf{u})_{xx}[\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx \\ &+ \gamma \varepsilon \int_{T^1} \left( (e(\mathbf{u})_{xx})^2 + \frac{1}{4}[\mathbf{u}_{xx}, \mathbf{u}_{xx}]^2 - \frac{1}{4}[\mathbf{u}, \mathbf{u}_{xxxx}]^2 \right) dx \\ &- \frac{2}{5} \varepsilon \int_{T^1} [\mathbf{u}_{xxxx}, \mathbf{u}_{xxxx}] dx - \frac{2}{5} \varepsilon \int_{T^1} [\mathbf{u}, \mathbf{u}_{xxxx}]^2 dx \\ &\leq \left( \left( \frac{6}{5} - \gamma \right) \eta + \gamma \right) \varepsilon \int_{T^1} (e(\mathbf{u})_{xx})^2 dx + \left( C(\gamma, \eta) + \frac{\gamma}{4} \right) \varepsilon \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}]^2 dx \\ &- \frac{2}{5} \varepsilon \int_{T^1} \left( [\mathbf{u}_{xxxx}, \mathbf{u}_{xxxx}] + \left( 1 + \frac{5}{8} \gamma \right) [\mathbf{u}, \mathbf{u}_{xxxx}] \right) dx \end{aligned}$$

where  $\gamma$  and  $\eta$  are arbitrary positive constants. Since

$$u_{3xx} = \frac{1}{u_3} (u_1 u_{1xx} + u_2 u_{2xx} + e(\mathbf{u})),$$

by virtue of Lemma 7 and the fact that  $u_3 \geq 1$ , we have

$$\begin{aligned} (83) \quad \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}]^2 dx &\leq C \int_{T^1} (|u_{1xx}|^4 + |u_{2xx}|^4 + |u_{1x}|^4 + |u_{2x}|^4) dx \\ &\leq \mu \int_{T^1} (|u_{1xxxx}|^2 + |u_{2xxxx}|^2) dx + C(\mu, \|u_{03}\|_\infty, E_0, \hat{T}, \varepsilon_0) \end{aligned}$$

for any  $\mu \geq 0$ . In (82) we take  $\gamma$  and  $\eta$  so as to  $(6/5 - \gamma)\eta + \gamma = 1/10$ . Then, Lemma 3 and Lemma 8 give

$$\begin{aligned} (84) \quad I &\leq \frac{1}{10} \varepsilon \int_{T^1} (e(\mathbf{u})_{xx})^2 dx + C_1 \varepsilon \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}]^2 dx \\ &- C_2 \varepsilon \int_{T^1} (|u_{1xxxx}|^2 + |u_{2xxxx}|^2) dx \\ &\leq \frac{1}{10} \varepsilon \int_{T^1} (e(\mathbf{u})_{xx})^2 dx - C_3 \varepsilon \int_{T^1} (|u_{1xxxx}|^2 + |u_{2xxxx}|^2) dx + C_4 \end{aligned}$$

where  $C_i$  ( $i = 1, 2, 3$ ) denote positive constants independent of  $\varepsilon$ . From (80) and (84) it follows that

$$\begin{aligned}
& \frac{d}{dt} \left( \frac{1}{4} \int_{T^1} e(\mathbf{u})^2 dx + \frac{1}{5} \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx \right) \\
& \leq -\frac{17}{2} \varepsilon \int_{T^1} (e(\mathbf{u})_x)^2 e(\mathbf{u}) dx - \varepsilon \int_{T^1} (e(\mathbf{u})_{xx})^2 dx \\
& \quad - C_3 \varepsilon \int_{T^1} (|u_{1xxx}|^2 + |u_{2xxx}|^2) dx + C_4 \varepsilon_0 - \varepsilon \int_{T^1} e(\mathbf{u})^2 ([\mathbf{u}_{xx}, \mathbf{u}_{xx}] + e(\mathbf{u})^2) dx \\
& \quad + \varepsilon \int_{T^1} e(\mathbf{u})^4 dx - \varepsilon \int_{T^1} e(\mathbf{u}) ([\mathbf{u}_{xxx}, \mathbf{u}_{xxx}] + [\mathbf{u}, \mathbf{u}_{xxx}]^2) dx \\
& \quad + \frac{9}{4} \varepsilon \int_{T^1} (e(\mathbf{u})_x)^2 e(\mathbf{u}) dx - \frac{13}{10} \lambda \int_{T^1} (e(\mathbf{u})_x)^2 dx - \frac{2}{5} \lambda \int_{T^1} ([\mathbf{u}_{xx}, \mathbf{u}_{xx}] + e(\mathbf{u})^2) dx \\
& \quad - \frac{3}{5} \lambda \int_{T^1} e(\mathbf{u})^2 dx - \frac{7}{5} \lambda \int_{T^1} e(\mathbf{u}) ([\mathbf{u}_{xx}, \mathbf{u}_{xx}] + e(\mathbf{u})^2) dx + \lambda \int_{T^1} e(\mathbf{u})^3 dx.
\end{aligned}$$

The Gagliardo-Nirenberg inequality yields that for  $p \geq 1$

$$\int_{T^1} e(\mathbf{u})^p dx \leq C \|e(\mathbf{u})_{xx}\|_{L^2(T^1)^3}^{2(p-1)/5} \|e(\mathbf{u})\|_{L^1(T^1)^3}^{(3p+2)/5}.$$

Noting that  $2(p-1)/5 < 2$  for  $p < 6$ , we obtain

$$\begin{aligned}
& \frac{d}{dt} \left( \frac{1}{4} \int_{T^1} e(\mathbf{u})^2 dx + \frac{1}{5} \int_{T^1} [\mathbf{u}_{xx}, \mathbf{u}_{xx}] dx \right) \\
& \leq -\frac{25}{4} \varepsilon \int_{T^1} (e(\mathbf{u})_x)^2 e(\mathbf{u}) dx - \frac{1}{2} \varepsilon \int_{T^1} (e(\mathbf{u})_{xx})^2 dx \\
& \quad - C_3 \varepsilon \int_{T^1} (|u_{1xxx}|^2 + |u_{2xxx}|^2) dx + C \varepsilon_0 - \lambda \int_{T^1} (e(\mathbf{u})_x)^2 dx + C \lambda
\end{aligned}$$

from which it follows that for  $0 \leq t \leq \hat{T}$

$$\begin{aligned}
& \frac{1}{4} \int_{T^1} e(\mathbf{u}^\varepsilon(x, t))^2 dx + \frac{1}{5} \int_{T^1} [\mathbf{u}_{xx}^\varepsilon(x, t), \mathbf{u}_{xx}^\varepsilon(x, t)] dx \\
& \leq \frac{1}{4} \int_{T^1} e(\mathbf{u}_0^\varepsilon)^2 dx + \frac{1}{5} \int_{T^1} [\mathbf{u}_{0xx}^\varepsilon, \mathbf{u}_{0xx}^\varepsilon] dx + C \hat{T} \leq C (\|\mathbf{u}_0\|_{H^2(T^1)^3}, \hat{T}).
\end{aligned}$$

Hence, Lemma 3 gives

$$\sup_{0 \leq t \leq \hat{T}} \int_{T^1} e(\mathbf{u}^\varepsilon(x, t))^2 dx \leq C (\|\mathbf{u}_0\|_{H^2(T^1)^3}, \hat{T})$$

and

$$\sup_{0 \leq t \leq \hat{T}} (\|u_{1xx}^\varepsilon(\cdot, t)\|_{L^2(\mathbf{T}^1)} + \|u_{2xx}^\varepsilon(\cdot, t)\|_{L^2(\mathbf{T}^1)}) \leq C(\|\mathbf{u}_0\|_{H^2(\mathbf{T}^1)^3}, \hat{T}).$$

Using

$$u_{3xx} = \frac{u_1 u_{1xx} + u_2 u_{2xx} + e(\mathbf{u})}{u_3},$$

we get

$$\sup_{0 \leq t \leq \hat{T}} \|u_{3xx}^\varepsilon(\cdot, t)\|_{L^2(\mathbf{T}^1)^3} \leq C(\|\mathbf{u}_0\|_{H^2(\mathbf{T}^1)^3}, \hat{T}).$$

Thus, we obtain

$$\sup_{0 \leq t \leq \hat{T}} \|\mathbf{u}_{xx}^\varepsilon(\cdot, t)\|_{L^2(\mathbf{T}^1)^3} \leq C(\|\mathbf{u}_0\|_{H^2(\mathbf{T}^1)^3}, \hat{T})$$

In view of Eq. (8), we have

$$\sup_{0 \leq t \leq \hat{T}} \|\mathbf{u}_t^\varepsilon(\cdot, t)\|_{L^2(\mathbf{T}^1)^3} \leq C(\|\mathbf{u}_0\|_{H^2(\mathbf{T}^1)^3}, \hat{T})$$

Taking the limit as  $\varepsilon$  goes to zero, we can construct a strong solution locally. Continuation of strong solutions in the large can be established in the same way as before. This completes the proof.

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